Weak convergence in l^1. Shur's theorem. A Project in Functional Analysis

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The goal of this project is to show that weak and strong convergence coincide on l^1 and that this is not true for $E = L^1, E^* = L^\infty$. Let $E = l^1$ so that $E^* = l^\infty$ and for $x \in E$ write $x = (x_1, x_2, \ldots, x_i, \ldots)$ and for $f \in E^*$ write $f = (f_1, f_2, \ldots, f_i, \ldots)$. Both spaces are equipped with their usual norms. Proposition 3.5 (ii) states that strong convergence implies weak convergence so one direction is already clear. The other direction will first be proved for weak convergence to zero and then the proof will be generalized to apply for all limits in E. In other words, let (x^n) be a sequence in l^1 such that $x^n \overset{weakly}{\longrightarrow} 0$ (weakly $\sigma(E, E^*)$), then what first needs to be shown is that $||x^n||_1 \longrightarrow 0$. At the very end of this discussion a counter example in $E = L^1, E^* = L^\infty$ will show that the conclusion does not generalise to the big-l counterpart of l^1.

Before we start, lets recollect some results from Brezis *Functional Analysis, Sobolov Spaces and Partial Differential Equations*, Springer (2011).

Baire Category Theorem, Remark 1

Let (X_n) be a sequence of closed sets in X such that $\cup_{i=1}^{\infty} X_i = X$ then there exists n_0 such that $Int X_{n_0} \neq \emptyset$.

Proposition 3.5, parts (i-ii)

Let (x_n) be a sequence in E, then

(i) $x_n \overset{weakly}{\longrightarrow} x$ in $\sigma(E, E^*)$ iff $\langle f, x_n \rangle \longrightarrow \langle f, x \rangle \quad \forall f \in E^*$

(ii) $x_n \longrightarrow x$ strongly then $x_n \overset{weakly}{\longrightarrow} x$ in $\sigma(E, E^*)$

Theorem 3.16 (Banach-Alaougly-Bourbaki)

The closed unit ball $B_{E^*} = \{f \in E^* : \; ||f|| \leq 1\}$ is compact in the weak* topology $\sigma(E^*, E)$

Part I

Let $f, g \in B_{E^*}$ so that $||f||_\infty \leq 1, ||g||_\infty \leq 1$ and define

$$d(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} |f_i - g_i|$$

The first task is to show that d is a metric on the closed unit ball B_{E^*} and that B_{E^*} is compact in the topology induced by this metric.

To see that d is a metric we check that it satisfies the definition of a metric. Let $f, g, h \in B_{E^*}$, then

1. $d(f, g) \geq 0$ since it is a sum of non-negative elements

2. $d(f,g)=0$ iff $\sum_{i=1}^{\infty}\frac{1}{2^i}|f_i-g_i|=0$ iff $|f_i-g_i|=0\ \forall i\geq 1$ i.e. $f_i=g_i\ \forall i\geq 1$ which is equivalent with $||f-g||_\infty=0$ and thus $f=g$

3. $d(f,g)=d(g,f)$ since $\sum_{i=1}^{\infty}\frac{1}{2^i}|f_i-g_i|=\sum_{i=1}^{\infty}\frac{1}{2^i}|g_i-f_i|$

4. $d(f,g)\leq d(f,h)+d(h,g)$ since $\sum_{i=1}^{\infty}\frac{1}{2^i}|f_i-g_i|=\sum_{i=1}^{\infty}\frac{1}{2^i}|f_i-h_i+h_i-g_i|\leq\sum_{i=1}^{\infty}\frac{1}{2^i}|f_i-h_i|+\sum_{i=1}^{\infty}\frac{1}{2^i}|h_i-g_i|$ by applying the triangle inequality for real numbers for each i.

