The de Finetti problem with uncertain competition

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Abstract

We consider a resource extraction problem which extends the classical de Finetti problem for a Wiener process to include the case when a competitor, who is equipped with the possibility to extract all the remaining resources in one piece, may exist. This situation is modelled as a non-zero-sum controller-and-stopper game with incomplete information. For this stochastic game we provide a Nash equilibrium with an explicit structure. In equilibrium, the agent and the competitor use singular strategies in such a way that a two-dimensional process, which represents available resources and the filtering estimate of active competition, reflects in a specific direction along a given boundary.

1 Introduction

In the classical single-player de Finetti problem for a Wiener process, the value of a limited resource evolves, in the absence of extraction, as

$$Y_t = x + \mu t + \sigma W_t,$$

where μ and σ are positive constants and W is a standard Brownian motion. The de Finetti problem – also known as *the dividend problem* – then consists of maximising

$$\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right]$$

over all adapted, non-decreasing, and right-continuous processes D with $D_{0-} = 0$, where $\tau_0 := \inf\{t \geq 0 : Y_t - D_t \leq 0\}$ is the extinction time (or bankruptcy time). It is well-known (see, e.g., Asmussen and Taksar [1] and Jeanblanc and Shiryaev [12]) that the optimal strategy $\tilde{D}$ is given by $\tilde{D}_t = \sup_{0\leq s\leq t}(Y_s - B)^+$, where $(x)^+ := \max\{x, 0\}$ and B is a constant that can be calculated explicitly.

In the current article, we study the de Finetti problem under an additional threat of competition. One interpretation of this uncertain competition is that the agent, who exerts the control D to extract from the source Y, is subject to possible fraud. Another interpretation is that Y represents the value of a common resource, but where currently only one agent is extracting; unknown competition then corresponds to the possibility that other agents will decide to extract as well. We thus include the possibility that a competitor exists, and we assume that the competitor has the capacity to extract all the remaining resources at once at a random time γ of his choice.

To model uncertain competition, we use a Bernoulli random variable θ indicating whether the competitor exists ($\theta = 1$) or not ($\theta = 0$), and we consider the maximisation of

$$\mathbb{E}\left[\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t\right]$$

over singular controls D, where $\hat{\gamma} := \gamma 1_{\{\theta=1\}} + \infty 1_{\{\theta=0\}}$. At the same time, the competitor chooses γ to maximise the expected payoff

$$\mathbb{E}\left[e^{-r(\tau_0\wedge\gamma)}X^D_{\tau_0\wedge\gamma}\right],$$

where $X^D = Y - D$ represents the remaining resources after extraction.

The above game is a controller-and-stopper non-zero-sum stochastic game and we thus extend the stream of literature on stochastic games of control and stopping; see, e.g., [13], [14], [3], [15], [10], [4] and [7]. However, in contrast to most of the literature on stochastic games of control and stopping, focusing on zero-sum games, we formulate and solve a *non-zero-sum* game. Moreover, an important feature that distinguishes our game from the works mentioned above is *incomplete information*, which in our framework stems from the fact that the existence of the competitor is uncertain. We thus complement existing literature investigating the role of uncertain competition in stochastic games. This strand of research can be traced back to the non-dynamic setting of an auction game with uncertain competition, see Hirshleifer and Riley [11, pages 386-389], and was more recently extended to a dynamic setting in [6] where a stopping game with uncertain competition was studied. In [6] the term "ghost" was also introduced to represent the players that may not exist. In Ekström et al. [8], the authors proposed and studied a ghost game in a setting related to fraud detection and so called "salami slicing" fraudulence. As in the current paper, a controller-and-stopper non-zero-sum game of ghost type is studied in [8], but with the "ghost" role inverted. More precisely, in [8] the controller is a ghost whereas in the current paper the stopper is a ghost. Our aim is thus to investigate the role of uncertain competition in a singular stochastic control problem. To set the ground for potential further studies, we have chosen the setting of the de Finetti problem, which is a well-known problem in the singular control literature.

Since the competitor is equipped with a binary stopping control, inference about the existence of competition is based on observations of the events $\{\hat{\gamma} \leq t\}$. Indeed, the strategies that we consider are based on observations/calculations of the two-dimensional process (X, Π): $X = X^D = Y - D$ is observed and represents the value of resources after extraction whereas Π is calculated and represents the adjusted belief of active competition, i.e., the conditional probability that $\theta = 1$ given that stopping has not yet occurred (see Section 3.2). Remarkably, our controller-and-stopper non-zero-sum game with incomplete information has an equilibrium with a rather explicit structure. In this equilibrium the controller extracts resources and the competitor stops at a randomised stopping time, specified in terms of a generalised intensity, in such a way that the corresponding two-dimensional process (X, Π) reflects obliquely at a given monotone boundary $x = b(p)$ (see Figure 1). To construct this two-dimensional reflected process, including a carefully specified reflection direction, we use the notion of perturbed Brownian motion, (see, e.g., Carmona et al. [5] and Perman and Werner [16]). To the best of our knowledge, it is the first time that a perturbed Brownian motion has been used as part of the solution in a stochastic control problem. We also remark how the structure of the equilibrium strategy, determined by this two-dimensional reflection, differs completely from the equilibrium found in the controller-and-stopper "ghost" game in [8]. In fact, in equilibrium, the two players act simultaneously when the sufficient statistic (X, Π) hits a certain boundary: the controller exerts control in the (negative) x-direction and the ghost stopper exerts control in the (negative) π-direction in such a way that the two-dimensional process (X, Π) reflects obliquely along the boundary.

The paper is organized as follows. In Section 2 we provide the precise game formulation of the de Finetti problem under uncertain competition. In Section 3 we review the standard single-player de Finetti problem and we provide properties of its game version that should hold in equilibrium using heuristic arguments. Section 4 uses the notion of perturbed Brownian motion to construct the candidate equilibrium. Our main result Theorem 11, in which the candidate equilibrium is verified, is presented in Section 5. Finally, Section 6 illustrates our findings with a numerical study.

