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Algebraic structures

First sheet of exercises

1. Let $\mathcal{P}(X)$ be the set of all subsets of a given set X. Show that $\mathcal{P}(X)$ is a monoid under the binary operation $\cup$. Describe those sets X for which $\mathcal{P}(X)$ is a group.

2. Find the multiplication table for S_3.

3. List the elements of S_4 and find the order of each element.

4. If G and H are groups, then the cartesian product $G \times H$, with binary operation

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$$

is again a group, called the *product* of G and H. Show that $G \times H \xrightarrow{\sim} H \times G$.

5. Let $\varphi : G \to G'$ be an isomorphism of groups. Show that the inverse bijection $\varphi^{-1} : G' \to G$ also is an isomorphism of groups.

6. Let φ be a monomorphism of groups. Show that if α, β are group morphisms with $\varphi\alpha = \varphi\beta$, then $\alpha = \beta$.

7. Let φ be an epimorphism of groups. Show that if α, β are group morphisms with $\alpha\varphi = \beta\varphi$, then $\alpha = \beta$.

8. Prove that a group morphism φ is injective if and only if $\ker\varphi = \{e\}$.

9. Let $\varphi : G \to G'$ be a morphism of groups. Let $x \in G$, and $y = \varphi(x)$. Prove that $o(y) \leq o(x)$, and more precisely $o(y) | o(x)$ in case $o(x) < \infty$.

10. Prove that $\mathrm{Aut}(\mathrm{C}_2 \times \mathrm{C}_2) \xrightarrow{\sim} \mathrm{S}_3$.

Given a natural number $n \geq 2$, the *dihedral group* D_n of index n is the group of all isometries (i.e. distance preserving linear operators) on $\mathbb{R}^2$ leaving the regular n-gon with vertices $\left(\cos\frac{2\pi\nu}{n}, \sin\frac{2\pi\nu}{n}\right)$, $0 \leq \nu \leq n-1$, invariant. It turns out that $\mathrm{D}_n = \{\varrho_0, \ldots, \varrho_{n-1}, \sigma_0, \ldots, \sigma_{n-1}\}$, where ϱ_ν denotes the rotation by angle $\frac{2\pi\nu}{n}$ about O and σ_ν denotes the reflection about the line L_ν through O and $\left(\cos\frac{\pi\nu}{n}, \sin\frac{\pi\nu}{n}\right)$. The multiplication in D_n is given by

$$\varrho_i\varrho_j = \varrho_{i+j}, \qquad \varrho_i\sigma_j = \sigma_{i+j}, \qquad \sigma_i\sigma_j = \varrho_{i-j}, \qquad \sigma_i\varrho_j = \sigma_{i-j} \ .$$

11. Find all morphisms $\mathrm{D}_2 \to \mathrm{D}_3$ and all morphisms $\mathrm{D}_3 \to \mathrm{D}_2$.

12. Show that $\mathrm{D}_2 \xrightarrow{\sim} \mathrm{C}_2 \times \mathrm{C}_2$ and $\mathrm{D}_3 \xrightarrow{\sim} \mathrm{S}_3$.

13. Determine $\mathrm{Aut}(\mathrm{D}_3)$.

Please turn over!

14. Prove that the following statements hold true for all elements x, y, z in a group G, and for all $m, n \in \mathbb{Z}$.

(a) If $xz = yz$, then $x = y$.

(b) If $zx = zy$, then $x = y$.

(c) If $xy = e$, then $x = y^{-1}$ and $y = x^{-1}$.

(d) $(x^{-1})^{-1} = x$.

(e) If $o(x) = \infty$, then $x^m = x^n \;\Leftrightarrow\; m = n$.

(f) If $o(x) = \ell < \infty$, then $x^m = x^n \;\Leftrightarrow\; m \equiv n \pmod{\ell}$.