Now, let $\mathbb{B}$ denote the topology induced by d. If X is a compact space and $I:X\longrightarrow Y$ is a continuous function then $I(X)\subset Y$ is compact in the topology on Y, compare with Munkres, James R. *Topology, A First Course*, Prentice-Hall (1974) theorem 5.5. If, moreover, I is surjective then $I(X)=Y$ so Y is compact. Now, let

$$I:(B_{E^*},\sigma(E^*,E))\longrightarrow(B_{E^*},\mathbb{B})$$

be the canonical injection $Ix=x$ for $x\in B_{E^*}$. It is clearly both injective and surjective. Now, by theorem 3.16 the ball B_{E^*} is compact in the weak* topology $\sigma(E^*,E)$, so to show that B_{E^*} is compact in $\mathbb{B}$ it is enough to show that I is continuous. This will be done by showing continuity at every "point" $f^0\in B_{E^*}$, again compare with Munkres (1974). Let $U_{f^0}\subset B_{E^*}$ and $U_{f^0}\in\mathbb{B}$ be a neighbourhood of f_0 in the metric topology, then, to show continuity at f_0, one must show that there exists $V_{f^0}\subset B_{E^*}$, $V_{f^0}\in\sigma(E^*,E)$ such that

$$I(V_{f^0})\subset U_{f^0}\quad\text{i.e.}\quad V_{f^0}\subset U_{f^0}$$

To do this, first note that a basis of neighbourhoods around $f^0\in B_{E^*}$ in $\mathbb{B}$ can be written on the form

$$U_{f^0}(\epsilon)=\{f\in B_{E^*}:d(f,f^0)<\epsilon\}$$

for arbitrary $\epsilon>0$. Using the canonical basis (e^i) on l^1, then by proposition 3.12, a basis of neighbourhoods around $f^0\in B_{E^*}$ in $\sigma(E^*,E)$ can be written on the form

$$V_{f^0}(\delta,n)=\{f\in B_{E^*}:|\langle f-f^0,e^i\rangle|<\delta\ \forall i=1,\ldots,n\}$$

Note that, given $f^0\in B_{E^*}$ and $U_{f^0}\in\mathbb{B}$, we may find $\epsilon>0$ such that $U_{f^0}(\epsilon)\subset U_{f^0}$, with $U_{f^0}(\epsilon)$ as above, being an open ball around f^0 in $\mathbb{B}$. Next, also note that

$$|\langle f-f^0,e^i\rangle|=|\sum_{j=1}^{\infty}(f_j-f_j^0)e_j^i|=|f_i-f_i^0|$$

so $V_{f^0}(\delta,n)$ can be written on the form

$$V_{f^0}(\delta,n)=\{f\in B_{E^*}:|f_i-f_i^0|<\delta,\quad i=1,\ldots,n\}$$

with $\delta>0$ and $n\in\mathbb{N}$ both arbitrary and vary independently of each other. Now, take an arbitrary $f\in V_{f^0}(\delta,n)$, then

$$d(f,f^0)=\sum_{i=1}^{\infty}\frac{|f_i-f_i^0|}{2^i}<\sum_{i=1}^{n}\frac{\delta}{2^i}+\sum_{i=n+1}^{\infty}\frac{|f_i-f_i^0|}{2^i}<\sum_{i=1}^{\infty}\frac{\delta}{2^i}+\sum_{i=n+1}^{\infty}\frac{|f_i|+|f_i^0|}{2^i}$$

$$<\delta+\sum_{i=n+1}^{\infty}\frac{||f||_1+||f^0||_1}{2^i}<\delta+2\sum_{i=n+1}^{\infty}1/2^i=\delta+2/2^n$$

So if we choose n big enough and δ small enough such that $\delta+2/2^n<\epsilon$. Then $d(f,f^0)<\epsilon$ so $f\in U_f^0(\epsilon)$. Hence we have found a $V_{f^0}(\delta,n)\in\sigma(E^*,E)$ contained in $U_f^0(\epsilon)\in\mathbb{B}$, so

$$I(V_{f^0}(\delta,n))=V_{f^0}(\delta,n)\subset U_{f^0}(\epsilon)\subset U_{f^0}$$

Hence $\forall f^0\in B_{E^*}$ and every U_{f^0} we can find $V_{f^0}\in\sigma(E^*,E)$ containing f^0 such that $I(V_{f^0})\subset U_{f^0}$, and therefore I is continuous at every $f^0\in B_{E^*}$. Hence the unit ball is compact in the metric topology.

