# Auctions with an invitation cost

Erik Ekström
Department of Mathematics, Uppsala University
Carl Lindberg
The Second Swedish National Pension Fund
*e-mail:* ekstrom@math.uu.se, carl.lindberg@ap2.se

July 7, 2020

**Abstract**

We consider an auction in which a seller invites potential buyers to a sealed-bid first-price auction, without disclosing to the buyers the number of extended invitations. In the presence of a fixed invitation cost for each invited bidder, the whole auction can be described as a game, where the set of players consists of all bidders together with the seller. In a setting with fully observable common values we show the existence of a Nash equilibrium in mixed strategies. In this equilibrium, the seller should invite precisely one or two potential buyers with certain probabilities, and each invited buyer should place a randomized bid according to a certain distribution.

**Key words:** game theory; auctions; unknown competition; randomized strategies.

**Mathematics Subject Classification (2010):** 91B26, 91A05, 91A06

## 1 Introduction

In sealed-bid auctions there is often a natural uncertainty about the number of bidders. For example, consider a house owner who wants to contract a painter to repaint a house. Naturally, to get a good price, the house owner should ask as many painters as possible. However, if each invitation has an associated cost, then there will be a trade-off between inviting many contractors to obtain a competitive price and fewer ones to keep the total invitation cost small. From the painters' perspective, this yields an uncertainty about the number of invited bidders. Another example is provided by option trading by large financial institutions. The standard industry trading method is that a seller asks a number of counterparties - typically investment banks or market makers - to come in with prices at which they are willing to do a specific trade. Again, if each invitation comes with a cost (for example, a loss in goodwill among bidders who do not win the auction), then there is from an invited bidder's perspective an uncertainty about how many other bidders are invited.

A standard setting in auction theory is the model where bidders have independent private values (IPV), in which each bidder assigns a private value to the good. While the distribution of the value for each bidder is common knowledge, the actual value (the realisation of the random variable) is known only to each individual bidder. Another standard auction setting is the one with interdependent values where each bidder receives a random signal, and the value of the good for a given bidder is a function of his own (known) signal and the remaining bidders' (unknown) signals; if all players have the same value function, and if this function is symmetric, then all bidders agree on a common value of the good. For an introduction to and discussion of different auction models we refer to the textbook [8].

There is a growing strand of literature that study auctions with unknown competition. In [10] it is shown that concealing the number of bidders is, on average, beneficial for the seller if bidders have constant risk aversion in an IPV setting. In [4] different auctions with interdependent values and with an unknown number of bidders are analysed. In particular, revenue equivalence for five different types of auctions is derived. In contrast to [10] and [4], the article [9] instead examines the buyer's point of view by comparing how the risk aversion affects the buyer's preferences regarding the auction type. The article [2] shows that an increase in the number of potential bidders in a procurement auction may lead to higher procurement costs, and the authors of [5] examine information aggregation in common value auctions. Finally, [1] considers estimation of first-price auctions for a set-up where the number of potential bidders is random, and in [7] a statistical study is performed to test for possible deviations of individual behaviour from theory.

In the current paper we study the effect of uncertain competition on sealed-bid first-price auctions under the assumption of a fully observable common value of the good. While combining the assumptions of an observable value and a common value of the good would lead to degenerate bidding strategies in the case of known competition, it turns out that the setting with uncertain competition is rich enough to obtain a non-degenerate equilibrium in mixed (randomized) strategies. Our choice to work within this rather simplistic framework allows us to fully analyse a setting where the seller pays a fixed cost for each invited bidder. In this game, the set of players consists of the bidders *and* the seller. In particular, we derive conditions under which a Nash equilibrium exists; this Nash equilibrium is symmetric for invited bidders, whereas the equilibrium strategy for the seller is specified by a distribution on the number of bidders to be invited. Interestingly, this distribution assigns mass only to one and two, meaning that the seller should never invite more than two bidders.

Moreover, we study asymmetric versions of the auction game. Such situations occur if invited bidders assign different values of the good or estimate the probability of competition differently. Again we provide a Nash equilibrium in mixed strategies, both among bidders and in situations when the seller is also included in the set of players.

The paper is organised as follows. In Section 2 we study the basic symmetric version of the sealed-bid first-price auction in which the number of bidders is unknown. In particular, we prove existence and uniqueness of a symmetric equilibrium for the bidders in mixed strategies. In Section 3, we show that in the presence of an additive invitation cost for the seller, there is an auction equilibrium which includes also the seller as one of the players. In particular, we show that there exists a probability distribution on the number of invited bidders that in equilibrium should be used by the seller, and that the seller should not invite more than two bidders. Sections 4-6 extend the basic version studied in Sections 2-3 by treating asymmetric cases. Section 4 contains a study of bidders with asymmetric, but still fully observable, private values and probability distributions of the number of bidders. This is used to study two examples of asymmetric auction games: an auction game where one of the bidders is given priority (Section 5), and an auction game where bidders have different values (Section 6).

# 2 The auction from the bidders' perspective

In this section we study a sealed-bid first-price auction from the perspective of a bidder who does not know the exact number of competing bidders. We assume that there are $n-1$ additional *potential* bidders, and we refer to the bidder under consideration as Player 1. We also assume that $n \geq 2$ so that there is at least one additional potential bidder.

We distinguish between the number of *potential* bidders, $n$, and the number of (*actual*) bidders, which is a random variable with values in $\{1, 2, ..., n\}$. More presicely, Player 1 estimates the probability that exactly $k$ bidders exist to be $p_k$, $k = 1, ..., n$. Here $p_k \geq 0$, $\sum_{k=1}^{n} p_k = 1$ and $0 < p_1 < 1$ so that the probability of being the only bidder and the probability of at least one more bidder are both strictly positive. We also assume that the game is symmetric in the sense that each of the other bidders (if any) estimates the probability that there are exactly $k-1$ additional bidders to be $p_k$.

