Uppsala University
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Algebraic number theory
Spring term 2012
April 19

Home assignments

Fourth set

In exercises 13–15, $p \geq 5$ is a prime number, $\zeta = e^{\frac{2\pi}{p}i}$, s is a natural number, and $Z_s = \{\zeta^s, \zeta^{s-1}, \zeta^{-s}, \zeta^{1-s}\}$.

13. Prove that if $\zeta^{p-1} \in Z_s$, then ζ^{p-1} equals precisely one of the numbers $\zeta^s, \zeta^{s-1}, \zeta^{-s}, \zeta^{1-s}$.

14. Prove that if $\zeta^{p-1} \notin Z_s$ and $|Z_s| \leq 3$, then $\zeta^s = \zeta^{1-s}$ and $\zeta^{s-1} = \zeta^{-s}$.

15. Show that the numbers $\zeta^{\frac{p+1}{2}}, \zeta^{\frac{p-1}{2}}, \zeta^{p-1}$ are distinct.

16. Let p be a prime number, $\zeta = e^{\frac{2\pi}{p}i}$, and $1 \leq i \leq p-1$ be a natural number.

(a) Show that the map $\varepsilon_i : \mathbb{Z}[X] \to \mathbb{Z}[\zeta],\ \varepsilon_i(f(X)) = f(\zeta^i)$ is a surjective ring morphism, with $\ker(\varepsilon_i) = (\Phi_p(X))$.

(b) Conclude that the map $\sigma_i : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta],\ \sigma_i(\sum_{j=0}^{p-2} a_j \zeta^j) = \sum_{j=0}^{p-2} a_j \zeta^{ij}$ is a ring automorphism.

Every exercise gives at most 5 points. Your assignments should to be handed in to me or my mailbox not later than Thursday, April 26, 10 a.m. Delayed exercises will in general be ignored. Exceptions are possible, but they require your explanation and my approval in advance.