Part II

Let $\epsilon > 0$ be given and define

$$F_k := \{f \in B_{E^*} : |\langle f, x^n \rangle| \leq \epsilon \; \forall n \geq k\} = \{f \in B_{E^*} : |\sum_{i=1}^{\infty} f_i x_i^n| \leq \epsilon \; \forall n \geq k\}$$

The next task will be to show that there exists some $f^0 \in B_{E^*}$, a constant $\rho > 0$, and an integer k_0 such that if $f \in B_{E^*}$ and $d(f, f^0) > \rho$ then $f \in F_{k_0}$.

Firstly, by compactness of B_{E^*} in the metric topology we see that B_{E^*} must be complete w.r.t d by Heine-Borel theorem, theorem 3.1 in Munkres (1974)

Theorem 1 (Heine-Borel)**.** A metric space is compact iff it is complete and totally bounded

Secondly, since B_{E^*} equipped with d is a complete metric space, we may use the Baire Category Theorem as in remark 1 if we can find a suitable sequence of closed subsets. The subsets F_k are the obvious choice since, as k increases, the restriction on the sets gets milder

$$f \in F_k \implies |\sum_{i=1}^{\infty} f_i x_i^n| \leq \epsilon \; \forall n \geq k \implies |\sum_{i=1}^{\infty} f_i x_i^n| \leq \epsilon \; \forall n \geq k+1$$

so $F_k \subset F_{k+1} \subset \ldots$. Furthermore $\cup_{i=1}^{\infty} F_i = B_{E^*}$, which follows from the fact that x^n converges weakly to zero, i.e. $\forall f \in E^* \; \exists N$ s.t. $\forall n \geq N$ we have $|\langle f, x^n \rangle| \leq \epsilon$, which means that for each $f \in B_{E^*}$ there is a k s.t. $f \in F_k$.

Now F_k is closed by the following argument: assume $f^m \in F_k$ and $f_m \xrightarrow{d} f^0 \in B_{E^*}$, i.e.

$$d(f^0, F^m) = \sum_{i=1}^{\infty} \frac{|f_i^0 - f_i^m|}{2^i} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow \infty$$

then, since each term is non-negative, we must have convergence of components $|f_i^0 - f_i^m| \longrightarrow 0$ as $m \longrightarrow \infty$. Therefore, since $f^m \in F_k$ we have for all $n \geq k$

$$|\sum_{i=1}^{\infty} f_i^0 x_i^n| = \lim_{m \longrightarrow \infty} |\sum_{i=1}^{\infty} f_i^0 x_i^n| \leq \lim_{m \longrightarrow \infty} |\sum_{i=1}^{\infty} f_i^m x_i^n| + \sum_{i=1}^{\infty} |f_i^0 - f_i^m||x_i^n| \leq \epsilon + \sum_{i=1}^{\infty} \lim_{m \longrightarrow \infty} |f_i^0 - f_i^m||x_i^n| = \epsilon$$

by Lebesgue Dominated Convergence theorem with the appropriate counting measure and dominating function $2|x_i^n|$, i.e.

$$f^0, f^m \in B_{E^*} \quad \text{so} \quad |f_i^0 - f_i^m||x_i^n| \leq (||f^0||_\infty + ||f^m||_\infty)|x_i^n| \leq 2|x_i^n| \quad \forall n \geq 1 \quad \text{with} \quad \sum_{i=1}^{n} 2|x_i^n| < \infty$$

since $x^n \in l^1$. Thus $|\langle f^0, x^n \rangle| \leq \epsilon \; \forall n \geq k$ so $f^0 \in F_k$ and we conclude that F_k is closed with respect to the metric topology on B_{E^*}.

Finally, by Baire Category Theorem, there exists some k_0 such that $IntF_{k_0} \neq \emptyset$ which means that $\exists f^0 \in IntF_{k_0} \in F_{k_0}$ and $IntF_{k_0}$ is by definition open with respect to the metric topology, so $\exists \rho > 0$ such that if $f \in B_{E^*}$ and $d(f^0, f) < \rho$ then $f \in F_{k_0}$.

