HADAMARD-PERRON THEOREM

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1. Invariant manifold of a fixed point

He we will discuss the simplest possible case in which the existence of invariant manifolds arises: the Hadamard-Perron theorem.

Definition 1. *Given a smooth map $T : X \to X$, X being a Riemannian manifold, and a fixed point $p \in X$ (i.e. $Tp = p$) we call* (local) stable manifold *(of size δ) a manifold $W^s(p)$ such that*[1]

$$W^s(p) = \{x \in B_\delta(x) \subset X \mid \lim_{n\to\infty} d(T^n x, p) = 0\}.$$

Analogously, we will call (local) unstable manifold *(of size δ) a manifold $W^u(p)$ such that*

$$W^u(p) = \{x \in B_\delta(x) \subset X \mid \lim_{n\to\infty} d(T^{-n} x, p) = 0\}.$$

It is quite clear that $TW^s(p) \subset W^s(p)$ and $TW^u(p) \supset W^u(p)$ (Problem 1). Less clear is that these sets deserve the name "manifold." Yet, if one thinks of the Arnold cat at the point zero (which is a fixed point) it is obvious that the stable and unstable manifolds at zero are just segments in the stable and unstable direction, the next Theorem shows that this is a quite general situation.

Theorem 1.1 (Hadamard-Perron)**.** *Consider an invertible map $T : U \subset \mathbb{R}^2 \to \mathbb{R}^2$, $T \in \mathcal{C}^1(U, \mathbb{R}^2)$, such that $T0 = 0$ and*

$$D_0 T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \tag{1.1}$$

where $0 < \mu < 1 < \lambda$.[2] *That is, the map T is hyperbolic at the fixed point 0. Then there exists stable and unstable manifolds at 0. Moreover, $\mathcal{T}_0 W^{s(u)}(0) = E^{s(u)}(0)$ where $E^{s(u)}(0)$ are the expanding and contracting subspaces of D_0T.*

Proof. We will deal explicitly only with the unstable manifold since the stable one can be treated exactly in the same way by considering T^{-1} instead of T.

Since the map is continuously differentiable for each $\varepsilon > 0$ we can choose $\delta > 0$ so that, in a 2δ-neighborhood of zero, we can write

$$T(x) = D_0 T x + R(x) \tag{1.2}$$

where $\|R(x)\| \le \varepsilon \|x\|$, $\|D_x R\| \le \varepsilon$.

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[1] Sometime we will write $W^s_\delta(p)$ when the size really matters. By $B_\delta(x)$ we will always mean the open ball of radius δ centered at x.

[2] Notice that if D_0T has eigenvalues $0 < \mu < 1 < \lambda$ then one can always perform a change of variables such that (1.1) holds. Also I am assuming real positive eigenvalues just for simplicity, complex eigenvalues with $|\Re(\mu)| < 1 < |\Re(\lambda)|$ would as well. Also note that the dimension is rather irrelevant in the following, an extension to operators on Banach space would hold almost verbatim.

1.0.1. ***Existence–a fixed point argument.*** The first step is to decide how to represent manifolds. In the present case, since we deal only with curves, it seems very reasonable to consider the set of curves $\Gamma_{\delta,c}$ passing through zero and "close" to being horizontal, that is the differentiable functions $\gamma : [-\delta,\, \delta] \to \mathbb{R}^2$ of the form

$$\gamma(t) = \begin{pmatrix} t \\ u(t) \end{pmatrix}$$

and such that $\gamma(0) = 0$; $\|(1,0) - \gamma'\|_\infty \le c$. It is immediately clear that any smooth curve passing through zero and with tangent vector, at each point, in the cone $\mathcal{C} := \{(a,\, b) \in \mathbb{R}^2 \mid |\frac{b}{a}| \le c\}$, can be associated to a unique element of $\Gamma_{\delta,c}$, just consider the part of the curve contained in the strip $\{(x, y) \in \mathbb{R}^2 \mid |x| \le \delta\}$. Moreover, if $\gamma \in \Gamma_{\delta,c}$ then $\gamma \subset B_{2\delta}(0)$, provided $c \le 1/2$.

Notice that it suffices to specify the function u in order to identify uniquely an element in $\Gamma_{\delta,c}$. It is then natural to study the evolution of a curve through the change in the associated function.