Furthermore, we assume that all bidders agree on the value of the good and that this value is a known constant $v > 0$. If the highest bid is $x \in [0, v]$ (and is unique), then the profit for the bidder with the highest bid is $v - x$, and the profit is 0 for all other bidders. If the highest bid $x$ is not unique, then the profit $v - x$ is split evenly between the corresponding bidders. Thus, if Player 1 bids $x$, and there are $k$ additional bidders with bids $y_1, y_2, ..., y_k$, then the profit for Player 1 is

$$\frac{v-x}{1+\sum_{i=1}^{k} 1_{\{x=y_i\}}} 1_{\{x \geq \max_{1 \leq i \leq k} y_i\}}.$$

**Definition 1** *A* mixed strategy *for a bidder is a distribution funtion $F$ on $[0, v]$, i.e. a non-decreasing and right-continuous function such that $F(0) \geq 0$ and $F(v) = 1$. We denote by $\mathcal{F}$ the collection of such functions.*

For a pair $(F, G) \in \mathcal{F}^2$ of mixed strategies, let $(X, Y_1, ..., Y_{n-1})$ be a vector of independent random variables with $F(x) = \mathbb{P}(X \leq x)$ and $G(y) = \mathbb{P}(Y_i \leq y)$,

$i = 1, ..., n-1$. If Player 1 bids $X$ and all other bidders (if any) use independent bids with distribution $G$, then the corresponding expected profit for Player 1 is

$$\mathcal{J}(F,G) := p_1 \mathbb{E}[v - X] + \sum_{k=1}^{n-1} p_{k+1} \mathbb{E}\left[ \frac{v - X}{1 + \sum_{i=1}^{k} 1_{\{X = Y_i\}}} 1_{\{X \geq \max_{1 \leq i \leq k} Y_i\}} \right]. \quad (1)$$

We refer to the game described above between bidders, but with no seller, as the *bidders' game.*

**Definition 2** *A pair $(G,G) \in \mathcal{F}^2$ of mixed strategies is a symmetric Nash equilibrium in the bidders' game if*

$$\mathcal{J}(F,G) \leq \mathcal{J}(G,G)$$

*for all $F \in \mathcal{F}$.*

**Lemma 3** *If $(G,G) \in \mathcal{F}^2$ is a symmetric Nash equilibrium, then $G$ is continuous with $G(0) = 0$ and $G(v-) = 1$.*

**Proof.** Let $G(0-) := 0$ and assume that $G(a) - G(a-) = \eta > 0$ for some $a \in [0, v]$.

First, if $a = v$ so that $G(v) - G(v-) = \eta > 0$, then consider the strategy given by

$$\tilde{G}(x) = \begin{cases} G(x) + \eta & \text{if } x \in [0, v) \\ 1 & \text{if } x = v. \end{cases}$$

We claim that $\tilde{G}$ is a strictly better response to $G$ than $G$. In fact, the expected profit for Player 1 increases with

$$\mathcal{J}(\tilde{G}, G) - \mathcal{J}(G,G) = \eta v \sum_{k=0}^{n-1} p_{k+1} \frac{G^k(0)}{k+1} \geq \eta v p_1 > 0,$$

compare (1). Consequently, we cannot have $a = v$.

Similarly, if $a \in [0, v)$, let $\epsilon \in (0, v-a)$ be a small number such that $G$ is continuous at $a + \epsilon$, and define $G^\epsilon$ by

$$G^\epsilon(x) = \begin{cases} G(x) & \text{if } x \notin [a, a+\epsilon) \\ G(x) - \eta & \text{if } x \in [a, a+\epsilon). \end{cases}$$

Then, given a random variable $X$ with distribution function $G$, a random variable $X^\epsilon$ with distribution $G^\epsilon$ can be constructed as $X^\epsilon = X 1_{\{X \neq a\}} + (X +$

$\epsilon)1_{\{X=a\}}$. Consequently, the expected profit for Player 1 increases with

$$\begin{aligned}
\mathcal{J}(G^\epsilon, G) - \mathcal{J}(G, G) &= \eta(v-a-\epsilon)\sum_{k=0}^{n-1} p_{k+1} G^k(a+\epsilon) \\
&\quad -\eta(v-a)\sum_{k=0}^{n-1} p_{k+1}\mathbb{E}\left[\frac{1_{\{X\geq \max_{1\leq i\leq k} Y_i\}}}{1+\sum_{i=1}^{k} 1_{\{X=Y_i\}}}\right] \\
&\geq \eta(v-a-\epsilon)\sum_{k=0}^{n-1} p_{k+1} G^k(a) \\
&\quad -\eta(v-a)p_1 - \eta(v-a)\sum_{k=1}^{n-1} p_{k+1}\mathbb{E}\left[\frac{1_{\{X\geq \max_{1\leq i\leq k} Y_i\}}}{1+1_{\{X=Y_1\}}}\right] \\
&= \frac{\eta^2}{2}(v-a)\sum_{k=1}^{n-1} p_{k+1} G^{k-1}(a) - \eta\epsilon\sum_{k=0}^{n-1} p_{k+1} G^k(a),
\end{aligned}$$

which is strictly positive if $\epsilon$ is small enough. Consequently, with such an $\epsilon$, $G^\epsilon$ is a strictly better response to $G$ than $G$ is, so $(G, G)$ is not an equilibrium. Therefore a symmetric equilibrium $(G, G)$ has to have a continuous distribution function with $G(0) = 0$ and $G(v-) = G(v) = 1$. ■