Part III

Fix an integer N such that $\frac{1}{2^{N-1}} < \rho$. The aim is to prove that the following inequality holds

$$||x^n||_1 \leq \epsilon + 2\sum_{i=1}^{N} |x_i^n| \quad \forall n \geq k_0$$

By part 2 we may take an arbitrary $f^0 \in F_{k_0}$ and find a $\rho > 0$ such that if $d(f, f^0) < \rho$ then $f \in F_{k_0}$. A special case of this is if f is on the form $f = (f_1^0, \dots, f_N^0, \pm 1, \pm 1, \dots)$. Then, since $f, f^0 \in B_{E^*}$ and

$$d(f, f^0) = \sum_{i=N+1}^{\infty} \frac{|\pm 1|}{2^i} = \frac{1}{2^{N-1}} < \rho$$

we have that $f \in F_{k_0}$. Therefore, since the signs of f after index N were arbitrary, we have

$$|\sum_{i=1}^{\infty} f_i x_i^n| = |\sum_{i=1}^{N} f_i^0 x_i^n + \sum_{i=N+1}^{\infty} \pm x_i^n| \leq \epsilon \implies |\sum_{i=1}^{N} f_i^0 x_i^n - \sum_{i=N+1}^{\infty} |x_i^n|\,| \leq \epsilon$$

for all $n \geq k$, and since $|f_i^0| \leq ||f^0||_\infty \leq 1$ we have that

$$\sum_{i=N+1}^{\infty} |x_i^n| \leq \epsilon + |\sum_{i=1}^{N} f_i^0 x_i^n| \leq \epsilon + \sum_{i=1}^{N} |f_i^0||x_i^n| \leq \epsilon + \sum_{i=1}^{N} |f_i^0||x_i^n| \leq \epsilon + \sum_{i=1}^{N} |x_i^n|$$

for all $n \geq k$. Adding $\sum_{i=1}^{N} |x_i^n|$ to both sides yields

$$||x^n||_1 = \sum_{i=1}^{\infty} |x_i^n| \leq \epsilon + 2\sum_{i=1}^{N} |x_i^n| \quad \forall n \geq k$$

Part IV

Note that $x^n \overset{weakly}{\longrightarrow} 0$ and the natural basis (e^i) is in E^*, so proposition 3.5 implies that we have component-wise convergence $|x_i^n| \longrightarrow 0$ as $n \longrightarrow \infty$. Applying this to the conclusion of part 3, we have

$$\lim_{n \longrightarrow \infty} ||x^n||_1 \leq \epsilon + \lim_{n \longrightarrow \infty} 2\sum_{i=1}^{N} |x_i^n| = \epsilon$$

since the sum on the right hand side is finite where N was chosen independently of n. The conclusion is that weak convergence to zero implies convergence in norm to zero, i.e. strong convergence to zero, when $E = l^1, E^* = l^\infty$.

Part V

This conclusion does carry over to the general case, when (x^n) is a sequence in l^1 converging weakly to x, i.e. for every $f \in l^\infty$ the sequence $(\langle f, x^n \rangle)$ converges to some limit, then it is true that (x^n) converges to some limit strongly in l^1. This follows from the fact that l^1 is complete, since all of the arguments in parts 2 and 3 are valid no matter how n, m tends to infinity and hold when replacing $\forall n \geq k$ with $\forall n, m \geq k$. Since $\langle f, x^n \rangle$ converges to some limit it is reasonable that for n, m big enough $|\langle f, x^n \rangle - \langle f, x^m \rangle| = |\langle f, x^n - x^m \rangle|$ will be small. Indeed, let $\epsilon > 0$ and define

$$F_k = \{f \in B_{E^*} : |\langle f, x^n - x^m \rangle| \leq \epsilon \quad \forall n, m \geq k\}$$

Observe that for each given f we have $\langle f, x^n - x^m \rangle = \langle f, x^n \rangle - \langle f, x^m \rangle = y_n - y_m$ for a real sequence (y_n). The Cauchy criterion of convergence states that for real sequences convergence is equivalent with Cauchy convergence, which in this case means that the convergence of $\langle f, x^n \rangle = y_n$ implies Cauchy convergence of $\langle f, x^n \rangle - \langle f, x^m \rangle$, which is exactly what defines F_k in this case.

