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Master Programme in Mathematics
Examination in mathematics
2010-05-31

Algebraic number theory

Time: 8.00-13.00. No tools are allowed except paper and pen. All solutions must be accompanied by explanatory text. Every problem gives at most 5 points.

1. The fundamental theorem of algebra asserts that for every monic polynomial $f(X) \in \mathbb{C}[X]$ there exist complex numbers $\alpha_1, \dots, \alpha_n$ such that

$$f(X) = \prod_{i=1}^{n} (X - \alpha_i).$$

Prove that if all coefficients of $f(X)$ are algebraic integers, then all roots α_i of $f(X)$ are algebraic integers.

2. Let p be an odd prime number, and $\zeta = e^{\frac{2\pi}{p} i}$. Let $s \in \mathbb{Z}$ be such that $\zeta^{p-1} \notin \{\zeta^{s-1}, \zeta^{-s}\}$. Show that $|\{\zeta^s, \zeta^{s-1}, \zeta^{-s}, \zeta^{1-s}\}| \neq 3$.

3. Let p be a prime number, and $\zeta = e^{\frac{2\pi}{p} i}$.

(a) What is a fractional $\mathbb{Z}[\zeta]$-ideal? Reproduce the definition.

(b) What is a principal fractional $\mathbb{Z}[\zeta]$-ideal? Reproduce the definition.

(c) What does it mean that p is regular? Reproduce the definition.

(d) Prove that if p is regular and A is a fractional $\mathbb{Z}[\zeta]$-ideal such that A^p is principal fractional, then A is principal fractional.

4. Let p be a prime number, and $\zeta = e^{\frac{2\pi}{p} i}$. We know that $\mathbb{Z}[\zeta]$ is a Dedekind domain. Hence every non-zero $\mathbb{Z}[\zeta]$-ideal A has a unique factorization into a product of prime $\mathbb{Z}[\zeta]$-ideals. Find this unique prime factorization for $A = p\mathbb{Z}[\zeta]$, and motivate your answer.

5. (a) Show that the ring $\mathbb{Z}[\sqrt{-3}\,]$ is an integral domain.

(b) Is $\mathbb{Z}[\sqrt{-3}\,]$ a noetherian domain? Motivate your answer!

(c) Show that $\mathbb{Q}[\sqrt{-3}\,]$ is the field of fractions of $\mathbb{Z}[\sqrt{-3}\,]$.

(d) Is $\mathbb{Z}[\sqrt{-3}\,]$ an integrally closed domain? Motivate your answer!

(e) Is $\mathbb{Z}[\sqrt{-3}\,]$ a Dedekind domain? Motivate your answer!

Please turn over!

6. Let R be a Dedekind domain, K its field of fractions, and $K \subset L$ a finite separable field extension. Show that L has a K-basis $(\beta_1, \ldots, \beta_n)$ such that all β_i are integral over R.

7. Let L be an algebraic number field. Every $\alpha \in L$ determines a $\mathbb{Q}$-linear operator $\mu_\alpha : L \to L$, $\mu_\alpha(\xi) = \alpha\xi$. Let M_α be the matrix of μ_α in some $\mathbb{Q}$-basis of L. From linear algebra we know that the determinant $\det(M_\alpha)$ does not depend on the chosen basis. Hence $\det(\mu_\alpha) = \det(M_\alpha)$ is well-defined. Moreover, the *norm* of α may be defined by $N(\alpha) = \det(\mu_\alpha)$. Derive from this definition that $N(\alpha) \in \mathbb{Z}$ whenever $\alpha \in L$ is an algebraic integer.

8. Let $p \geq 5$ be a regular prime number, and $\zeta = e^{\frac{2\pi}{p}i}$. Kummer's Theorem asserts that the equation $X^p + Y^p = Z^p$ has no non-trivial integral solution. Sketch a proof of the following statement.

If (x, y, z) is a minimal counterexample of the first kind to Kummer's Theorem, then for each $0 \leq i \leq p-1$ there exists a $\mathbb{Z}[\zeta]$-ideal I_i such that $I_i^p = (x + \zeta^i y)$.

Good luck!