Modelling Complex Systems: Project Sheet 1

April 4, 2011

The modelling complex systems course is assessed on the basis of two exercise sheets, of which this is the first one. Each exercise sheet has three 'projects', each consisting of a series of questions. After each question is a number of points associated with the question. The total points over all the questions is 100. To pass the course (grade 3) you must correctly answer questions amounting to at least half the points. In order to get grade 4 you must correctly answer 75% of the questions. In order to get a grade 5 you must correctly answer some of the questions labelled *grade 5 work* and have answered at least 75% correct over all the questions.

The lab sessions of this course are intended as a learning experience and we will provide extensive help with the exercises. To get the most out of this course you should attend these sessions where we will give valuable tips and advice on the problems. You are encouraged to work together in groups but each person should submit their own final set of answers to the questions. It is preferred that simulations should be implemented in Matlab, but it is acceptable to use other languages. We will try to help with programming problems in all languages, but 'guarantee' help only in Matlab.

David Sumpter (Å5401) has an office hour Wednesdays 11-12 (although not in weeks 15 and 16).

The deadline for the first exercise sheet is Monday 2nd of May 2011. Please email hand-ins to **david.sumpter@math.uu.se**. Written hand-ins can also be given to me on paper at the lecture on Monday the 9th. All code should be submitted as an appendix and not as part of the hand-ins.

1 Particles in boxes

Implement a model of particles which can be in one of two connected boxes X and Y. Assume that the total number of particles is N. The state of the model X is the number of particles in X. Assume that on each time step dt of the model there is a probability $p.dt$ per particle of it moving boxes. The probability of moving is the same for both boxes.

1. Implement the above model as a state-based simulation. Start with all the particles in box Y and run the simulation. Give example runs for $N = 10$, $N = 50$, $N = 100$ and $N = 1000$ particles. **(3)**

2. Give a mean field approximation of your model. Let x be the number of particles in box X and write down an equation of the form

   $$\frac{dx}{dt} = f(x)$$

   where $f(x)$ is a function determined by you. Set the initial number of particles $x(0] = 0$ and solve your equation to get an expression for how x changes over time t. Plot your solution on the same figure as the example runs of the previous question. **(4)**

3. Define a criteria for system equilibrium. Investigate how long the simulation takes to reach equilibrium as a function of N. In particular run the simulation repeatedly for different values of N and plot N vs time to equilibrium. Again, compare your answer to the mean-field approximation, for which you can solve to get an analytic expression for $x(t)$. **(3)**

2 Fashion and fads

Consider a group of students Y of whom own iPhones and $X = N - Y$ others who don't. We now investigate various models for how iPhone ownership changes over time.

1. First consider the case where each individual has a probability p per time step of deciding to buy an iPhone. There is also a probability q per time step that an iPhone breaks down. If it breaks down the individual chooses another brand of phone (although we assume for simplicity that they have a short memory and may choose an iPhone again later). Find, by relating this model to that in the 'Particles in boxes' question, the proportion of iPhone owners as a function of p and q. **(2)**

2. Now consider a model where iPhone usage is determined by peer pressure from other students. For non-iPhone users the probability of buying an iPhone per time step is given by

   $$p(Y) = s + (0.2 - s)\frac{Y^2}{Y^2 + (kN)^2}$$

   where k and s are constants. Plot the function $p(Y)$ for various values of k and s, explain what these parameters mean in terms of an individual's decision to buy a phone. **(4)**

3. Implement a state-based simulation using $p(Y)$ and assuming iPhones break down with a constant rate q. Start by investigating iPhone ownership for a small group of 15 students (i.e. $N = 15$), with $s = 0.001$ and $k = 0.58$. Use the simulation to investigate the role of q on the dynamics of this system. In particular, for a values q ranging between 0.01 and 0.2 investigate how the number of individuals in state Y depends upon this parameter. Find a value of q where you get switching backwards and forwards between about half of individuals owning an iPhone to only small numbers owning one. **(4)**

4. Plot a bifurcation diagram showing how the number of iPhone owners depends on q. In particular, run the simulation for a large number of steps for q in the range $[0.01 : 0.01 : 0.2]$ and plot a 'phase transition' diagram showing how the distribution of the number of iPhone owners depends on q. **(4)**

5. Derive a mean-field equation for the model. Set $N = 1000$ and compare the simulation to the mean-field model. Investigate what happens if Apple start by making very reliable phones and gradually lower reliability so break downs or more frequent. *(grade 5 work)* **(3)**

6. Further investigate the mean-field equation in the case where

$$p(Y) = s + (0.2 - s)\frac{Y}{Y + (kN)}$$

   Draw bifurcation diagrams for this case. What is the qualitative difference between this case and that in question 2? *(grade 5 work)* **(3)**

3 Population dynamics

In this project we implement a stochastic model of population dynamics model similar to that presented in lecture 2. Imagine an environment consisting of n resource sites and a population of X_t individuals. We assign individuals to sites uniformly at random, so each site has an equal probability of being chosen. We then apply the following rule. If an individual is alone at a site it produces B offspring, where B is a binomialy distributed random variable with parameters b and p, i.e. with mean bp and variance $bp(1-p)$. These new individuals pass to the next generation. If two or more individuals occupy the same site then they all die and produce no offspring. Empty sites also produce no offspring. The new population X_{t+1} now consists of all the surviving offspring and which now choose sites at random and repeat the procedure.

1. Implement the above model and show how the population changes through time for $p = 0.5$ and $b = 10$, $b = 18$, $b = 30$ and $b = 40$, with n =100,1000 and 5000. **(3)**

2. By running the simulation for $p = 0.5$ and different values of b within a range [1:1:50] and $n = 1000$ draw a bifurcation diagram for your model. **(2)**

3. Write a program to calculate the Entropy and the Lyapunov exponent of your simulations. For example, you should run your simulation first for at least 100 steps then calculate these statistics over at least the next 1000 steps of the simulation. Plot b against these measurements for $n = 1000$. Explain how Entropy and the Lyapunov exponent can be used to characterise the dynamics of the population. **(5)**

4. Make a two-dimensional 'heatmap' of how the Entropy depends upon both b and p for $n = 100$. To make this heatmap run your simulation for different values of p and b and calculate the entropy, store the result in a matrix where the rows correspond to different p values and the columns to different b values. Then use the Matlab imagesc to plot your matrix. **(5)**

5. Extend the stochastic model so that individuals move only locally between sites. Assume that the sites are arranged on a one dimensional

ring of n sites. On each time step all offspring at site i move to a site chosen uniformly at random in the range $[i-d, i+d]$. The reproduction rule, that only sites occupied by single individuals produce B offspring is the same as before. Investigate the dynamics of this model for various values of d and b for $n = 1000$. What role does d play in the population dynamics? Determine for which values of d and b the population becomes extinct and for which it stabilises. *(grade 5 work)* **(5)**