By Lemma 3, if all bidders use a symmetric equilibrium strategy $G$, then the probability that two bids coincide is 0. Consequently, the expression for $\mathcal{J}$ then simplifies to

$$\mathcal{J}(F, G) = p_1\mathbb{E}[v-X] + \sum_{k=1}^{n-1} p_{k+1}\mathbb{E}\left[(v-X)1_{\{X\geq \max_{1\leq i\leq k} Y_i\}}\right]$$

(if Player 1 uses the strategy $F \in \mathcal{F}$). Furthermore, for a pair $(G, G) \in \mathcal{F}^2$ to be a symmetric Nash equilibrium, one therefore expects that

$$\begin{aligned}
& p_1(v-x) + \sum_{k=1}^{n-1} p_{k+1}\mathbb{E}\left[(v-x)1_{\{x\geq \max_{1\leq i\leq k} Y_i\}}\right] \\
=\; & p_1(v-x) + (v-x)\sum_{k=1}^{n-1} p_{k+1} G^k(x)
\end{aligned}$$

is constant on the support of $G$. This leads to a candidate equilibrium $(G, G)$ defined implicitly by

$$\begin{cases} (v-x)\sum_{k=1}^{n-1} p_{k+1} G^k(x) = p_1 x & x \in [0, v(1-p_1)) \\ G(x) = 1 & x \in [v(1-p_1), v]. \end{cases} \tag{2}$$

Note that for a fixed $x \in [0, v(1-p_1)]$ there is a unique solution $G(x) \in [0, 1]$ to (2). Moreover, the unique solution $G$ to (2) is continuous and non-decreasing on $[0, v]$ with $G(0) = 0$ and $G(v) = 1$, so $G \in \mathcal{F}$.

Furthermore, and for future reference, we note that in the case $n = 2$ we have $p_2 = 1 - p_1$, and the function $G$ in (2) can be explicitly written

$$G(x) = \begin{cases} \frac{p_1 x}{(1-p_1)(v-x)} & x \in [0,(1-p_1)v) \\ 1 & x \in [(1-p_1)v, v]. \end{cases} \tag{3}$$

**Theorem 4** *The pair $(G, G)$, where $G$ is defined in (2), is a Nash equilibrium in mixed strategies. Moreover, it is the unique symmetric equilibrium.*

**Proof.** First we check that $(G, G)$ is an equilibrium. If Player 1 bids $x$ and all other bidders use mixed strategies represented by the distribution function $G$, then the expected profit for Player 1 is

$$(v-x)\sum_{k=0}^{n-1} p_{k+1} G^k(x) = \begin{cases} p_1 v & x \in [0,(1-p_1)v) \\ v-x & x \in [(1-p_1)v, v]. \end{cases}$$

Thus the maximal expected reward is $p_1 v$, and since the strategy $G$ distributes all mass on $[0,(1-p_1)v]$ where the expected profit is constant (and maximal), there is no strategy giving a higher average profit than $G$. Thus the strategy $G$ is an optimal response if all other players use $G$, so $(G, G)$ is a symmetric Nash equilibrium.

For uniqueness, assume that $(H, H) \in \mathcal{F}^2$ is a symmetric Nash equilibrium in mixed strategies. Then $H$ is continuous on $[0, v]$ with $H(0) = 0$ and $H(v-) = 1$ by Lemma 3. Next consider the function

$$h(x) = (v-x)\sum_{k=0}^{n-1} p_{k+1} H^k(x),$$

i.e. the expected profit for Player 1 from bidding $x$ in case the other bidders use the strategy $H$. Then $h$ is continuous on $[0, v]$, and we denote its maximum $\overline{h}$.

We claim that $h \equiv \overline{h}$ on the support of $H$. To see this, assume that $x \in \operatorname{supp}(H)$ and that $h(x) < \overline{h}$. By continuity, $h < \overline{h}$ in some interval $(x-\epsilon, x+\epsilon)$, and since $x$ is in the support of $H$, we have $H(x+\epsilon) - H(x-\epsilon) > 0$. Let $x'$ be a point such that $h(x') = \overline{h}$. Then the strategy that puts all mass at $x'$ gives an average profit $\overline{h}$, which strictly exceeds the expected profit from playing $H$, so $(H, H)$ is not an equilibrium.

If $y \in (0, v)$ with $h(y) < \overline{h}$, then $y \notin \operatorname{supp}(H)$ so $H$ is constant in an open interval containing $y$. Therefore $h$ is strictly decreasing on that interval, and by continuity of $h$ it follows that the whole interval $(y, v]$ belongs to the complement of $\operatorname{supp}(H)$. This proves that $\operatorname{supp}(H) = [0, \overline{x}]$ for some $\overline{x} \in [0, v]$. Consequently, $h(x) = \overline{h}$ on $[0, \overline{x}]$, so

$$\overline{h} = h(0) = p_1 v$$

and

$$(v-x)\sum_{k=0}^{n-1} p_{k+1} H^k(x) = p_1 v \tag{4}$$

for $x \in [0, \overline{x}]$. The only continuous function $H \in \mathcal{F}$ with $H(0) = 0$ of the form (4) is $H = G$ specified in (2), which completes the proof. $\blacksquare$

**Remark 5** *It is easy to check that the game degenerates in the excluded cases* $p_1 = 0$ *and* $p_1 = 1$*. In fact, if* $p_1 = 1$*, i.e. if there is only one bidder, then there is no competition, and hence the optimal strategy for the bidder is to bid* $0$*. On the other hand, if* $p_1 = 0$*, then each bidder knows that there is at least one more bidder, and a situation with pre-emption appears. The symmetric equilibrium in this case consists of bids of size* $v$*, thus generating no profit at all.*