To this end let us investigate how the image of a curve in $\Gamma_{\delta,c}$ under T looks like.

$$T\gamma(t) = \begin{pmatrix} \lambda t + R_1(t, u(t)) \\ \mu u(t) + R_2(t, u(t)) \end{pmatrix} := \begin{pmatrix} \alpha_u(t) \\ \beta_u(t) \end{pmatrix}.$$

At this point the problem is clearly that the image it is not expressed in the way we have chosen to represent curves, yet this is easily fixed. First of all, $\alpha_u(0) = \beta_u(0) = 0$. Second, by choosing $\varepsilon < \lambda$, we have $\alpha_u'(t) > 0$, that is, α_u is invertible. In addition, $\alpha_u([-\delta,\, \delta]) \supset [-\lambda\delta + \varepsilon\delta,\, \lambda\delta - \varepsilon\delta] \supset [-\delta,\, \delta]$, provided $\varepsilon \le \lambda - 1$. Hence, α_u^{-1} is a well defined function from $[-\delta,\, \delta]$ to itself. Finally,

$$|\frac{d}{dt}\beta_u \circ \alpha_u^{-1}(t)| = \left|\frac{\beta_u'(\alpha_u^{-1}(t))}{\alpha_u'(\alpha_u^{-1}(t))}\right| \le \frac{\mu c + \varepsilon}{\lambda - \varepsilon} \le c$$

where, again, we have chosen $\varepsilon \le \frac{c(\lambda-\mu)}{1+c}$.

We can then consider the map $\tilde{T} : \Gamma_{\delta,c} \to \Gamma_{\delta,c}$ defined by

$$\tilde{T}\gamma(t) := \begin{pmatrix} t \\ \beta_u \circ \alpha_u^{-1}(t) \end{pmatrix} \tag{1.3}$$

which associates to a curve in $\Gamma_{\delta,c}$ its image under T written in the chosen representation. It is now natural to consider the set of functions $B_{\delta,c} = \{u \in \mathcal{C}^1([-\delta,\, \delta]) \mid u(0) = 0,\ |u'|_\infty \le c\}$ in the vector space $\mathrm{L}ip([-\delta,\, \delta])$.[3] As we already noticed $B_{\delta,c}$ is in one-one correspondence with $\Gamma_{\delta,c}$, we can thus consider the operator $\hat{T} : \mathrm{L}ip([-\delta,\, \delta]) \to \mathrm{L}ip([-\delta,\, \delta])$ defined by

$$\hat{T}u = \beta_u \circ \alpha_u^{-1} \tag{1.4}$$

From the above analysis follows that $\hat{T}(B_{\delta,c}) \subset B_{\delta,c}$ and that $\hat{T}u$ determines uniquely the image curve.

The problem is then reduced to studying the map $\hat{T}$. The easiest, although probably not the most productive, point of view is to show that $\hat{T}$ is a contraction in the sup norm. Note that this creates a little problem since $\mathcal{C}^1$ it is not closed in the sup norm (and not even $\mathrm{L}ip([-\delta,\, \delta])$ is closed). Yet, the set $B^*_{\delta,c} = \{u \in$

[3] This are the Lipschitz functions on $[-\delta,\, \delta]$, that is the functions such that $\sup_{t,s\in[-\delta,\,\delta]} \frac{|u(s)-u(t)|}{|t-s|} < \infty$.

$\mathrm{L}ip([-\delta,\,\delta])\mid u(0)=0, \sup_{t,s\in[-\delta,\,\delta]}\frac{|u(s)-u(t)|}{|t-s|}\leq c\}$ is closed (see Problem 2). Thus $\overline{B_{\delta,c}}\subset B^*_{\delta,c}$. This means that, if we can prove that the sup norm is contracting, then the fixed point will belong to $B^*_{\delta,c}$ and we will obtain only a Lipschitz curve. We will need a separate argument to prove that the curve is indeed smooth.

Let us start to verify the contraction property. Notice that

$$\alpha_u^{-1}(t)=\lambda^{-1}t+\lambda^{-1}R_1(\alpha_u^{-1}(t),u(\alpha_u^{-1}(t))),$$

thus, given $u_1,u_2\in B_{\delta,c}$, by Lagrange Theorem

$$\begin{aligned}|\alpha_{u_1}^{-1}(t)-\alpha_{u_2}^{-1}(t)|&\leq\lambda^{-1}|\langle\nabla_\zeta R_1,\,(\alpha_{u_1}^{-1}(t)-\alpha_{u_2}^{-1}(t),u_1(\alpha_{u_1}^{-1}(t))-u_2(\alpha_{u_2}^{-1}(t)))\rangle|\\&\leq\frac{\varepsilon}{\lambda}\left\{|\alpha_{u_1}^{-1}(t)-\alpha_{u_2}^{-1}(t)|+|u_1(\alpha_{u_2}^{-1}(t))-u_2(\alpha_{u_2}^{-1}(t))|\right\}.\end{aligned}$$