2 Problem set-up

We begin by setting the mathematical stage necessary for our analysis. Throughout the paper, we let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which a standard Brownian motion W, a Bernoulli random variable θ with $\mathbb{P}(\theta = 1) = 1 - \mathbb{P}(\theta = 0) = p \in [0, 1]$ and a Uniform-$(0, 1)$ random variable U are defined. Moreover, W, θ and U are assumed to be independent.

We consider a stochastic game between Player 1 and Player 2 in which both players seek to maximise certain quantities to be specified below. Let Y be a Brownian motion with drift given by

$$Y_t = x + \mu t + \sigma W_t,$$

where the initial condition satisfies $x \geq 0$ and μ and σ are given positive constants. Denote by $\mathbb{F}^W = (\mathcal{F}^W_t)_{0 \leq t < \infty}$ the augmentation of the filtration generated by the Brownian motion W; this filtration will represent the information that Player 1 (the "controller") is equipped with.

Definition 1 (Admissible controls for Player 1)**.** *An admissible control for Player 1 is a non-decreasing, right-continuous, $\mathbb{F}^W$-adapted processes $D = (D_t)_{t \geq 0}$ satisfying $D_{0-} = 0$ and $D_{\tau_0} \leq Y_{\tau_0}$ on $\{\tau_0 < \infty\}$, where $\tau_0 := \inf\{s \geq 0 : D_s \geq Y_s\}$. We denote by $\mathcal{A}_1$ the set of admissible controls for Player 1.*

For any strategy $D \in \mathcal{A}_1$, let $X = X^D := Y - D$ and define

$$\tau^X_0 := \inf\{t \geq 0 : X_t \leq 0\}. \tag{1}$$

To simplify the notation, we will often omit the superscript and simply write X instead of X^D and τ_0 instead of τ^X_0.

In order to let Player 2 (the "competitor") hide his existence, he will be equipped with randomized stopping times. To define the strategies of Player 2, we denote by $\mathcal{D}$ the Skorokhod space of cadlag paths on $[0, \infty)$.

Definition 2 (Admissible controls for Player 2)**.** *An admissible control $\Gamma = (\Gamma_t(X))_{t \geq 0}$ for Player 2 is a mapping $(t, X) \mapsto \Gamma_t(X)$ from $[0-, \infty) \times \mathcal{D}$ into $[0, 1]$ which is progressively measurable for the canonical filtration on $\mathcal{D}$, non-decreasing and right-continuous in t, and satisfying $\Gamma_{0-}(X) = 0$. We denote by $\mathcal{A}_2$ the set of admissible controls for Player 2.*

Given a pair of admissible strategies $(D, \Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$, we define a randomized stopping time γ as

$$\gamma := \gamma^\Gamma := \inf\{t \geq 0 : \Gamma_t(X^D) > U\}, \tag{2}$$

where we recall that U is a random variable which is Unif(0,1)-distributed and independent of θ and W. In accordance with the notation for $X = X^D$, we will often omit the superscript and simply write γ instead of γ^Γ.

Remark 3. *We note that Player 2 selects a universal map Γ that he will apply to any given path of $X = Y - D$ to generate his randomized stopping time $\gamma = \gamma^\Gamma$ in (2). In this way, Player 2 is equipped with feed-back controls, and we will obtain a Markovian game structure.*

Given a fixed discount rate $r > 0$ and a pair $(D, \Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$, we define the payoffs for Player 1 and Player 2 as

$$J_1(x, p, D, \Gamma) := \mathbb{E}\left[\int_0^{\tau_0 \wedge \hat{\gamma}} e^{-rt} dD_t\right] \tag{3}$$

and

$$J_2(x,p,D,\Gamma) := \mathbb{E}\left[e^{-r(\tau_0\wedge\gamma)}X_{\tau_0\wedge\gamma}\right], \tag{4}$$

respectively, where $\tau_0 = \tau_0^X$ and $\gamma = \gamma^\Gamma$ are defined as in (1)-(2), and

$$\hat{\gamma} := \begin{cases} \gamma & \text{if } \theta = 1 \\ \infty & \text{if } \theta = 0. \end{cases}$$

The integral in (3) is interpreted in the Lebesgue-Stieltjes sense, with

$$\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t := \int_{[0,\tau_0\wedge\hat{\gamma}]} e^{-rt}dD_t.$$

The inclusion of the lower limit 0 of integration thus accounts for the contribution to Player 1 from an initial push $dD_0 = D_0 > 0$.

Each player seeks to maximise their respective profit, and we are looking for a Nash equilibrium to this non-zero-sum game in the sense of the following definition.

Definition 4. *A pair $(D^*,\Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium (NE) if*

$$J_1(x,p,D^*,\Gamma^*) \geq J_1(x,p,D,\Gamma^*)$$
$$J_2(x,p,D^*,\Gamma^*) \geq J_2(x,p,D^*,\Gamma)$$

for any pair $(D,\Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$.