**Remark 6** *Recall the classical IPV auction in which each bidder has an independent private value drawn from a distribution with a continuous density; the case with a known number of bidders can be found in the classical reference [11], and for extensions to an unknown number of bidders, see [4], [9] and [10]. In that setting, an equilibrium strategy is obtained if each bidder bids a certain deterministic function of her/his private value.*

*Assume that the private values are drawn from a distribution with a continuous distribution function* $H : [0, v] \to [0, v]$*, and that any invited bidder estimates the probability of being the only bidder to be* $p$ *and the probability that there is exactly one more bidder to* $1 - p$*. Then the expected profit for Player 1 if bidding* $y$ *is*

$$p(x - y) + (1 - p)(x - y)H(\beta^{-1}(y)).$$

*Here* $\beta : [0, v] \to [0, v]$ *is a strictly increasing function that is used by the second bidder (if invited) to map his/her private value into a bid, and* $x$ *is the private value of Player 1. Optimizing over the bid* $y$ *and using* $\beta(x) = y$ *for a symmetric equilibrium (for details, see [4] or [8, Chapter 3.2]), one obtains*

$$\beta(x) = \frac{(1 - p)(xH(x) - \int_0^x H(z)\, dz)}{p + (1 - p)H(x)}. \tag{5}$$

*Now assume that private values are distributed uniformly on* $[v - \epsilon, v]$*. By* (5)*, the symmetric equilibrium strategy consists of bidding* $\beta(x)$ *if the private value is* $x \in [v - \epsilon, v]$*, where*

$$\beta(x) = \frac{(1 - p)(x^2 - (v - \epsilon)^2)}{2\epsilon - 2(1 - p)(v - x)}.$$

*Take* $y < (1 - p)v$*, and let* $X$ *be a random variable that represents the private value of an invited bidder. Then straightforward calculations show that for* $\epsilon < 2((1 - p)v - y)/(1 - p)$ *we have*

$$\mathbb{P}(\beta(X) \leq y) \quad = \quad \epsilon^{-1} \left( y - v + \epsilon + \sqrt{y^2 - 2(v - \frac{\epsilon}{1 - p})y + (v - \epsilon)^2} \right),$$

*so*

$$\mathbb{P}(\beta(X) \leq y) \to \frac{py}{(1 - p)(v - y)}$$

*as $\epsilon \to 0$. Thus, as the support of the distribution of private values collapses to $\{v\}$, the equilibrium bid in the IPV setting converges in distribution to the corresponding Nash equilibrium determined in Theorem 4; to wit, the equilibrium in the auction with a known value $v$ coincides with the equilibrium in the IPV setting where values are uniformly distributed on an infinitesimal interval* $[v-, v]$.

**Remark 7** *Our setting with a known common value $v$ but with unknown competition is strategically equivalent to a setting with known competition and independent private values with a discrete distribution $p\delta_0 + (1-p)\delta_v$, where $\delta_a$ is a point mass at $a$. In fact, for $n = 2$ the equilibrium in* (3) *was obtained in [6, pages 386-389], but with no discussion about uniqueness.*

# 3 The seller's perspective

In this section we view the whole auction as a game, thus including also the seller as a player. The seller's strategy amounts to determining how many bidders to invite, but with the assumption that each invitation is made at a cost $c \in (0, v)$. Naturally, the seller wants to invite many bidders to ensure that the winning bid is sufficiently high, but not too many in order to control the total cost.

If the total number of potential bidders is $n$, then a mixed strategy of the seller consists of a distribution on the set $\{1, 2, ..., n\}$, which is represented by an $n$-tuple $q = (q_1, q_2, ..., q_n)$ with $q_k \geq 0$ and $\sum_{k=1}^{n} q_k = 1$. Here $q_k$ represents the probability that the seller invites exactly $k$ bidders, thereby infering a total cost $ck$. We also assume that the seller has no preferences between various potential bidders. More precisely, if the seller decides to invite $k$ bidders, then any of the $\binom{n}{k}$ possible configurations of bidders is equally likely (asymmetric cases are studied in Sections 4-6 below).

Note that the fact that a bidder receives an invitation affects the probability distribution of the number of invited bidders. Denote by

$$\Pi = \{q \in [0,1]^n : q_k \geq 0 \text{ and } \sum_{k=1}^{n} q_k = 1\}$$

the set of strategies for the seller. The following proposition provides the link between a mixed strategy $q$ of the seller and the distribution $p$ of the number of invited bidders *from an invited bidder's perspective.*

**Proposition 8** *Let $q \in \Pi$. Then, from an invited player's perspective, the distribution of invited players is given by $p = (p_1, ..., p_n)$, where*

$$p_k = \frac{kq_k}{\sum_{k=1}^{n} kq_k}, \quad k = 1, ..., n. \tag{6}$$

**Proof.** Straightforward calculations give

$$\begin{aligned} p_k &= \mathbb{P}(\text{exactly } k \text{ invited bidders}|\text{Player 1 invited}) \\ &= \frac{\mathbb{P}(\text{exactly } k \text{ invited bidders and Player 1 invited})}{P(\text{Player 1 invited})} \\ &= \frac{q_k k/n}{(\sum_{k=1}^n kq_k)/n} = \frac{kq_k}{\sum_{k=1}^n kq_k}, \end{aligned}$$

which proves the claim. ■

It follows from Proposition 8 that if the seller uses a strategy $q \in \Pi$, then an invited bidder will use the strategy $G = G^q$ specified in (2) where $p = (p_1, ..., p_n)$ is given by (6). For the relation (6) we write $p = p(q)$.

