Siegel's theorem

An Application of KAM Theory to Complex Dynamics

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Basic definitions of complex dynamics

Definition

A map is called ***conformal*** *if it is analytic and one-to-one.*

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Let f be an analytic function and z_0 a fixed point, $f(z_0) = z_0$. We define the ***multiplier*** *of f at z_0 as*

$$\lambda := f'(z_0).$$

We distinguish between the following types of multipliers:

1. ***Attracting:*** $|\lambda| < 1$. (***Superattracting*** *if $\lambda = 0$).*
2. ***Repelling:*** $|\lambda| > 1$.
3. ***Rationally neutral:*** $|\lambda| = 1$ *and* $\lambda^n = 1$ *for some* $n \in \mathbb{Z}$.
4. ***Irrationally neutral:*** $|\lambda| = 1$ *but never* $\lambda^n = 1$.

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*A function $f : U \to U$ is said to be (**conformally**) **conjugated** to a function $g : V \to V$ if there exists a conformal mapping $\varphi : U \to V$ such that*

$$g = \varphi \circ f \circ \varphi^{-1} \quad \Leftrightarrow \quad \varphi(f(z)) = g(\varphi(z)).$$

*This expression is called **Schröder Equation**. Sometimes it is more convenient to define $h = \varphi^{-1}$ so $h^{-1} \circ f \circ h = g$.*

Remark

φ maps fixed points into fixed points and keeps the multipliers.

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Statement of the problem

Given an analytic function f defined near a fixed point z_0,

$$f(z) = z_0 + \lambda(z - z_0) + a(z - z_0)^p + \ldots$$

we know that f "looks like"

$$\begin{cases} g(\zeta) = \lambda\zeta & \text{if } \lambda \neq 0 \\ g(\zeta) = a\zeta^p & \text{if } \lambda = 0 \ a \neq 0, \end{cases}$$

where $\zeta = z - z_0$. But is there always a φ conjugating f to g?

On the uniqueness of conjugation

Lemma

If $\lambda \neq 0$ and λ is not a root of unity, then the conjugation φ is unique up to a scale factor.

Proof.

It suffices to show that any conjugation from $f(z) = \lambda z$ to itself is a constant multiple of z. Indeed, if $\varphi(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots$ and $\varphi(\lambda z) = \lambda \varphi(z)$ then

$$a_1 \lambda z + a_2 \lambda^2 z^2 + a_3 \lambda^3 z^3 + \ldots = a_1 \lambda z + a_2 \lambda z^2 + a_3 \lambda z^3 + \ldots,$$

so we have $a_n \lambda^n = a_n \lambda$. As a conclusion, $a_n = 0$ for $n \geq 2$ and $\varphi(z) = a_1 z$. □

A similar result holds in the superattracting case.

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On the existence of solutions

Regarding the existence of solutions:

- **Attracting case:** True.
- **Superattracting case:** True.
- **Repelling case:** True.
- **Rationally neutral:** In general false. Something can be done though.
- **Irrationally neutral:** Let's see.

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Bad news

Theorem

There exists a $\lambda = e^{2\pi i\theta}$ such that the Schröder Equation has no solution for any polynomial f.

Definition

A real number is called ***Diophantine*** *if it is badly approximable by rational numbers, in the sense that there exist $c > 0$ and $\mu < \infty$ so that*

$$\left|\theta - \frac{p}{q}\right| \geq \frac{c}{q^{\mu}} \quad \forall p, q \in \mathbb{Z}, q \neq 0,$$

which occurs if and only if

$$|\lambda^j - 1| \geq c j^{1-\mu}, \qquad \forall j \geq 1.$$

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Good news

It can be shown that *almost all* real numbers are Diophantine, and:

Theorem (Siegel)

If θ is Diophantine and f is an analytic map with a fixed point in 0 with multiplier $\lambda = e^{2\pi i\theta}$, then there exists a solution to the Schröder Equation, that is, f can be conjugated near 0 to multiplication by λ.