Remark 5. *Note that it is a consequence of our set-up that Player 1 has precedence over Player 2 in the sense that if a lump sum $dD_t > 0$ is paid out at the same time $t = \hat{\gamma}$ as Player 2 stops, then Player 1 receives the lump sum, whereas Player 2 receives the reduced amount $Y_t - D_t$. Consequently, since Player 1 may choose a strategy with $D_0 = x$, for any Nash equilibrium $(D^*,\Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ we must have*

$$J_1(x,p,D^*,\Gamma^*) = \sup_{D\in\mathcal{A}_1} J_1(x,p,D,\Gamma^*) \geq x.$$

Remark 6. *Notice that for Player 2 we have chosen to maximise his expected payoff when he is active, i.e., when $\theta = 1$. Alternatively, one could set Player 2 to maximise*

$$\hat{J}_2(x,p,D,\Gamma) := \mathbb{E}\left[\theta e^{-r(\tau_0\wedge\hat{\gamma})}X_{\tau_0\wedge\hat{\gamma}}\right].$$

The formulations for J_2 and $\hat{J}_2$ have the following interpretations. Imagine that before the game starts, at time $t = 0-$, neither player knows θ and that the value of θ will be revealed to Player 2 at time $t = 0$. Then, $\hat{J}_2$ is the expected payoff for Player 2 at time $t = 0-$, whereas J_2 is the expected payoff at time $t = 0$ when $\theta = 1$. These games are referred to as the ex-ante version of the game and the interim version of the game, respectively (see [2, 9] for classical theory of games under incomplete information). Also notice that the two formulations are equivalent as by independence one obtains $\hat{J}_2(x,p,D,\Gamma) = pJ_2(x,p,D,\Gamma)$ and so the second inequality in Definition 4 can be equivalently replaced by $\hat{J}_2(x,p,D^*,\Gamma^*) \geq \hat{J}_2(x,p,D^*,\Gamma)$ for $p > 0$.

Proposition 7. *For a given pair $(D,\Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$, we have*

$$J_1(x,p,D,\Gamma) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1 - p\Gamma_{t-})dD_t\right],$$

where $\Gamma_t := \Gamma_t(X^D)$.

Proof. By conditioning, we have

$$\begin{aligned}
J_1(x,p,D,\Gamma) &= \mathbb{E}\Big[\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t\Big] \\
&= p\mathbb{E}\Big[\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t\Big|\theta=1\Big] + (1-p)\mathbb{E}\Big[\int_0^{\tau_0\wedge\hat{\gamma}} e^{-rt}dD_t\Big|\theta=0\Big] \\
&= p\mathbb{E}\Big[\int_0^{\tau_0\wedge\gamma} e^{-rt}dD_t\Big] + (1-p)\mathbb{E}\Big[\int_0^{\tau_0} e^{-rt}dD_t\Big] \qquad (5)
\end{aligned}$$

For every $\rho \in [0,1)$, let $\gamma(\rho) := \inf\{t \geq 0 : \Gamma_t(X) > \rho\}$. Then, by Fubini's theorem, we have that

$$\mathbb{E}\Big[\int_0^{\tau_0\wedge\gamma} e^{-rt}dD_t\Big] = \mathbb{E}\Big[\int_0^1\int_0^{\tau_0\wedge\gamma(\rho)} e^{-rt}dD_t d\rho\Big] = \mathbb{E}\Big[\int_0^{\tau_0} e^{-rt}\Big\{\int_0^1 \mathbb{1}_{\{t\leq\gamma(\rho)\}}d\rho\Big\}dD_t\Big]. \quad (6)$$

Note that

$$\{\Gamma_{t-} \leq \rho\} = \{t \leq \gamma(\rho)\},$$

so

$$\int_0^1 \mathbb{1}_{\{t\leq\gamma(\rho)\}}d\rho = \int_0^1 \mathbb{1}_{\{\Gamma_{t-}\leq\rho\}}d\rho = 1-\Gamma_{t-}. \qquad (7)$$

Combining (5), (6) and (7), we obtain

$$J_1(x,p,D,\Gamma) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1-p\Gamma_{t-})dD_t\right].$$

□

3 Background material and heuristics

3.1 The single-player de Finetti problem

Note that if $p = 0$, then Player 1 acts under no competition and thus faces the standard de Finetti problem for which the value function

$$V(x) := \sup_{D\in\mathcal{A}_1} \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}dD_t\right] \qquad (8)$$

and the optimal strategy $\tilde{D}$ are well known (see, e.g., [12]). To describe this solution in more detail, let ψ be the unique increasing solution of

$$\mathcal{L}\psi(x) = 0, \quad x \geq 0,$$

with $\psi(0) = 0$ and $\psi'(0) = 1$, where $\mathcal{L}$ denotes the differential operator

$$\mathcal{L} := \frac{\sigma^2}{2}\partial_x^2 + \mu\partial_x - r. \qquad (9)$$

More explicitly,

$$\psi(x) = \frac{e^{\zeta_2 x} - e^{\zeta_1 x}}{\zeta_2 - \zeta_1}, \qquad (10)$$

where ζ_i, $i = 1, 2$ are the solutions of the quadratic equation

$$\zeta^2 + \frac{2\mu}{\sigma^2}\zeta - \frac{2r}{\sigma^2} = 0$$

with $\zeta_1 < 0 < \zeta_2$. Setting

$$B := \frac{\ln(\zeta_1^2) - \ln(\zeta_2^2)}{\zeta_2 - \zeta_1}, \tag{11}$$

we have that ψ is concave on $[0, B]$ and convex on (B, ∞), and

$$V(x) = \begin{cases} \frac{\psi(x)}{\psi'(B)}, & x \leq B, \\ x - B + V(B), & x > B. \end{cases} \tag{12}$$

Moreover,

$$\tilde{D}_t = \sup_{s \in [0,t]} \left(Y_s - B\right)^+ \tag{13}$$

is an optimal strategy in (8), i.e.,

$$V(x) = \mathbb{E}\left[\int_0^{\tilde{\tau}_0} e^{-rt} d\tilde{D}_t\right],$$

where $\tilde{X} := X^{\tilde{D}}$ and $\tilde{\tau}_0 := \tau_0^{\tilde{X}}$. We also remark that $(\tilde{X}, \tilde{D})$ is the solution of a Skorokhod reflection problem with reflection at the barrier B.

