Homework 2

Problem 1. Lienard System
i) Show that the equation

$$x'' + \mu(x^2 - 1)x' + \tanh x = 0, \mu > 0,$$

has exactly one periodic solution, and classify its stability.

ii) Consider the equation

$$x'' + \mu(x^4 - 1)x' + x = 0.$$

a) Prove that the system has a unique stable limit cycle if $\mu > 0$.

b) Using a computer, plot the phase portrait for the case $\mu = 1$.

c) If $\mu < 0$, does the system still have a limit cycle? If so, is it stable or unstable?

Problem 2. Oscillating Chemical Reactions
The **Brusselator** is a simple model of a hypothetical chemical oscillator, named after the home of the scientist who proposed it. In dimensionless form, its dynamics is

$$\begin{aligned}\dot{x} &= 1 - (b+1)x + ax^2y,\\ \dot{y} &= bx - ax^2y.\end{aligned}$$

where $a, b > 0$ are parameters and $x, y \geq 0$ are dimensionless concentrations.

a) Find all fixed points and use the Jacobian to classify them.

b) Sketch the nullclines, and thereby construct a trapping region for the flow.

c) Show that a Hopf bifurcation occurs at some parameter value $b = b_c$, where b_c is to be determined.

d) Does the limit cycle exist for $b > b_c$ or $b < b_c$? Explain, using the Poincaré-Bendixson theorem.

e) Find the approximate period of the limit cycle for $b = b_c$.

Problem 3. Lyapunov exponents in Lorenz

Using numerical integration of two nearby trajectories, estimate the largest Lyapunov exponent in the Lorenz system with the classical parameters $r = 28, \sigma = 10, b = 8/3$. Try several sets of initial conditions.