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Strategy

$$\zeta = h(z) \xrightarrow{\quad f \quad} f(\zeta)$$

$$g_1 = {\psi_1}^{-1} \circ f \circ \psi_1$$

$$g_2 = {\psi_2}^{-1} \circ {\psi_1}^{-1} \circ f \circ \psi_1 \circ \psi_2$$

$$g_3 = {\psi_3}^{-1} \circ {\psi_2}^{-1} \circ {\psi_1}^{-1} \circ f \circ \psi_1 \circ \psi_2 \circ \psi_3$$

$$\psi_1, \psi_2, \psi_3, \ldots \qquad {\psi_1}^{-1}, {\psi_2}^{-1}, {\psi_3}^{-1}, \ldots$$

$$h \uparrow \qquad \downarrow h^{-1}$$

$$z \xrightarrow{\quad \lambda\cdot \quad} \lambda z$$

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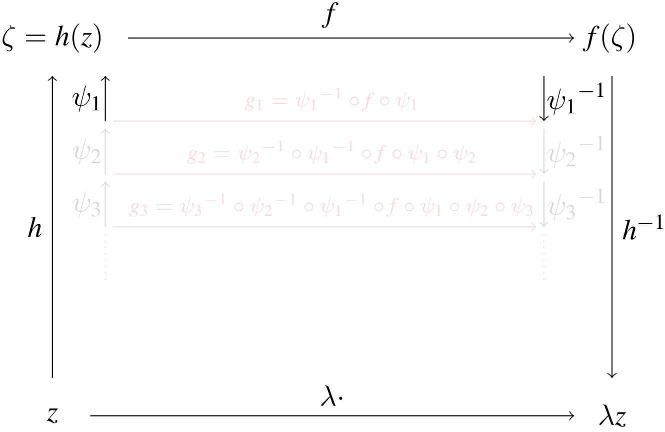
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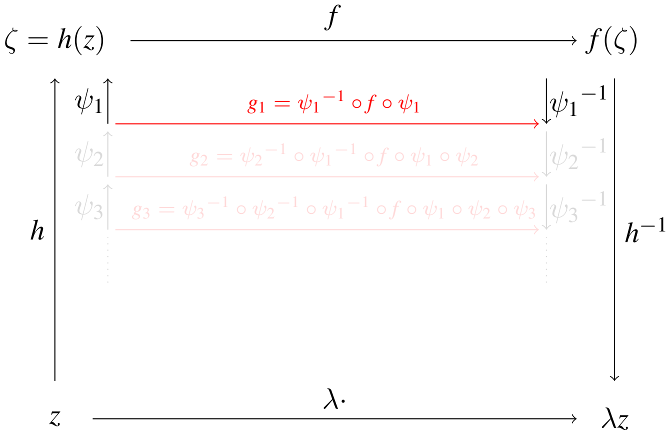
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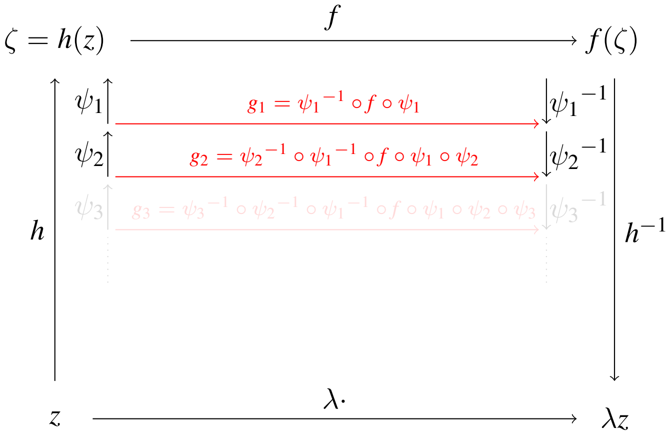
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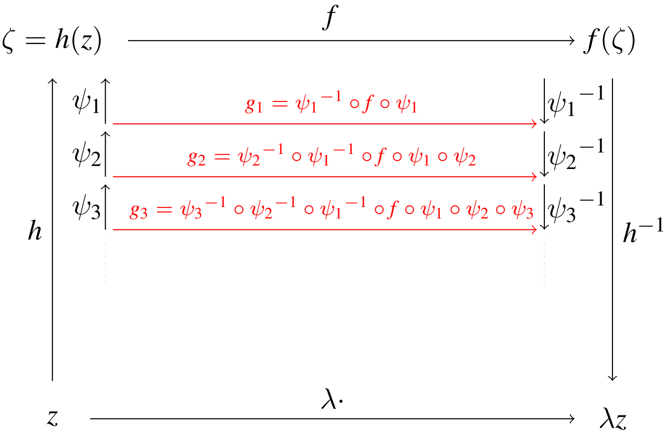
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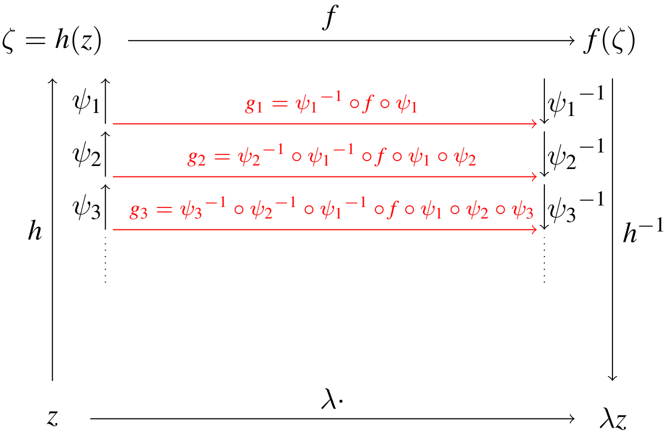
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Proof