3.2 Adjusted beliefs

We now return to our version of the game including a ghost feature as described in Section 2. At the beginning of the game, from the perspective of Player 1 there is active competition (i.e., $\theta = 1$) with probability p. As time passes, and if no stopping occurs, Player 1's conditional probability of competition Π will decrease. More precisely, at time $t \geq 0$, assuming that the strategy pair $(D, \Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$ is played, we have

$$\begin{aligned} \Pi_t = \Pi_t^{\Gamma} \quad &:= \quad \mathbb{P}(\theta = 1 | \mathcal{F}_t^W, \hat{\gamma} > t) = \frac{\mathbb{P}(\theta = 1, \hat{\gamma} > t | \mathcal{F}_t^W)}{\mathbb{P}(\hat{\gamma} > t | \mathcal{F}_t^W)} \\ &= \quad \frac{p\mathbb{P}(\gamma > t | \mathcal{F}_t^W)}{(1-p) + p\mathbb{P}(\gamma > t | \mathcal{F}_t^W)} = \frac{p(1 - \Gamma_t(X^D))}{1 - p\Gamma_t(X^D)} \end{aligned} \tag{14}$$

since $\mathbb{P}(\gamma > t | \mathcal{F}_t^W) = 1 - \mathbb{P}(U \leq \Gamma_t | \mathcal{F}_t^W) = 1 - \Gamma_t$ for $\Gamma = \Gamma(X^D)$. Moreover, since the initial probability of the event $\{\theta = 1\}$ is p, we also have $\Pi_{0-} := p$. Also note that solving for Γ_t in the equation above gives

$$\Gamma_t = \Gamma_t^{\Pi} = \frac{p - \Pi_t}{p(1 - \Pi_t)}, \tag{15}$$

so there is a bijection between Π and Γ.

3.3 Heuristics

This section is intended to illustrate the heuristic arguments which lead to the formulation of a Nash equilibrium for our problem. These heuristics will be rigorously supported in Theorem 11.

Since

$$J_1(x, p, D, \Gamma) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1 - p\Gamma_{t-}) dD_t\right] \leq \mathbb{E}\left[\int_0^{\tau_0} e^{-rt} dD_t\right] \leq V(x)$$

for any strategy pair $(D, \Gamma) \in \mathcal{A}_1 \times \mathcal{A}_2$, it is clear that the risk of competition decreases the value from the perspective of Player 1. On the other hand, to obtain a lower bound, let $\tilde{D}$ denote the optimal control of the single-player de Finetti problem, see (13). Then,

$$J_1(x, p, \tilde{D}, \Gamma) = \mathbb{E}\left[\int_0^{\tilde{\tau}_0} e^{-rt}(1 - p\Gamma_{t-}) d\tilde{D}_t\right] \geq (1-p)\mathbb{E}\left[\int_0^{\tilde{\tau}_0} e^{-rt} d\tilde{D}_t\right] = (1-p)V(x)$$

for any $\Gamma \in \mathcal{A}_2$. It is thus clear that

$$(1-p)V(x) \leq J_1(x, p, D^*, \Gamma^*) \leq V(x) \tag{16}$$

if $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ is a Nash equilibrium.

In this section we will provide heuristic arguments to obtain a candidate Nash equilibrium $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$. To do that, we make the Ansatz that

(a) there exists a non-increasing continuous boundary $p = c(x)$ such that the overall effect of the equilibrium strategy $(D^*, \Gamma^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ amounts to reflection of the two-dimensional process $(X^*, \Pi^*) = (Y - D^*, \Pi^{\Gamma^*})$ along this boundary (see Figure 1);

(b) the corresponding equilibrium value v of Player 1 satisfies

$$v(x, p) = (1-p)V(x), \qquad \text{for } p \leq c(x). \tag{17}$$

Note that by the bijection between Γ and Π we have that $\Pi^* = \Pi^*(X^D)$ for every $D \in \mathcal{A}_1$ and to obtain the reflection of (X^*, Π^*) along the monotone boundary c we need that

$$\Pi_t^* = \Pi_t^*(X^D) = p \wedge c\Big(\sup_{0 \leq s \leq t} X_s^D\Big), \qquad \text{for } t \geq 0. \tag{18}$$

With a slight abuse of notation, Π^* will be used to indicate both $\Pi^*(X^D)$ and $\Pi^*(X^*)$ but this will be clear from the context as it will depend on whether Player 1 plays an arbitrary admissible strategy $D \in \mathcal{A}_1$ or the equilibrium strategy D^*.

Notice also that the Ansatz (17) coincides with the lower bound in (16) and that it thus bears some resemblance with the equilibrium obtained in the ghost Dynkin game studied in [6].

Given this Ansatz, we further need to determine

(i) the boundary c;

(ii) the direction of reflection when the process (X^*, Π^*) is at the boundary;

(iii) the strategy pair (D^*, Γ^*) corresponding to the reflected process (X^*, Π^*).

(iv) the strategy for starting points (x, p) with $p > c(x)$;

We do this below, and then the candidate Nash equilibrium that we produce is verified in Section 5. Notice that we will not discuss item (iv) here as it is not relevant at this stage, but it will be considered in Theorem 11.

First, let us consider a starting point $(x, p) \in [0, \infty) \times (0, 1)$ with $p \leq c(x)$, and recall that we expect in equilibrium that

$$(X_t^*, \Pi_t^*) = \left(Y_t - D_t^*, p \wedge c\Big(\sup_{0 \leq s \leq t} (Y_s - D_s^*)\Big)\right),$$

for $D^* \in \mathcal{A}_1$ to be specified. Since c is assumed to be continuous and non-increasing, we see that

$$p \wedge c\Big(\sup_{0 \leq s \leq t} (Y_s - D_s)\Big) \leq c(Y_t - D_t) \tag{19}$$

for any choice $D \in \mathcal{A}_1$. By construction, Π^* is continuous and we have

$$\Gamma_t^* = \frac{p - \Pi_t^*}{p(1 - \Pi_t^*)}$$

and

$$d\Pi_t^* = -\frac{1}{1 - \Gamma_t^*} \Pi_t^* (1 - \Pi_t^*) d\Gamma_t^* \tag{20}$$

on $\{t \geq 0 : \Gamma^*_t < 1\}$, cf. (14) and (15).

