Problem 2.1.5 done in class

Problem 2.1.7

$u_{xx} + u_{yy} = 0$

$u(x,0) = 0, \; u_y(x,0) = -k^{-1} \sin kx, \; k > 0$

Similarly to how we worked out spherical harmonics in class, ~~[illegible]~~ ~~[illegible]~~ ~~[illegible]~~ we look for a solution:

$$u(x,y) = X(x) \cdot Y(y), \text{ then}$$

$$Y \cdot X'' + X \cdot Y'' = 0 \quad \text{or}$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

$\boxed{\lambda \text{ can be either real or } \cancel{\ldots} \text{ complex}}$

(1) $X'' + \lambda^2 X = 0 \Rightarrow X = a \cos \lambda x + b \sin \lambda x$

(2) $Y'' - \lambda^2 Y = 0 \Rightarrow Y = c e^{\lambda y} + d e^{-\lambda y}$

Initial conditions:

$u(x,0) = X(x) Y(0) = (a \cos \lambda x + b \sin \lambda x) \cdot$

$\cdot (c + d) = 0$

$u_y(x,0) = X(x) \cdot (c\lambda - d\lambda) = k^{-1} \sin kx$

i.e. $c = -d$ and

$(a \cos \lambda x + b \sin \lambda x) \cdot \lambda \, 2 \cdot c = k^{-1} \sin kx$

$\Rightarrow a = 0, \; \lambda = k, \; b \cdot \lambda \cdot 2 \cdot c =$

$= b \cdot k \cdot 2 \cdot c = k^{-1}$

$\Rightarrow 2 b \cdot c = k^{-2} \Rightarrow b \cdot c = \frac{1}{2} k^{-2}$

$u(x,y) = b \cdot \sin kx \cdot c \cdot (e^{ky} - e^{-ky})$

$= \frac{1}{2} k^{-2} \sin kx \cdot (e^{ky} - e^{-ky})_{z}$

$$= \boxed{k^{-2} \cdot \sin kx \cdot \sinh ky}$$

Notice, as $k \to \infty$ $\sin kx$ is bounded, while $\sinh ky \to \frac{1}{2} e^{ky}$ $y > 0$ and $\sinh ky \to -\frac{1}{2} e^{-ky}$, $y < 0$

Since $\lim\limits_{k \to \infty} \frac{e^{ky}}{k^2} = \infty$ (exponential grows faster than any power) for any $y > 0$.

and similarly for $\frac{e^{-ky}}{k^2}$, $y < 0$, we have that the problem is not well posed, since the initial data $\to 0$.

Also, see Section V4.1.a

Problem Find characteristics of
$x^2 u_{xx} - 2x u_{xy} + \frac{3}{4} u_{yy} + \frac{1}{2} u_y = 0$

Read Sections 2.2.a and 2.2.b

Equation for characteristics (eq-n (10) in Section 2.2.a):

$$\frac{dy}{dx} = \frac{-2x \pm \sqrt{4x^2 - 4x^2 \cdot \frac{3}{4}}}{2x^2} =$$

3

$$= \frac{-2x \pm |x|}{2x^2} \Rightarrow \quad x \geq 0: \begin{cases} -\frac{3}{2}\frac{1}{x} \\ -\frac{1}{2}\frac{1}{x} \end{cases} \quad x < 0: \begin{cases} -\frac{1}{2}\frac{1}{x} \\ -\frac{3}{2}\frac{1}{x} \end{cases}$$

$x \geq 0$: $dy = c\frac{1}{x}dx$, $c < 0 = -\frac{3}{2}$ or $-\frac{1}{2}$

$$y = c\ln(x) + const$$

$$y = \begin{cases} -\frac{3}{2}\ln(x) + const \\ -\frac{1}{2}\ln(x) + const \end{cases}$$

$x < 0$: $dy = c\frac{1}{x}dx$, $c < 0$ or

$$dy = c\frac{1}{(-x)}d(-x)$$

$$y = c\ln(-x) + const$$

$$y = \begin{cases} -\frac{3}{2}\ln(-x) + const \\ -\frac{1}{2}\ln(-x) + const \end{cases}$$

Characteristics: $y = \begin{cases} -\frac{1}{2}\ln(|x|) + const \\ -\frac{3}{2}\ln(|x|) + const \end{cases}$

The coord change:

$\mu(x,y) = y + \frac{1}{2}\ln(|x|)$
$\eta(x,y) = y + \frac{3}{2}\ln(|x|)$ reduce this

eq-n to the canonical form
$u_{\mu\eta} = d(\mu, \eta, u, u_\mu, u_\eta)$.