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z & \xrightarrow{\; g = \psi^{-1} \circ f \circ \psi \;} & g(z)
\end{array}
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We will use the following notation:

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\begin{cases}
f(z) = \lambda z + \hat{f}(z) = \lambda z + \sum_{j=2}^{\infty} b_j z^j \\
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Construction of $\hat{\psi}(z)$

From Schröder Equation

$$\lambda z + \hat{\psi}(\lambda z) =$$

$$\psi(\lambda z) \simeq h(\lambda z) = f(h(z)) \simeq f(\psi(z)) =$$

$$f(z + \hat{\psi}(z)) = \lambda z + \lambda \hat{\psi}(z) + \hat{f}(z + \hat{\psi}(z)) \simeq \lambda z + \lambda \hat{\psi}(z) + \hat{f}(z),$$

so we obtain

$$\hat{\psi}(\lambda z) - \lambda \hat{\psi}(z) = \hat{f}(z),$$

which can be solved easily

$$\sum_{j=2}^{\infty} a_j (\lambda z)^j - \lambda \sum_{j=2}^{\infty} a_j z^j = \sum_{j=2}^{\infty} b_j z^j \quad \Rightarrow \quad a_j = \frac{b_j}{\lambda^j - \lambda}$$

$$\Rightarrow \hat{\psi}(z) = \sum_{j=2}^{\infty} \frac{b_j}{\lambda^j - \lambda} z^j.$$

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Estimation of $\hat{g}(z)$

We want now to estimate the size of $\hat{g}(z)$. First, we make a list of assumptions:

- $|\lambda^j - 1| \geq c j^{1-\mu}$ so $\dfrac{1}{|\lambda^j - 1|} \leq \dfrac{j^\mu}{cj} \leq \dfrac{\mu!}{c}\dfrac{j^\mu}{\mu!} = c_0 \dfrac{j^\mu}{\mu!}$.
- f defined in $\Delta(0, r)$ and $|\hat{f}'(z)| < \delta$ there, given an $\delta > 0$.
- We define some $0 < \eta < 1/5$.
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- $|\lambda^j - 1| \geq cj^{1-\mu}$ so $\dfrac{1}{|\lambda^j - 1|} \leq \dfrac{j^\mu}{cj} \leq \dfrac{\mu!}{c}\dfrac{j^\mu}{\mu!} = c_0\dfrac{j^\mu}{\mu!}$.
- f defined in $\Delta(0, r)$ and $|\hat{f}'(z)| < \delta$ there, given an $\delta > 0$.
- We define some $0 < \eta < 1/5$.
- $|b_j|\, j\, r^{j-1} \leq \delta$.