Let $\hat{\gamma}^* := \gamma^* 1_{\{\theta=1\}} + \infty 1_{\{\theta=0\}}$. Note that by the dynamic programming principle one would expect that the process $M = M^D$ given by

$$M_t := \int_0^{t\wedge\hat{\gamma}^*} e^{-rs} dD_s + e^{-rt} v(X_t, \Pi^*_t) \mathbb{1}_{\{t<\hat{\gamma}^*\}}$$

is an $\mathbb{F}^{W,\hat{\gamma}^*}$-martingale if $D = D^* \in \mathcal{A}_1$ is an optimal response to $\Gamma^* \in \mathcal{A}_2$, and an $\mathbb{F}^{W,\hat{\gamma}^*}$-supermartingale if $D \in \mathcal{A}_1$ is any admissible response. Here, $\mathbb{F}^{W,\hat{\gamma}^*} = (\mathcal{F}^{W,\hat{\gamma}^*})_{0\leq t<\infty}$ is the smallest right-continuous filtration to which W and $\mathbb{1}_{\{\cdot\geq\hat{\gamma}^*\}}$ are adapted, augmented with the $\mathbb{P}$-null sets of Ω. Moreover, by conditioning (cf. Proposition 7), M is an $\mathbb{F}^{W,\hat{\gamma}^*}$-(super)martingale if and only if

$$\hat{M}_t := \int_0^t e^{-rs}(1 - p\Gamma^*_{s-})\, dD_s + e^{-rt}(1 - p\Gamma^*_t) v(X_t, \Pi^*_t)$$

is an $\mathbb{F}^W$-(super)martingale.

Thus, by an application of Ito's formula, we see that when Player 2 plays the equilibrium strategy Γ^* and (X^*, Π^*) is at the boundary we need that

$$(1 - v_x)\, dD^*_t - \frac{\Pi^*_t}{1 - \Gamma^*_t}\big((1 - \Pi^*_t)v_p \ + v\big)\, d\Gamma^*_t = 0 \qquad \text{(optimality)};$$

whereas, when Player 2 plays the equilibrium strategy Γ^* and Player 1 plays any admissible strategy $D \in \mathcal{A}_1$, we need that

$$(1 - v_x)\, dD_t - \frac{\Pi^*_t}{1 - \Gamma^*_t}\big((1 - \Pi^*_t)v_p \ + v\big)\, d\Gamma^*_t \leq 0 \qquad \text{(suboptimality)},$$

We stress that Π^* here stands for $\Pi^*(X^*)$ in the optimality condition and $\Pi^*(X^D)$ in the suboptimality condition. Note that we obtain from (17) that

$$(1 - p)v_p(x, p) + v(x, p) = 0$$

when $p \leq c(x)$. Thus, to satisfy the optimality condition we need to have $v_x(x, p) = 1$ at the boundary, and consequently the boundary $p = c(x)$ should be defined by

$$(1 - c(x))V'(x) = 1$$

for $x \in [0, B]$ where B is as specified in (11). Hence, for $x \in [0, B]$ we have

$$c(x) = \frac{V'(x) - 1}{V'(x)}, \tag{21}$$

from which it follows immediately that $c(B) = 0$, $c'(x) < 0$, and $c'(x) \to 0$ as $x \nearrow B$ by (12). Let $\hat{p} := (V'(0) - 1)/V'(0)$. Then $c : [0, B] \to [0, \hat{p}]$ is a continuous strictly decreasing bijection and we denote its inverse by $b : [0, \hat{p}] \to [0, B]$, i.e.,

$$b(c(x)) = x, \qquad \forall\, x \in [0, B]. \tag{22}$$

From here on, we will refer to b (instead of c) as the boundary when it is more convenient to do so. By convention, we also extend b and c by continuity and define $b(p) = 0$ for every $p \in (\hat{p}, 1]$, and $c(x) = 0$ for $x \in (B, \infty)$.

Moreover, notice that since $\Pi^*_t \leq c(X^D_t)$, for every admissible strategy $D \in \mathcal{A}_1$, we also have that

$$v_x(X^D_t, \Pi^*_t) = (1 - \Pi^*_t)V'(X^D_t) \geq (1 - c(X^D_t))V'(X^D_t) = 1,$$

so that the suboptimality condition is verified as well.

Since Player 2 in equilibrium only stops at time points when (X^*, Π^*) is at the boundary, we expect his equilibrium value u to be of the form $u(x,p) = g(p)\psi(x)$, for some function g, and to satisfy the condition $u(b(p),p) = b(p)$. Consequently,

$$u(x,p) = b(p)\frac{\psi(x)}{\psi(b(p))} \tag{23}$$

for $x \le b(p)$. Furthermore, by dynamic programming principle arguments, the process

$$N_t = e^{-rt}u(X_t^*, \Pi_t^*)$$

should be a martingale when Player 1 plays the equilibrium strategy D^*. After applying Ito's formula this yields

$$-u_x\, dD_t^* + u_p\, d\Pi_t^* = 0 \tag{24}$$

on the boundary, so the reflection direction of (X^*, Π^*) needs to be $(u_p, -u_x)$.

We now show how to construct the candidate Nash equilibrium (D^*, Γ^*) so that the corresponding process (X^*, Π^*) reflects along the boundary c in the direction $(u_p, -u_x)$. To do that, we first specify Γ^* by setting

$$\Gamma_t^*(X^D) = \frac{p - \Pi_t^*}{p(1-\Pi_t^*)}, \quad \text{for } t \ge 0,$$

(cf. (15)), where $\Pi_t^* = \Pi_t^*(X^D) = p \wedge c(\sup_{0\le s\le t}(X_s^D))$ for an arbitrary strategy $D \in \mathcal{A}_1$. The process (X^D, Π^*) then reflects at the boundary c but the direction of reflection is, for an arbitrary strategy $D \in \mathcal{A}_1$, not necessarily equal to $(u_p, -u_x)$.