Let $Y_1, ..., Y_k$ be independent random variables with distribution function $G^q$. The expected profit for the seller if inviting precisely $k$ bidders is then

$$\beta_k^q := \mathbb{E}[Y_1 \vee Y_2 \vee ... \vee Y_k] - ck \tag{7}$$

for $k = 1, ..., n$. Consequently, the expected profit for the seller associated with a strategy $q \in \Pi$ is

$$\beta^q := \sum_{k=1}^n q_k \beta_k^q.$$

A pair $(G, q) \in \mathcal{F} \times \Pi$ is a *Nash equilibrium for the full auction game* if $(G, G)$ is a symmetric equilibrium for the bidders' game with $p = p(q)$ and if for any $k$ with $q_k > 0$ we have $\beta_i^q \le \beta_k^q$ for all $i \in \{1, ..., n\}$.

**Lemma 9** *Assume that $Y_1, ..., Y_k$ are independent and identically distributed with distribution function $G^q$, and that the support of the distribution contains at least two points, and let $\beta_k^q := \mathbb{E}[Y_1 \vee Y_2 \vee ... \vee Y_k] - ck$ as above. Then*

$$\beta_k^q - \beta_{k-1}^q < \beta_2^q - \beta_1^q, \quad k = 3, ..., n. \tag{8}$$

**Proof.** Using that $x \vee y - x = 0 \vee (y - x)$ is decreasing in $x$, we have

$$(Y_1 \vee ... \vee Y_{k-1}) \vee Y_k - Y_1 \vee ... \vee Y_{k-1} \le Y_{k-1} \vee Y_k - Y_{k-1}.$$

Moreover, the inequality is strict if $Y_1 \vee ... \vee Y_{k-1} \ge Y_k > Y_{k-1}$, which happens with positive probability. Taking expected values thus yields

$$\mathbb{E}[Y_1 \vee ... \vee Y_k - Y_1 \vee ... \vee Y_{k-1}] < \mathbb{E}[Y_{k-1} \vee Y_k - Y_{k-1}],$$

which implies (8). ■

**Proposition 10** *Assume that $n \ge 3$ and that $q^*$ is an equilibrium strategy for the seller. Then $q_k^* = 0$ for $k = 3, ..., n$.*

**Proof.** Assume that $q^*$ is an equilibrium strategy. First we claim that

$$q_1^* > 0. \tag{9}$$

Indeed, if $q_1^* = 0$, then there are at least two invited bidders. Therefore, by pre-emption, every invited bidder will bid $v$, and the expected profit for the seller is then at most $v - 2c$. However, given that all invited bidders bid $v$, then an optimal response for the seller is to invite only one bidder, which has an expected profit $v - c$. Since $v - c > v - 2c$, this strategy is strictly better than $q^*$, which is a contradiction. This proves (9).

Now assume that $q_k^* > 0$ for some $k \geq 3$ (if $n = 2$ there is nothing to prove; therefore we may without loss of generality assume that $n \geq 3$). The expected profit for the seller is

$$\sum_{i=1}^{n} q_i^* \beta_i.$$

However, modifying the strategy $q$ by defining $q^\epsilon$ by $q_i^\epsilon = q_i^*$ for $i \notin \{1, 2, k-1, k\}$ and

$$\begin{cases} q_1^\epsilon = q_1^* - \epsilon \\ q_2^\epsilon = q_2^* + \epsilon \\ q_{k-1}^\epsilon = q_{k-1}^* + \epsilon \\ q_k^\epsilon = q_k^* - \epsilon \end{cases}$$

for small $\epsilon > 0$ (in the case $k = 3$ we set $q_2^\epsilon = q_2^* + 2\epsilon$), the expected profit for the seller becomes

$$\sum_{i=1}^{n} q_i^\epsilon \beta_i = \sum_{i=1}^{n} q_i^* \beta_i + \epsilon(\beta_2 - \beta_1 - (\beta_k - \beta_{k-1})) > \sum_{i=1}^{n} q_i^* \beta_i,$$

where the inequality comes from Lemma 9. This contradicts the optimality of $q^*$, so $q_k^* = 0$ for $k \geq 3$. ■

In view of Proposition 10, we only need to look for equilibrium strategies for the seller on the form $q^* = (q, 1-q, 0, ..., 0)$ with $q \in (0, 1)$. With a slight abuse of notation, we will therefore refer to a number $q \in (0, 1)$ as a strategy of the seller. The corresponding belief for an invited bidder is then given by $p = (\frac{q}{2-q}, \frac{2-2q}{2-q})$ (see Proposition 8). Recalling (3) above, the equilibrium strategy of an invited bidder is then

$$\begin{aligned} G(x) &= \begin{cases} \frac{p_1 x}{(1-p_1)(v-x)} & x \in [0, (1-p_1)v) \\ 1 & x \in [(1-p_1)v, 1] \end{cases} \\ &= \begin{cases} \frac{qx}{2(1-q)(v-x)} & x \in [0, \frac{2(1-q)}{2-q} v) \\ 1 & x \in [\frac{2(1-q)}{2-q} v, 1]. \end{cases} \end{aligned} \tag{10}$$

The corresponding expected profit for the seller if inviting precisely one bidder is then

$$\beta_1 = \int_0^{\frac{2(1-q)}{2-q} v} x G'(x)\, dx - c = v \left(1 + \frac{q}{2(1-q)} \ln \frac{q}{2-q}\right) - c. \tag{11}$$

Similarly, the expected benefit for the seller if inviting two bidders is

$$\begin{aligned}\beta_2 &= \int_0^{\frac{2(1-q)}{2-q}v} x(G^2)'(x)\,dx - 2c \\ &= v\left(\frac{1-2q}{1-q} + \frac{q^2}{2(1-q)^2}\ln\frac{2-q}{q}\right) - 2c.\end{aligned} \tag{12}$$

Consequently, the seller is indifferent between inviting one and inviting two bidders precisely when $\beta_1 = \beta_2$, i.e. when $q$ satisfies

$$\frac{c}{v} = \frac{-q}{1-q} + \frac{q}{2(1-q)^2}\ln\frac{2-q}{q}. \tag{13}$$

Defining

$$f(q) := \frac{-q}{1-q} + \frac{q}{2(1-q)^2}\ln\frac{2-q}{q},$$

one can check that $f > 0$ on $(0,1)$ with $f(0+) = 0$ and $f(1-) = 0$. By continuity, there exists a maximum $\overline{q} \in (0,1)$, and we denote by $\overline{f} = f(\overline{q})$ the maximal value of $f$.