Estimation of $\hat{g}(z)$

Lemma (Cauchy estimates)

If f is analytic inside a disk Δ containing a circle C of radius r and centre a, with $|f(z)| \leq M$ on C, then

$$|f^{(n)}(a)| \leq M\, n!\, r^{-n}.$$

In particular, applying Cauchy estimates to $\hat{f}'$ we get

$$\hat{f}(z) = \sum_{j=2}^{\infty} b_j z^j \ \Rightarrow \ \hat{f}'(z) = \sum_{j=2}^{\infty} j b_j z^{j-1} \ \Rightarrow$$

$$|\hat{f}'^{(j-1)}(0)| = |b_j|\, j! \leq \delta (j-1)! r^{1-j}.$$

Estimation of $\hat{\psi}(z)$

Now, for $z \in \Delta(0, r(1-\eta))$:

$$|\hat{\psi}'(z)| = \left| \sum_{j=2}^{\infty} \frac{jb_j}{\lambda^j - \lambda} z^{j-1} \right| \leq \sum_{j=2}^{\infty} \frac{j|b_j|}{|\lambda^j - \lambda|} |z|^{j-1}$$

$$\leq \sum_{j=2}^{\infty} \frac{j|b_j|}{|\lambda^j - \lambda|} r^{j-1}(1-\eta)^{j-1} \leq \frac{\delta}{|\lambda|} \sum_{j=2}^{\infty} \frac{1}{|\lambda^{j-1} - 1|}(1-\eta)^{j-1}$$

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and if we pick $\delta > 0$ small enough so that $c_o \delta < \eta^{\mu+2}$ we can assume $|\hat{\psi}'(z)| \leq \eta$.

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Some remarks on $\hat{\psi}(z)$

We saw that $|\hat{\psi}'(z)| \leq \eta$. From here:

Remark

$\psi(\Delta(0, r(1-4\eta))) \subseteq \Delta(0, r(1-3\eta))$.

Remark

Assuming $\delta < \eta$, $f(\Delta(0, r(1-3\eta))) \subseteq \Delta(0, r(1-2\eta))$.

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ψ maps $\Delta(0, r(1-\eta))$ into $\Delta(0, r(1-2\eta))$ one-to-one.

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Proof.

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$$|\psi(z)| = |\psi'(\hat{z})||z| < (1+\eta)r(1-4\eta) = r(1-3\eta-4\eta^2) < r(1-3\eta).$$

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Injectivity:

- Let $z_1, z_2 \in \Delta(0, r(1-\eta))$ such that $\psi(z_1) = \psi(z_2)$. Then

$$0 = |\psi(z_1) - \psi(z_2)| = |\psi'(\hat{z})||z_1 - z_2| \geq (1-\eta)|z_1 - z_2| \quad \Rightarrow \quad z_1 = z_2.$$

Surjectivity:

- For $z \in \partial\Delta(0, r(1-\eta))$, $|\psi(z)| \geq r(1-2\eta)$, since
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- Given a $y \in \Delta(0, r(1-2\eta))$, we define $F(z) = \psi(z) - y$ and $G(z) = \psi(z)$. We see that
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Theorem (Rouché)

Some remarks on $\hat{\psi}(z)$

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Theorem (Rouché)

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Some remarks on $\hat{\psi}(z)$

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Theorem (Rouché)

Let $K \subset O \subset \mathbb{C}$, with ∂K continuous and F, G holomorphic functions in O, O open. Then if
$$|F(z) - G(z)| < |G(z)| \qquad \forall z \in \partial K,$$
f and g have the same number of roots in K.

Sets and mappings

Thus, we have seen:

$$
\begin{array}{ccc}
\Delta(0, r(1-3\eta)) & \xrightarrow{\quad f \quad} & \Delta(0, r(1-2\eta)) \\
\uparrow \psi & & \downarrow \psi^{-1} \\
\Delta(0, r(1-4\eta)) & \xrightarrow{\quad g \quad} & \Delta(0, r(1-\eta))
\end{array}
$$

So $g = \psi^{-1} \circ f \circ \psi$ maps $\Delta(0, r(1-4\eta))$ into $\Delta(0, r(1-\eta))$.