One should only push in $X = Y - D$ when the process is at its current maximum (after the first time it hits the boundary). Therefore, one would expect to choose D^* so as to satisfy

$$dD_t^* = \lambda(\bar{X}_t^*)\, d\bar{X}_t^*,$$

where $\bar{X}_t^* := b(p) \vee \sup_{0\le s\le t} X_s^*$ and $X^* = Y - D^*$, for some function λ to be determined. Moreover, from (18) we have that, when Player 1 plays the equilibrium strategy D^*, $\Pi_t^* = c(\bar{X}_t^*)$, so (24) gives

$$\lambda(x) = \frac{c'(x)u_p(x, c(x))}{u_x(x, c(x))}. \tag{25}$$

Using (23), we then get

$$u_x(x, c(x)) = \frac{\psi'(x)}{\psi(x)}x$$

and

$$u_p(x, c(x)) = \frac{\psi(x) - x\psi'(x)}{\psi(x)c'(x)},$$

so

$$\lambda(x) = \frac{\psi(x) - x\psi'(x)}{x\psi'(x)}. \tag{26}$$

and since $\psi(0) = 0$ and ψ is concave on $[0, B]$, we have $\psi(x) \ge x\psi'(x)$ and so $\lambda \ge 0$ on $(0, B]$.

In the next section we study in detail the solvability of the equation

$$X_t^* = Y_t - \int_0^t \lambda(\bar{X}_s^*)\, d\bar{X}_s^*$$

using the notion of *perturbed Brownian motion*, which will allow us to obtain the equilibrium strategy D^* for Player 1.

4 A perturbed Brownian motion with drift

To construct the equilibrium strategy D^* for Player 1 we will use the notion of perturbed Brownian motion, which is a linear Brownian motion that gets an extra push when it hits its current maximum. Here we provide what is needed for the study of our problem, and refer to [5], [16] and the references therein for further details on such processes. First, define $\Lambda : [b(p), B] \to [0, \infty)$ by

$$\Lambda(x) := \int_{b(p)}^{x} \lambda(y)\, dy, \tag{27}$$

where

$$\lambda(x) = \frac{\psi(x)}{x\psi'(x)} - 1$$

as in (26) and the boundary b defined as in (22). Since $\lambda \geq 0$ on $(0, B]$, we note that Λ is increasing. Note also that $\lambda(x)$ is a bounded function for $x \in [0, B]$ so Λ is well-defined. For $x \leq b(p)$ we now consider the equation

$$X_t = Y_t - \Lambda(\bar{X}_t), \quad t \in [0, \tau_B], \tag{28}$$

where $Y_t = x + \mu t + \sigma W_t$, $\bar{X}_t := b(p) \vee \sup_{0 \leq s \leq t} X_s$, and $\tau_B = \tau_B^X := \inf\{t \geq 0 : X_t \geq B\}$. The process X is then a perturbed Brownian motion with drift.

To construct a solution of (28), let

$$\bar{Y}_t := b(p) \vee \sup_{0 \leq s \leq t} Y_s. \tag{29}$$

Define the function $f : [b(p), \infty) \to [b(p), B]$ by the relations

$$\begin{aligned} \Lambda(f(y)) + f(y) &= y, \quad y \in [b(p), \Lambda(B) + B], \\ f(y) &= B, \quad y > \Lambda(B) + B, \end{aligned} \tag{30}$$

i.e., f is the inverse of the increasing function $x \mapsto y := \Lambda(x) + x$ for $y \in [b(p), \Lambda(B) + B]$ and then extended constantly for $y > \Lambda(B) + B$. Now define

$$X_t := Y_t - \bar{Y}_t + f(\bar{Y}_t). \tag{31}$$

Proposition 8. *Assume that $x \leq b(p)$. Then the process X in (31) solves equation (28).*

Proof. Let $t \in [0, \tau_B]$. Since $\bar{X}_t := b(p) \vee \sup_{s \in [0,t]} X_s$ we obtain, from (31), that $\bar{X}_t = f(\bar{Y}_t)$ as $f(b(p)) = b(p)$. Consequently $\tau_B = \inf\{t \geq 0 : Y_t \geq \Lambda(B) + B\}$ and so, by (30), we have $f(\bar{Y}_t) - \bar{Y}_t = -\Lambda(f(\bar{Y}_t))$. This leads to

$$X_t = Y_t - \Lambda(\bar{X}_t),$$

which proves the claim. □

Remark 9. *The set-up in (28) of a perturbed Brownian motion is slightly more general than what is used in most literature on perturbed Brownian motions; in fact, the typical choice of perturbation used in the literature is linear, corresponding to a linear function Λ in (28). On the other hand, we only deal with one-sided perturbation, in which case the solution can be constructed explicitly as in (31) above. It is straightforward to check that the argument for pathwise uniqueness of solutions of (28), cf. [5, Proposition 2.1], carries over to our setting.*

Remark 10. *The function f defined in (30) is constructed in such a way that the process $X_t = Y_t - \bar{Y}_t + f(\bar{Y}_t)$ is a perturbed Brownian motion with drift for $t \in [0, \tau_B]$ (as proved in Proposition 8) and it is the Skorokhod reflection of the process Y_t at the barrier B for $t \in (\tau_B, \infty)$. Indeed, for $t \in (\tau_B, \infty)$, we have*

$$\begin{aligned} X_t &= Y_t - \bar{Y}_t + f(\bar{Y}_t) = Y_t - \bar{Y}_t + B \\ &= Y_t - \sup_{s\in[0,t]} (Y_s - B) = Y_t - \sup_{s\in[0,t]} (Y_s - B)^+, \end{aligned} \tag{32}$$

i.e., we have $X_t = X_t^{\tilde{D}}$ for $t \in (\tau_B, \infty)$ where $\tilde{D}$ is defined as in (13).