**Theorem 11** *For $c \in (0, \overline{f}]$, denote by $q^*$ a solution to (13) in $(0,1)$, and let $G^*$ be given by (10) with $q = q^*$. Then $(G^*, G^*, q^*)$ is a Nash equilibrium.*

**Proof.** First assume that all invited bidders use the mixed strategy $G^*$. By construction, the seller is then indifferent between inviting one or two bidders, so there is no strategy for the seller that strictly dominates the strategy $q^*$.

Next, assume that Player 1 is invited to the auction, that the seller uses the strategy $q^*$ and that Player 2 uses the mixed strategy $G^*$ if invited. By the same argument as in the proof of Theorem 4, $G^*$ is an optimal strategy for Player 1. From the above, and by symmetry, $(G^*, G^*, q^*)$ is a Nash equilibrium. ∎

**Remark 12** *Since $f > 0$ on $(0,1)$ with $f(0+) = 0$ and $f(1-) = 0$, the equation $f(q) = c/v$ has multiple solutions if $c < \overline{f}$. Note that the expected benefit $\beta_i$ for the seller if inviting exactly $i$ bidders, $i = 1, 2$ (see (11) and (12), respectively) is decreasing in the probability $q$ to invite only one bidder. Therefore, the seller prefers the equilibrium $(G^*, G^*, q^*)$ where $q^*$ is the* smallest *solution of (13) to any other equilibrium on this form. Also, if $\hat{q} \in (q^*, 1)$ solves (13) and all bidders use $\hat{G}$ defined as in (10) with $q = \hat{q}$, then the seller is indifferent between inviting one or two bidders. Consequently, the strategy $q^*$ is an optimal strategy for the seller also if all invited bidders use $\hat{G}$, so the equilibrium $(\hat{G}, \hat{G}, \hat{q})$ will not be used.*

**Remark 13** *We end this section with pointing out the existence of a degenerate Nash equilibrium, which exists for any invitation cost $c \in (0, v)$. Denote by $\overline{q} = 1$ the seller strategy of inviting precisely one bidder, and denote by $G(x) = 1$ for $x \geq 0$ the degenerate strategy of always bidding zero. Then $(G, G, \overline{q})$ is a Nash equilibrium. Indeed, first assume that all invited bidders bid 0. Then the* *expected profit for the seller with a strategy $q \in [0,1]$ is $-cq - 2c(1-q)$, which is maximized for $q = 1$, so $\overline{q}$ is optimal for the seller. Similarly, if the seller uses $\overline{q}$, then the optimal response for an invited bidder is to bid 0, so $(G, G, \overline{q})$ is a Nash equilibrium. However, this equilibrium is of minor practical importance since it yields a negative value for the seller.*

# 4 An asymmetric auction set-up

In this section we consider an auction with asymmetric views on the value of the good and/or on the probability of competition. We first consider the bidders' perspective, and then extend to include the seller's perspective in two examples in Sections 5-6 below.

Assume in a two-player setting that Player 1 has value $v_1$ and estimates the probability of being alone in the auction to be $p_1$, whereas Player 2 has value $v_2$ and estimates the probability of being alone in the auction to be $p_2$. However, we still assume complete information in the sense that all parameters $p_i, v_i$, $i = 1, 2$ are known to both players.

**Theorem 14 (Bidder perspective.)** *Assume that $(1-p_1)v_1 \leq (1-p_2)v_2$. Define the distribution functions $F_1$ and $F_2$ with support on $[0, (1-p_1)v_1]$ by*

$$F_1(x) = \begin{cases} \frac{(1-p_2)v_2 - (1-p_1)v_1 + p_2 x}{(1-p_2)(v_2 - x)} & x \in [0, (1-p_1)v_1) \\ 1 & x \in [(1-p_1)v_1, v_1] \end{cases} \tag{14}$$

*and*

$$F_2(x) = \begin{cases} \frac{p_1 x}{(1-p_1)(v_1 - x)} & x \in [0, (1-p_1)v_1) \\ 1 & x \in [(1-p_1)v_1, v_2]. \end{cases} \tag{15}$$

*Then the pair $(F_1, F_2)$ of mixed strategies forms a Nash equilibrium.*

**Proof.** Assume that Bidder 2 uses the mixed strategy $F_2$ if invited. Then any bid $x \in [0, (1-p_1)v_1)$ gives Bidder 1 the constant average payoff

$$p_1(v_1 - x) + (1-p_1)(v_1 - x)F_2(x) = p_1 v_1,$$

and any bid $x \geq (1-p_1)v_1$ gives a payoff

$$v_1 - x \leq p_1 v_1.$$

Consequently, there is no strategy for Bidder 1 that is strictly better than the mixed strategy $F_1$.