Estimation of $\hat{g}(z)$

We recall the relation $g(z) = (\psi^{-1} \circ f \circ \psi)(z) = \lambda z + \hat{g}(z)$. Applying ψ to it we get

$$\lambda z + \lambda\hat{\psi}(z) + \hat{f}(z + \hat{\psi}(z)) = f(z + \hat{\psi}(z)) =$$

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Let now be $z \in \Delta(0, r(1-4\eta))$,

$$|\hat{f}(z + \hat{\psi}(z)) - \hat{f}(z)| = |\hat{f}'(\hat{z})||\hat{\psi}(z)| \leq \delta|\hat{\psi}(z)| = \delta|\hat{\psi}'(\hat{z}')||z| < \delta\frac{c_0\delta}{\eta^{\mu+1}}r$$

and therefore, if $C = \sup_\Delta |\hat{g}|$,

$$C \leq \sup_\Delta |\hat{\psi}'| C + \sup_\Delta |\hat{f}(z + \hat{\psi}(z)) - \hat{f}(z)| < \eta C + \delta\frac{c_0\delta}{\eta^{\mu+1}}r \quad \Rightarrow \quad C < \frac{c_0\delta^2 r}{\eta^{\mu+1}}\frac{1}{1-\mu}.$$

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Finally, since for every $z \in \Delta(0, r(1-5\eta))$ there is always a disk of radius η contained in $\Delta(0, r(1-4\eta))$, we can apply again the Cauchy estimates:

$$|\hat{g}'| \leq \frac{c_0\delta^2 r}{\eta^{\mu+1}} \frac{1}{1-\mu} \frac{1}{\mu} = \frac{c_0\delta^2 r}{\eta^{\mu+2}} \frac{1}{1-\mu}.$$

Conclusion: For an f satisfying $|\hat{f}'| < \delta$ in $\Delta(0, r(1-\eta))$ we found a g satisfying $|\hat{g}'| \leq c_0\delta^2 r\eta^{-(\mu+2)}(1-\eta)^{-1}$ in $\Delta(0, r(1-5\eta))$. The following assumptions were made:

- f is defined in $\Delta(0, r_0)$.
- $r \leq r_0$.
- $0 < \eta < 1/5$.
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These can be reduced to choosing η small enough and $\delta < \eta$.

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The iteration

We have now all the ingredients to do the iteration. We define:

- $r_{n+1} = r_n(1 - 5\eta_n)$
- $\eta_{n+1} = \eta_n/2$
- $\delta_{n+1} = c_0 \delta_n{}^2 \eta_n{}^{-(\mu+2)} 2^{-(\mu+2)}$

Lemma

$\forall n \in \mathbb{N}$ *we have* $c_0\delta_n \leq \eta_n{}^{\mu+2}$.

Proof.

We can prove this by induction. For $n = 1$ is true. Now, assuming it true for n,

$$c_0\delta_{n+1} \frac{c_0{}^2 \delta_n{}^2}{\eta_n{}^{\mu+2} 2^{\mu+2}} \leq \frac{\eta_n{}^{2(\mu+2)}}{\eta_n{}^{\mu+2} 2^{\mu+2}} = \left(\frac{\eta_n}{2}\right)^{\mu+2} = \eta_{n+1}{}^{\mu+2}.$$

□

Iteration

By these means we can build sequences ψ_n and g_n, where

$$g_n = {\psi_n}^{-1} \circ {\psi_{n-1}}^{-1} \circ \ldots \circ {\psi_1}^{-1} \circ f \circ \psi_1 \circ \ldots \circ \psi_{n-1} \circ \psi_n.$$

The last step is just to prove that all this makes sense:
$R = r_0 \prod_{n=1}^{\infty}(1 - 5\eta_n) > 0$?
This product is known as the *Q-Pochhammer symbol* $R = r(5\eta, 1/2)_\infty$ and is positive indeed.

Q.E.D.

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