5 Main result

In this section, we state and prove our main result: an explicit Nash equilibrium for our game. To do that, let us fix $(x, p) \in [0, \infty) \times [0, 1]$ and recall that Y is given by

$$Y_t = x + \mu t + \sigma W_t.$$

First, define a new process $Y^\wedge$ by

$$Y_t^\wedge := x \wedge b(p) + \mu t + \sigma W_t = Y_t - (x - b(p))^+,$$

so that $Y^\wedge$ starts below the boundary $b(p)$ (recall definition (22)). Then define $\bar{Y}^\wedge$ as in (29) but with $Y^\wedge$ instead of Y, i.e.,

$$\bar{Y}_t^\wedge := b(p) \vee \sup_{0\le s\le t} Y_s^\wedge.$$

Also, recall the definitions of $\Lambda : [b(p), B] \to [0, \infty)$ in (27) and $f : [b(p), \infty) \to [b(p), B]$ in (30), and define $D^* \in \mathcal{A}_1$ by $D^*_{0-} = 0$ and

$$D_t^* := (x - b(p))^+ + \bar{Y}_t^\wedge - f(\bar{Y}_t^\wedge), \quad t \ge 0. \tag{33}$$

Setting

$$X_t^* := Y_t - D_t^*,$$

Proposition 8 applied with $Y^\wedge$ in place of Y yields

$$X_t^* = Y_t^\wedge - \bar{Y}_t^\wedge + f(\bar{Y}_t^\wedge) = Y_t^\wedge - \Lambda(\bar{X}_t^*), \qquad t \in [0, \tau_B^*], \tag{34}$$

where $\tau_B^* = \tau_B^{X^*} := \inf\{t \ge 0 : X_t^* \ge B\}$. Note that by construction we have $dD_t^* = \Lambda(X_t^*) d\bar{X}_t^*$ for $t \in (0, \tau_B]$.

Moreover, for a given path $X = X^D \in \mathcal{D}$ (with $D \in \mathcal{A}_1$), define $Z^* = Z^*(X)$ by $Z^*_{0-} := p$ and

$$Z_t^* := p \wedge c\Big(\sup_{0\le s\le t} X_s\Big), \quad t \ge 0 \tag{35}$$

(cf. (18)), and define $\Gamma^* \in \mathcal{A}_2$ by

$$\Gamma_t^*(X) := \begin{cases} \mathbb{1}_{\{t\ge\tau_B\}}, & p = 0, \\ \frac{p - Z_t^*}{p(1 - Z_t^*)}, & p > 0, \end{cases} \tag{36}$$

where we recall that $\tau_B := \inf\{t \ge 0 : X_t \ge B\}$.

Theorem 11. *Let $(x, p) \in [0, \infty) \times [0, 1]$. The pair (D^*, Γ^*) defined above is a NE for the stochastic game (3)-(4), with equilibrium values*

$$J_1(x, p, D^*, \Gamma^*) = v(x, p) := \begin{cases} (1 - p)V(x), & x \le b(p), \\ (1 - p)V(b(p)) + x - b(p), & x > b(p), \end{cases}$$

$$J_2(x, p, D^*, \Gamma^*) = u(x, p) := \begin{cases} b(p) \frac{\psi(x)}{\psi(b(p))}, & x \le b(p), \\ b(p), & x > b(p), \end{cases}$$

(with the understanding that $b(p)\psi(x)/\psi(b(p)) = 0$ for $x = 0$ also when $b(p) = 0$). Here, V is the value of the single-player de Finetti problem given in (12), b is defined in (22) and ψ is given by (10).

Proof. Step 1. We first prove that D^* is an optimal response to Γ^*. Let $D \in \mathcal{A}_1$ be an arbitrary strategy for Player 1 and set $X := Y - D$. Let Z^* be defined as in (35) and $\Gamma^*_t := \Gamma^*_t(X)$ as in (36) accordingly .

If $p = 0$, then $\theta = 0$ a.s. and so

$$J_1(x, 0, D, \Gamma^*) = \mathbb{E}\bigg[\int_0^{\tau_0} e^{-rt} dD_t\bigg].$$

Namely, the optimization problem for Player 1 degenerates into the single-player de Finetti problem, and D^* coincides with its optimal solution $\tilde{D}$, as highlighted in Remark 10. Hence, also $v(x, 0) = J_1(x, 0, D^*, \Gamma^*) \geq J_1(x, 0, D, \Gamma^*)$ for every $D \in \mathcal{A}_1$.

If $x = 0$, then $J_1(0, p, D, \Gamma^*) = 0$ for every $p \in [0, 1]$, $D \in \mathcal{A}_1$ and so, in particular, $v(0, p) = J_1(0, p, D^*, \Gamma^*)$ for every $p \in [0, 1]$.

Now let $p \in (0, 1]$ and let us first consider $0 < x \leq b(p)$ (note that this implies that $p \in (0, \hat{p})$ as $b(p) = 0$ for every $p \in [\hat{p}, 1]$). By (36), we have

$$Z^*_t = \frac{p(1 - \Gamma^*_t)}{1 - p\Gamma^*_t}, \quad t \geq 0.$$

Since Z^* and Γ^* are continuous and of finite variation, we obtain

$$dZ^*_t = -\frac{p(1 - Z^*_t)}{1 - p\Gamma^*_t}\, d\Gamma^*_t, \quad t \geq 0.$$