Now, by the same method as above applied to the double indexed sequence $x^n - x^m$ converging weakly to zero, we see that F_k is closed in the metric topology (again, take a sequence f^m converging to f and show $f \in F_k$ as above with $x^n - x^m$) and that Baire Category theorem implies there is an index for which the interior is non-empty. Therefore, as in part 3, $\exists k_0, N$ integers such that

$$||x^n - x^m||_1 \leq \epsilon + \sum_{i=1}^{N} |x_i^n - x_i^m| \quad \forall n, m \geq k_0$$

and again, by using the standard basis of l^∞, it is easy to see that $\lim\limits_{n,m \longrightarrow \infty} |x_i^n - x_i^m| = 0$ for each i. Hence

$$\lim_{n,m \longrightarrow \infty} ||x^n - x^m||_1 \leq \epsilon + \lim_{n,m \longrightarrow \infty} 2 \sum_{i=1}^{N} |x_i^n - x_i^m| = \epsilon$$

This means that given $\hat{\epsilon} > 0$ we can find $\epsilon = \hat{\epsilon}/2$ and an N (to be used as in the previous displays) to derive and M (maximum of all lower bounds on the top-indices of the N terms in the previous display) such that $\lim\limits_{n,m \longrightarrow \infty} ||x^n - x^m||_1 < \hat{\epsilon} \; \forall n, m \geq M$. So (x^n) is Cauchy in l^1 so (x^n) converges strongly to some limit by completeness.

Part VI

Instead of looking at sequences, now consider $E = L^1(0,1)$ so that $E^* = L^\infty(0,1)$. The goal is to show that weak convergence does not imply strong convergence by constructing a sequence u^n in E such that $u^n \overset{weakly}{\longrightarrow} 0$ weakly $\sigma(E, E^*)$ and such that $||u^n||_1 = 1$ for all n.

Define $U_n(x) = \frac{\pi}{2} \sin(2\pi n x)$ and let $f \in L^\infty(0,1)$. Note that $|f| \leq c$ a.e. for some $c > 0$, so $\int_0^1 |f| \leq \int_0^1 c = c$ and thus $f \in L^1(0,1)$. The Fourier coefficients, and especially the sinusoidal part of the Fourier coefficients, go to zero as $n \longrightarrow \infty$ by the Riemann-Lebesgue lemma, compare with theorem 2.2 in Vretblad, Anders, *Fourier Analysis and Its Applications*, Springer (2003).

Theorem 2 (Riemann-Lebesgue lemma)**.** Let $f \in L^1(I)$ for an interval I, then

$$\lim_{n \longrightarrow \infty} \int_I f(u) sin(nu) du = 0$$

In this case the Fourier coefficients for f are on the form $b_n = \langle f, dU_n \rangle = d\langle f, U_n \rangle \longrightarrow 0$ as $n \longrightarrow \infty$, for some normalizing constant $d > 0$. So weak convergence of U_n follows by proposition 3.5 since f was an arbitrary element of the dual space. On the other hand, since sin is a periodic function

$$||U_n||_1 = \int_0^1 \frac{\pi}{2} |\sin(2\pi n x)| dx = \frac{\pi}{2} \frac{1}{2\pi n} \int_0^{2\pi n} |\sin(x)| dx = \frac{\pi}{2} \frac{n}{2\pi n} \int_0^{2\pi} |\sin(x)| dx =$$

$$\frac{\pi}{2} \frac{2n}{2\pi n} \int_0^{\pi} |\sin(x)| dx = \frac{\pi}{2} \frac{2n}{2\pi n} 2 = 1 \;, \quad \forall n$$

This shows that $||U_n||_1 = 1 \; \forall n$ but $U_n \overset{weakly}{\longrightarrow} 0$ so weak convergence does not imply strong convergence in this setting.