Similarly, if Bidder 1 uses $F_1$, then any bid $x \in (0, (1-p_1)v_1)$ gives Bidder 2 an average payoff

$$p_2(v_2 - x) + (1-p_2)(v_2 - x)F_1(x) = v_2 - (1-p_1)v_1,$$

whereas a bid $x = 0$ gives

$$p_2 v_2 + \frac{(1-p_2)v_2 F_1(0)}{2} < v_2 - (1-p_1)v_1$$

and a bid $x \geq (1-p_1)v_1$ gives

$$(v_2 - x) \leq v_2 - (1-p_1)v_1.$$

Thus the mixed strategy $F_2$, which distributes all mass on $(0, (1-p_1)v_1)$, is optimal for Bidder 2, and hence $(F_1, F_2)$ is an equilibrium. ■

**Remark 15** *Note that $F_1(0) = 1 - \frac{(1-p_1)v_1}{(1-p_2)v_2} \geq 0$, so Bidder 1 may bid 0 with positive probability. Also note that in the special case that $v_1 = v_2$ and $p_1 = p_2$, the equilibrium in Theorem 4 is recovered.*

**Remark 16** *It is straightforward to check that if $p_1 = p_2$ and $v_1 \leq v_2$, then $F_1 \geq F_2$. Consequently, in such a setting the bidder with the smaller value will bid more aggressively in the sense that the probability for a bid smaller than any given threshold is bigger than for the other bidder. A similar feature has been observed in the case of asymmetry among bidders in the independent private value setting, e.g. see [8, Proposition 4.4].*

**Remark 17** *Interestingly, the equilibrium strategy $F_2$ of Bidder 2 does not depend on the estimated probability $p_2$ of being the only player (as long as $(1-p_1)v_1 \leq (1-p_2)v_2$).*

# 5 The auction game with priority

In this section we consider a scenario with one seller and two potential bidders, in which the seller gives priority to one of the bidders. More precisely, we assume that Player 1 is known to be invited whereas Player 2 is invited only with a given probability. The players have the same value $v$, but Player 1 estimates the probability of being the only bidder to be $p \in (0,1)$, whereas Player 2 knows that Player 1 is active. Moreover, the number $p$ is known to both players.

In this setting, Theorem 14 applies (with $p_1 = p$ and $p_2 = 0$), so $(F_1, F_2)$ is an equilibrium, where

$$F_1(x) = \begin{cases} \frac{pv}{v-x} & x \in [0, v(1-p)) \\ 1 & x \in [v(1-p), v] \end{cases} \tag{16}$$

and

$$F_2(x) = \begin{cases} \frac{px}{(1-p)(v-x)} & x \in [0, v(1-p)) \\ 1 & x \in [v(1-p), v]. \end{cases} \tag{17}$$

Since Player 1 is given priority as described above, the seller either invites one bidder (which then is automatically Player 1), or invites both bidders. A mixed strategy for the seller is thus described by a number $q \in [0,1]$ which represents the probability of inviting precisely one bidder (Player 1). Note that if the seller uses a strategy $q \in [0,1]$, then the probability of competition from Player 1's perspective is $p = q$.

Denote by $X_i$ a random variable with distribution function $F_i$, $i = 1, 2$ so that $X_1$ and $X_2$ are independent. If the seller invites only one bidder, then the expected profit is

$$\beta_1 := \mathbb{E}[X_1] - c = v(1-p) + pv \ln p - c,$$

and the expected profit from inviting two bidders is

$$\beta_2 := \mathbb{E}[X_1 \vee X_2] - 2c = v(1-2p) - \frac{p^2 v}{1-p} \ln p - 2c.$$

The seller is indifferent between inviting one and inviting two bidders precisely when $\beta_1 = \beta_2$, i.e. when

$$p + \frac{p}{1-p} \ln p + \frac{c}{v} = 0. \tag{18}$$

Define the function $f(p) = -p - \frac{p}{1-p} \ln p$, and note that $f > 0$ on $(0,1)$ with $f(0+) = f(1-) = 0$, and $f$ is strictly concave. Denote by $\overline{f}$ the maximal value of $f$, which is attained at a point $\overline{p}$.

**Theorem 18 (The auction game with priority.)** *Assume that $c/v \leq \overline{f}$, and denote by $q^*$ the unique solution of $f(q) = c/v$ satisfying $q \leq \overline{p}$, and let $F_i$, $i = 1, 2$ be defined as in (16)-(17) with $p = q^*$. Then $(q^*, F_1, F_2)$ is a Nash equilibrium for the auction game.*

**Proof.** If the bidders use $(F_1, F_2)$ if invited, then the seller is indifferent between inviting one and two bidders, so $q^*$ is an optimal response. Similarly, if the seller uses $q^*$ and Player $i$ uses $F_i$ (if invited), then $F_{3-i}$ is optimal for Player $3-i$ by Theorem 14. Thus $(q^*, F_1, F_2)$ is a Nash equilibrium. ■

# 6 The auction game with different values

In this section we study an auction with two players who assign different values to the good. More precisely, we assume that Player $i$ assigns the value $v_i$, $i = 1, 2$, and without loss of generality we let $v_1 < v_2$. Furthermore, the seller has three possible actions, namely to invite only Player 1, to invite only Player 2, or to invite both Player 1 and Player 2. A mixed strategy for the seller is thus described by a pair $(q_1, q_2)$ with $q_i \geq 0$ and $q_1 + q_2 \leq 1$, where $q_i$ denotes the probability of inviting only Player $i$, $i = 1, 2$.

If the seller uses $(q_1, q_2)$, then the probability $p_i$ from the perspective of Player $i$ to be the only bidder (if invited) is

$$p_i = \frac{q_i}{1 - q_{3-i}}, \tag{19}$$

which can be inverted to yield

$$q_i = \frac{p_i(1 - p_{3-i})}{1 - p_i p_{3-i}}. \tag{20}$$

By Theorem 14, if $(1-p_1)v_1 \le (1-p_2)v_2$, then Player $i$ will use the strategy $F_i$, where $F_i$ is defined by (14)-(15) with $p_i$ as in (19).

