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Prov i matematik
Algebraiska strukturer
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Skrivtid: 8.00-13.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text. Varje uppgift ger maximalt 5 poäng.

1. (a) Define the *dihedral group* D_n for any natural number $n \geq 2$.

(b) Determine the kernel of any non-trivial morphism $\varphi : \mathrm{D}_3 \to \mathrm{D}_2$.

(c) Use (b) to find the number of morphisms from D_3 to D_2.

2. (a) What is a *subgroup* of a group G? Reproduce the definition!

(b) The set G of all invertible complex 2×2-matrices is a group under matrix multiplication. Is $H = \left\{ \left(\begin{array}{cc} w & -z \\ \overline{z} & \overline{w} \end{array} \right) \middle| \, w, z \in \mathbb{C}, \ (w, z) \neq (0, 0) \right\}$ a subgroup of G? Motivate your answer!

3. (a) Classify all abelian groups of order 360.

(b) Classify all groups of order 361.

4. (a) What is a *subring* of a ring R? Reproduce the definition!

The set R of all real 2×2-matrices is a ring under addition and multiplication of matrices. Motivate your answer to each of the following questions.

(b) Is $S = \{M \in R \mid M_{21} = 0\}$ a subring of R?

(c) Is $T = \{M \in R \mid M_{11} = M_{12} = M_{21} = 0\}$ a subring of R?

(d) Is T a ring under addition and multiplication of matrices?

5. (a) Let R be a commutative ring. What is meant by the *universal property of the polynomial ring* $R[X]$? Reproduce the statement!

(b) Use (a) to show that the map $\sigma : \mathbb{Z}[X] \to \mathbb{Z}[X], \ \sigma(f(X)) = f(X+1)$ is an automorphism of the polynomial ring $\mathbb{Z}[X]$.

(c) Use (b) to show that the polynomial $f(X) = 1 + X + X^2 + X^3 + X^4$ is irreducible in $\mathbb{Z}[X]$.

PLEASE TURN OVER!

6. (a) Reproduce the *cubic formula*, expressing the roots of a complex cubic $f(X) = X^3 + qX + r$ in terms of its coefficients q and r.

(b) Express the roots of the cubic $f(X) = X^3 - 3X + 6$ in terms of its coefficients -3 and 6.

7. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{11}i}$.

(a) When is a field extension called *separable*, when is it called *normal*, and when is it called *Galois*? Reproduce the definitions!

(b) Explain why $\mathbb{Q} \subset E$ is a finite Galois extension.

(c) Determine $\mathrm{Gal}(E/\mathbb{Q})$, up to isomorphism.

(d) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(e) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

8. Show that every field extension $K \subset E$ of degree 2 is Galois, provided that $\mathrm{char}(K) \neq 2$.

GOOD LUCK!