This implies immediately

$$|\alpha_{u_1}^{-1}(t)-\alpha_{u_2}^{-1}(t)|\leq\frac{\lambda^{-1}\varepsilon}{1-\lambda^{-1}\varepsilon}\|u_1-u_2\|_\infty. \tag{1.5}$$

On the other hand

$$\begin{aligned}|\beta_{u_1}(t)-\beta_{u_2}(t)|&\leq\mu|u_1(t)-u_2(t)|+|\langle\nabla_\zeta R_2,\,(0,u_1(t)-u_2(t))\rangle|\\&\leq(\mu+\varepsilon)\|u_1-u_2\|_\infty.\end{aligned} \tag{1.6}$$

Moreover,

$$|\beta_u'(t)|\leq\mu+\varepsilon. \tag{1.7}$$

Collecting the estimates (1.5, 1.6, 1.7) readily yields

$$\begin{aligned}\|\hat T u_1-\hat T u_2\|_\infty&\leq\|\beta_{u_1}\circ\alpha_{u_1}^{-1}-\beta_{u_1}\circ\alpha_{u_2}^{-1}\|_\infty+\|\beta_{u_1}\circ\alpha_{u_2}^{-1}-\beta_{u_2}\circ\alpha_{u_2}^{-1}\|_\infty\\&\leq\left\{[\mu+\varepsilon]\frac{\lambda^{-1}\varepsilon}{1-\lambda^{-1}\varepsilon}+(\mu+\varepsilon)\right\}\|u_1-u_2\|_\infty\\&\leq\sigma\|u_1-u_2\|_\infty,\end{aligned}$$

for some $\sigma\in(0,1)$, provided ε is chosen small enough.

Clearly, the above inequality immediately implies that there exists a unique element $\gamma_*\in\Gamma_{\gamma,c}$ such that $\tilde T\gamma_*=\gamma_*$, this is the *local* unstable manifold of 0.

1.0.2. ***Regularity–a cone field.*** As already mentioned, a separate argument is needed to prove that γ_* is indeed a $\mathcal{C}^1$ curve.

To prove this, one possibility could be to redo the previous fixed point argument trying to prove contraction in $\mathcal{C}^1_{Lip}$ (the $\mathcal{C}^1$ functions with Lipschitz derivative); yet this would require to increase the regularity requirements on T. A more geometrical, more instructive and more inspiring approach is the following.

Define the cone field $\mathcal{C}_{\theta,h}(x,u):=\{\xi\in B_h(x)\mid(a,b)=\xi-x;\ a\neq0;\ |\frac{b}{a}-u|\leq\theta\}$, with $|u|\leq c\delta$, $\theta\leq c\delta$ and $h\leq\delta$. By construction $B_h(x)\cap\gamma_*\subset\mathcal{C}_{c\delta,h}$ for each $x\in\gamma_*$. We will study the evolution of such a cone field on γ_*.

For all $\xi\in\mathcal{C}_{\theta,h}(x,u)$, if $(a,b)=\xi-x$ and $(\alpha,\beta)=T\xi-Tx$, it holds

$$(\alpha,\beta)=D_xT(a,b)+\mathcal{O}(C\|(a,b)\|^2).$$

Thus, setting $(\alpha',\beta')=D_xT(a,b)$ and $u'=\frac{\beta'}{\alpha'}$, one can compute

$$\left|\frac{\beta}{\alpha}-u'\right|\leq\mu\lambda^{-1}[c_1h+\theta],$$

for some constant c_1 depending only on T and δ. Accordingly, if $h \leq c_2\theta$, for same appropriate constant c_2, and δ is small enough, there exists $\sigma \in (0,1)$ such that

$$B_h(x) \cap T\mathcal{C}_{\theta,h}(x,u) \subset \mathcal{C}_{\sigma\theta,h}(Tx,u').$$

Hence, if $x \in \gamma_*$, $\gamma_* \cap B_{\sigma^n h}(T^{-n}x) \subset \mathcal{C}_{c\delta,\sigma^n h}(T^{-n}x,0)$ and, since $T^{-n}\gamma_* \subset \gamma_*$,

$$\gamma_* \cap B_{\sigma^n h}(x) \subset \mathcal{C}_{\sigma^n c,\sigma^n h}(x,v_n) \tag{1.8}$$

where $(a, av_n) = D_{T^{-n}x}T^n(1,0)$, for some $a \in \mathbb{R}_+$.