~~First:~~ $\cancel{3\mu = 3y + \frac{3}{2}\ln(|x|)} \Rightarrow$
$\cancel{\eta = y + \frac{3}{2}\ln(|x|)}$

We now compute this form:

$\mu_x = \frac{1}{2}\frac{1}{|x|}\,\mathrm{sign}(x) = \frac{1}{2}\frac{1}{x}$

$\mu_y = 1$

$\eta_x = \frac{3}{2}\frac{1}{x}$

$\eta_y = 1$

$u_x = \frac{\partial}{\partial x} u(\mu,\eta) = u_\mu \cdot \mu_x + u_\eta \cdot \eta_x = u_\mu \frac{1}{2x} +$
$+ u_\eta \frac{3}{2x}$

$u_y = \frac{\partial}{\partial y} u(\mu,\eta) = u_\mu \cdot \mu_y + u_\eta \cdot \eta_y = u_\mu + u_\eta$

$u_{xx} = u_\mu\left(-\frac{1}{2x^2}\right) + u_{\mu\mu}\frac{1}{4x^2} + u_{\mu\eta}\frac{3}{4x^2}$
$+ u_\eta\left(-\frac{3}{2x^2}\right) + u_{\eta\mu}\frac{3}{4x^2} + u_{\eta\eta}\frac{9}{4x^2}$

$x^2 u_{xx} = -\frac{1}{2}u_\mu - \frac{3}{2}u_\eta + \frac{1}{4}u_{\mu\mu} + \frac{9}{4}u_{\eta\eta} +$
$+ \frac{3}{2}u_{\eta\mu}$

$u_{yy} = u_{\mu\mu} + 2u_{\mu\eta} + u_{\eta\eta}$

$u_{xy} = u_{\mu\mu}\mu_x + u_{\mu\eta}\eta_x + u_{\eta\mu}\mu_x + u_{\eta\eta}\eta_x$

$= u_{\mu\mu}\frac{1}{2x} + u_{\eta\eta}\frac{3}{2x} + u_{\mu\eta}\, 2\frac{1}{x}$

$-2x \cdot u_{xy} = -u_{\mu\mu} - 3u_{\eta\eta} - 4u_{\mu\eta}$

$\frac{3}{4}u_{yy} = \frac{3}{4}u_{\mu\mu} + \frac{3}{2}u_{\mu\eta} + \frac{3}{4}u_{\eta\eta}$

$x^2 u_{xx} - 2x\,u_{xy} + \frac{3}{4}u_{yy} = u_{\mu\mu}\left(\frac{1}{4} - \cancel{1} + \frac{3}{4}\right)$

$+ u_{\eta\eta}\left(\frac{9}{4} - \cancel{3} + \frac{3}{4}\right) + u_{\mu\eta}\left(\frac{3}{2} - 4 + \frac{3}{2}\right) = \frac{1}{2}u_\mu - \frac{3}{2}u_\eta =$

5

$= -u_{\mu\eta}$, i.e. the eq-n becomes:
$-\frac{1}{2}u_\mu - \frac{3}{2}u_\eta$

$-u_{\mu\eta} + \left[-\frac{1}{2}u_\mu - \frac{3}{2}u_\eta + \frac{1}{2}u_y\right] =$

$= -u_{\mu\eta} + \left[-\cancel{\frac{1}{2}u_\mu} - \frac{3}{2}u_\eta + \cancel{\frac{1}{2}u_\mu} + \frac{1}{2}u_\eta\right] =$

$= -u_{\mu\eta} - u_\eta = 0$, i.e.

$u_{\mu\eta} = -u_\eta$ integrate once:

$u_\mu = -u + F(\mu)$, integrate again:

$\boxed{u = G(\eta)\cdot e^{-\mu} + \tilde{F}(\mu)}$,

$u_\mu = -G(\eta)\cdot e^{-\mu} + \frac{d\tilde{F}(\mu)}{d\mu} = -G(\eta)e^{-\mu} + \tilde{F}'(\mu)$

and this has to be equal to
$-u + F(\mu) = -G(\eta)\cdot e^{-\mu} - \tilde{F}(\mu) + F(\mu)$, i.e. we must have

$\tilde{F}'(\mu) = -\tilde{F}(\mu) + F(\mu) \Rightarrow$

$\Rightarrow \tilde{F}(\mu) = e^{-\mu} + \hat{F}(\mu)$ where

$\hat{F}(\mu)$ is an antiderivative of $F(\mu)$. Since $F(\mu)$ is and arbitrary function, so is $\hat{F}(\mu)$

$$u = G(\eta) e^{-\mu} + e^{-\mu} + \hat{F}(\mu)$$

$$\boxed{u = \hat{G}(\eta) e^{-\mu} + \hat{F}(\mu)}$$

where $\hat{G}(\eta)$ and $\hat{F}(\mu)$ are arbitrary C^1 functions is a general solution

Problem 3.4(a)

$$u_{tt} - c^2 u_{xx} = 0$$

$$u(x,0) = x^3, \quad u_t(x,0) = \sin x$$

A straightforward application of d'Alembert:

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) +$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)\,dz = \frac{1}{2}\left((x+ct)^3 + (x-ct)^3 +\right.$$

$$\left.+ \frac{1}{2c}\int_{x-ct}^{x+ct} \sin z\,dz = \frac{1}{2}(2x^3 + 6xc^2t^2\right) -$$

$$- \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right)$$

$$= x^3 + 3xc^2t^2 - \frac{1}{2c}\left(\cos(x+ct) - \cos(x-ct)\right)$$