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Prov i matematik
Algebraiska strukturer
2007-12-11

Skrivtid: 9.00-14.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text! Varje uppgift ger maximalt 5 poäng.

1. Consider the strictly ascending chain

$$\mathbb{Z}\frac{1}{2} \subset \mathbb{Z}\frac{1}{4} \subset \ldots \subset \mathbb{Z}\frac{1}{2^n} \subset \mathbb{Z}\frac{1}{2^{n+1}} \subset \ldots$$

of cyclic subgroups of $\mathbb{Q} = (\mathbb{Q}, +)$. Show that $A = \bigcup_{n \geq 1} \mathbb{Z}\frac{1}{2^n}$ is a subgroup of $\mathbb{Q} = (\mathbb{Q}, +)$ which is not finitely generated.

2. (a) Reproduce the statements of the three Sylow theorems.

(b) Show that every group of order 18 has a nontrivial proper normal subgroup.

3. (a) Reproduce the definition of a solvable group.

(b) Show that $V = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of the symmetric group S_4.

(c) Show that S_4 is solvable.

4. Let K be a field.

(a) Explain why the polynomial ring $K[X, Y]$ is a factorial domain.

(b) Prove that $K[X, Y]$ is not a principal ideal domain.

5. (a) Reproduce the definition of an irreducible element p in a domain R.

(b) Give an explicit example of a ring extension $R \subset S$ and an element $p \in R$ such that R and S are domains, and p is irreducible in R but not irreducible in S.

(c) Show that if p is an irreducible element in a domain R, then p is irreducible even in the polynomial ring $R[X]$.

PLEASE TURN OVER!

6. (a) Reproduce the definition of an algebraic field extension $K \subset E$.

(b) Let $K \subset F$ be a field extension, and let α, β be elements in F which are algebraic over K. Prove that the field extension $K \subset K(\alpha, \beta)$ is algebraic.

7. Determine $\mathrm{Gal}(E/\mathbb{Q})$, where $E = \mathbb{Q}(\sqrt{3}, \sqrt{5})$.

8. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{13}i}$.

(a) Explain why $\mathbb{Q} \subset E$ is a finite Galois extension.

(b) Determine $\mathrm{Gal}(E/\mathbb{Q})$, up to isomorphism.

(c) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(d) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

Lycka till!