The estimate (1.8) clearly implies

$$\gamma_*'(x) = (1, \lim_{n\to\infty} v_n) \tag{1.9}$$

which indeed exists (see Problem 3). □

There is an issue not completely addresses in our formulation of Hadamard-Perron theorem: the uniqueness of the manifolds.[4] It is not hard to prove that $W^{s(u)}(p)$ are indeed unique (see Problem 4).

There is another point of view that can be adopted in the study of stable and unstable manifolds: to "grow" the manifolds. This is done by starting with a very short curve in $\Gamma_{\delta,c}$, e.g. $\gamma_0(t) = (t,0)$ for $t \in [\lambda^{-n}\delta,\, \lambda^n\delta]$, and showing that the sequence $\gamma_n := T^n\gamma_0$ converges to a curve in the strip $[-\delta,\, \delta]$, independent of γ_0. From a mathematical point of view, in the present case, it corresponds to spell out explicitly the proof of the fixed point theorem. Nevertheless, it is a more suggestive point of view and it is more convenient when the hyperbolicity is non uniform. For example consider the map[5].

$$T\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 2x - \sin x + y \\ x - \sin x + y \end{pmatrix} \tag{1.10}$$

then 0 is a fixed point of the map but

$$D_0T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not hyperbolic, yet, due to the higher order terms, there exist stable and unstable manifolds (see Problems 6, 7, 8).

Problems

1 Show that, if p is a fixed point, then $TW^s(p) \subset W^s(p)$ and $TW^u(p) \supset W^u(p)$.

2 Prove that the set $B^*_{\delta,c}$ in section 1 is closed with respect to the sup norm $\|u\|_\infty = \sup_{t\in[-\delta,\delta]} |u(t)|$.

3 Prove that the limit in (1.9) is well defined.

4 Prove that, in the setting of Theorem 1.1, the unstable manifold is unique. (Hint: This amounts to show that the set of points that are attracted to zero are exactly the manifolds constructed in Theorem 1.1. Use the local hyperbolicity to show that.)

5 Show that Theorem 1.1 holds assuming only $T \in \mathcal{C}^1(U,U)$.

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[4] Namely the doubt may remain that a less regular set satisfying Definition 1 exists.

[5] Some times this is called *Lewowicz map*

6 Consider the Lewowicz map (1.10), show that, given the set of curves $\Gamma_{\delta,c} := \{\gamma : [-\delta, \delta] \to \mathbb{R}^2 \mid \gamma(t) = (t, u(t));\ \gamma(0) = 0;\ |u'(t)| \in [c^{-1}t, ct]\}$, it is possible to construct the map $\tilde{T} : \Gamma_{\delta,c} \to \Gamma_{\delta(1+c^{-1}\delta),\, c}$ in analogy with (1.3).

7 In the case of the previous problem show that for each $\gamma_i \in \Gamma_{\delta,c}$ holds $d(\tilde{T}\gamma_1, \tilde{T}\gamma_2) \leq (1 - c\delta)d(\gamma_1, \gamma_2)$.

8 Show that for the Lewowicz map zero has a unique unstable manifold. (Hint: grow the manifolds, that is, for each $n > 1$ define $\delta_n := \frac{\rho}{n}$. Show that one can choose ρ such that $\delta_{n-1} \geq \delta_n(1+c^{-1}\delta_n)$. according to Problem 6 it follows that $\tilde{T} : \Gamma_{\delta_n,c} \to \Gamma_{\delta_{n-1},c}$. Moreover,

$$d(\tilde{T}^{n-1}\gamma_1, \tilde{T}^{n-1}\gamma_2) \leq \prod_{i=1}^{n}(1 - c\delta_i)d(\gamma_1, \gamma_2).$$

Finally, show that, setting $\gamma_n(t) = (0, t) \in \Gamma_{\delta_n,c}$, the sequence $\tilde{T}^{n-1}\gamma_n$ is a Cauchy sequence that converges in $\mathcal{C}^0$ to a curve in $\Gamma_{1,c}$ invariant under $\tilde{T}$.)

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