Let $X_i$ be a random variable with distribution $F_i$. For the seller to be indifferent between inviting only Player 1 and inviting only Player 2, we need that $\alpha_1 = \alpha_2$, where

$$\alpha_i := \mathbb{E}[X_i] - c$$

is the expected profit for the seller if inviting only Player $i$. Straightforward calculations show that

$$\mathbb{E}[X_1] = \frac{1-p_1}{1-p_2}v_1 - \frac{v_2-(1-p_1)v_1}{1-p_2}\ln\frac{v_2}{v_2-(1-p_1)v_1}$$

and

$$\mathbb{E}[X_2] = v_1 + \frac{p_1}{1-p_1}v_1 \ln p_1,$$

respectively. Thus $\alpha_1 = \alpha_2$ if

$$\frac{p_2-p_1}{1-p_2}v_1 = \frac{v_2-(1-p_1)v_1}{1-p_2}\ln\frac{v_2}{v_2-(1-p_1)v_1} - \frac{p_1}{1-p_1}v_1\ln\frac{1}{p_1},$$

which yields

$$p_2 = \frac{p_1v_1 + (v_2-(1-p_1)v_1)\ln\frac{v_2}{v_2-(1-p_1)v_1} - \frac{p_1}{1-p_1}v_1\ln\frac{1}{p_1}}{v_1(1-\frac{p_1}{1-p_1}\ln\frac{1}{p_1})} =: h_1(p_1,v_1,v_2).$$

Furthermore,

$$\begin{aligned}\mathbb{E}[X_1 \vee X_2] \quad &= \quad \frac{(1-p_1-p_2)v_1}{1-p_2} + \frac{p_1v_2(p_2(v_2-v_1)-v_2+(1-p_1)v_1)}{(v_2-v_1)(1-p_1)(1-p_2)}\ln\frac{1}{p_1} \\ &\quad + \frac{p_2v_2(v_2-(1-p_1)v_1)}{(1-p_1)(1-p_2)(v_2-v_1)}\ln\frac{v_2}{v_2-(1-p_1)v_2},\end{aligned}$$

so the seller is indifferent between inviting both bidders and inviting only Player 2 if

$$\begin{aligned} c \quad &= \quad \mathbb{E}[X_1 \vee X_2] - \mathbb{E}[X_2] \\ &= \quad \frac{-p_1}{1-p_2}v_1 + \frac{p_1}{1-p_1}\left(v_1 - v_2 - \frac{p_1v_1v_2}{(v_2-v_1)(1-p_2)}\right)\ln\frac{1}{p_1} \\ &\quad + \frac{p_2v_2(v_2-(1-p_1)v_1)}{(1-p_1)(1-p_2)(v_2-v_1)}\ln\frac{v_2}{v_2-(1-p_1)v_2} \\ &=: \quad h_2(p_1,p_2,v_1,v_2).\end{aligned}$$

We summarise our findings in the following theorem.

**Theorem 19 (The seller perspective)** *Assume that* $(p_1^*, p_2^*) \in (0,1)^2$ *satisfies* $h_2(p_1^*, p_2^*, v_1, v_2) = c$, $p_2^* := h_1(p_1^*, v_1, v_2)$ *and* $(1-p^*1)v_1 \le (1-p_2^*)v_2$. *Let* $F_1^*$ *and* $F_2^*$ *be defined as in* (14)-(15) *with* $(p_1,p_2) = (p_1^*, p_2^*)$, *and let*

$$(q_1^*, q_2^*) = (\frac{p_1^*(1-p_2^*)}{1-p_1^*p_2^*}, \frac{p_2^*(1-p_1^*)}{1-p_1^*p_2^*})$$

*(compare (20)). Then $(F_1^*, F_2^*, (q_1^*, q_2^*))$ is a Nash equilibrium for the auction game with different values.*

## References


[1] An, Y., Hu, Y. and Shum, M. Estimating first-price auctions with an unknown number of bidders: A misclassification approach. Journal of Econometrics 157 (2010), 328-341.

[2] De Silva, D. and Jeitschko, T. Entry and bidding in common and private value auctions with an unknown number of bidders. Rev. Ind. Organ. 35 (2009), 73-93.

[3] Fudenberg, D. and Tirole, J. Game Theory. MIT Press, Cambridge, MA, 1991.

[4] Harstad, R., Kagel, J. and Levin, D. Equilibrium bid functions for auctions with an uncertain number of bidders. Econom. Lett. 33 (1990), no. 1, 35-40.

[5] Harstad, R., Pekec, A. and Tsetlin, I. Information aggregation in auctions with an unknown number of bidders. Games Econom. Behav. 62 (2008), no. 2, 476-508.

[6] Hirshleifer, J. and Riley, J. The Analytics of Uncertainty and Information. Cambridge University Press, 1992.

[7] Isaac, M., Pevnitskaya, S. and Schnier, K. Individual behavior and bidding heterogeneity in sealed bid auctions where the number of bidders is unknown. Economic Inquiry 50 (2012), no. 2, 516-533.

[8] Krishna, V. Auction Theory. Academic Press, San Diego, 2002.

[9] Matthews, S. Comparing auctions for risk averse buyers: a buyer's point of view. Econometrica 55 (1987), no. 3, 633-646.

[10] McAfee, P. and McMillan, J. Auctions with a stochastic number of bidders. J. Econom. Theory 43 (1987), no. 1, 1-19.

[11] Vickrey, W. Counterspeculation, auctions, and competitive sealed tenders. J. Finance 16 (1961), no. 1, 8-37.