Now define

$$\tilde{v}(x, p) := (1 - p)V(x) \in C^2([0, \infty) \times [0, 1]).$$

By setting $\tau := \tau_0 \wedge T$ with $T \geq 0$ and applying Ito's formula to $e^{-rt}(1 - p\Gamma^*_t)\tilde{v}(X_t, Z^*_t)$, we have that

$$\begin{aligned} e^{-r\tau}(1 - p\Gamma^*_\tau)\tilde{v}(X_\tau, Z^*_\tau) = \tilde{v}(x, p) &+ \int_0^\tau e^{-rt}(1 - p\Gamma^*_t)\mathcal{L}\tilde{v}(X_{t-}, Z^*_t)\, dt \\ &- \int_0^\tau e^{-rt}(1 - p\Gamma^*_t)\tilde{v}_x(X_{t-}, Z^*_t)\, dD^c_t \\ &+ \int_0^\tau \sigma e^{-rt}(1 - p\Gamma^*_t)\tilde{v}_x(X_{t-}, Z^*_t)\, dW_t \\ &- \int_0^\tau e^{-rt} p\big[(1 - Z^*_t)\tilde{v}_p(X_{t-}, Z^*_t) + \tilde{v}(X_{t-}, Z^*_t)\big] d\Gamma^*_t \\ &+ \sum_{0 \leq t \leq \tau} e^{-rt}(1 - p\Gamma^*_t)\big(\tilde{v}(X_t, Z^*_t) - \tilde{v}(X_{t-}, Z^*_t)\big), \end{aligned} \tag{37}$$

where $\mathcal{L}$ is defined as in (9) and D^c denotes the continuous part of D. Notice that $\tilde{v}(x, p) = v(x, p)$ for $x \leq b(p)$ and that by definition of $\tilde{v}$, we have for every $t > 0$

$$\mathcal{L}\tilde{v}(X_{t-}, Z^*_t) = 0 \quad \text{and} \quad (1 - Z^*_t)\tilde{v}_p(X_{t-}, Z^*_t) + \tilde{v}(X_{t-}, Z^*_t) = 0.$$

Hence, equation (37) becomes

$$\begin{aligned} v(x, p) = e^{-r\tau}(1 - p\Gamma^*_\tau)\tilde{v}(X_\tau, Z^*_\tau) &+ \int_0^\tau e^{-rt}(1 - p\Gamma^*_t)\tilde{v}_x(X_{t-}, Z^*_t)\, dD^c_t \\ - \int_0^\tau \sigma e^{-rt}(1 - p\Gamma^*_t)\tilde{v}_x(X_{t-}, Z^*_t)\, dW_t& \\ - \sum_{0 \leq t \leq \tau} e^{-rt}(1 - p\Gamma^*_t)\big(\tilde{v}(X_t, Z^*_t) - \tilde{v}(X_{t-}, Z^*_t)\big).& \end{aligned} \tag{38}$$

For the summation term we have by the mean value theorem that

$$\sum_{0\le t\le\tau} e^{-rt}(1-p\Gamma_t^*)\big(\tilde v(X_t,Z_t^*)-\tilde v(X_{t-},Z_t^*)\big) = -\sum_{0\le t\le\tau} e^{-rt}(1-p\Gamma_t^*)\tilde v_x(\xi_t,Z_t^*)\Delta D_t \tag{39}$$

where $\xi_t \in (X_{t-},X_t)$ and $\Delta D_t := D_t - D_{t-}$. By plugging (39) into (38), and using that $\tilde v \ge 0$ and $\tilde v_x \ge 1$, we obtain

$$v(x,p) \ge \int_0^\tau e^{-rt}(1-p\Gamma_t^*)\,dD_t - \int_0^\tau \sigma e^{-rt}(1-p\Gamma_t^*)\tilde v_x(X_{t-},Z_t^*)\,dW_t. \tag{40}$$

Let

$$\mathcal{O} := \{(x,p)\in[0,\infty)\times[0,1] : x \le b(p)\} \cup ((B,\infty)\times\{0\}) \tag{41}$$

and note that $(X_{t-},Z_t^*)\in\mathcal{O}$ for every $t\ge0$ (by construction of Z_t) and that $\tilde v_x$ is bounded on $\mathcal{O}$ ($\tilde v_x(x,p)=1$ for $(x,p)\in(B,\infty)\times\{0\}$). Thus, the stochastic integral above is a martingale and by an application of the optional sampling theorem we have that

$$\tilde v(x,p) \ge \mathbb{E}\left[\int_0^{\tau_0\wedge T} e^{-rt}(1-p\Gamma_t^*)\,dD_t\right].$$

Letting $T\to\infty$ yields, by the monotone convergence theorem,

$$v(x,p) \ge \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1-p\Gamma_t^*)\,dD_t\right] = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(1-p\Gamma_{t-}^*)\,dD_t\right] = J_1(x,p,D,\Gamma^*)$$

for every $D\in\mathcal{D}$, where the last equality follows by Proposition 7.

Now notice that D_t^* defined in (33) is continuous for every $t\ge0$, when $x\le b(p)$, and that the same holds for $X_t^* := X_t^{D^*}$. Let $\tau_0^* := \tau_0^{X^*}$, then equation (38) for $D=D^*$ and $\tau^* := \tau_0^*\wedge T$ becomes

$$\begin{aligned}
v(x,p) &= e^{-r\tau^*}(1-p\Gamma_{\tau^*}^*)\tilde v(X_{\tau^*}^*,Z_{\tau^*}^*) + \int_0^{\tau^*} e^{-rt}(1-p\Gamma_t^*)\tilde v_x(X_t^*,Z_t^*)\,dD_t^* \\
&\quad - \int_0^{\tau^*} \sigma e^{-rt}(1-p\Gamma_t^*)\tilde v_x(X_t^*,Z_t^*)\,dW_t \\
&= e^{-r\tau^*}(1-p\Gamma_{\tau^*}^*)\tilde v(X_{\tau^*}^*,Z_{\tau^*}^*) + \int_0^{\tau^*} e^{-rt}(1-p\Gamma_t^*)\,dD_t^* \\
&\quad - \int_0^{\tau^*} \sigma e^{-rt}(1-p\Gamma_t^*)v_x(X_t^*,Z_t^*)\,dW_t,
\end{aligned}$$

where the last equality holds since $\tilde v_x(x,p)=1$ if $x\ge b(p)$ and $dD_t^*=0$ if $X_t^*<b(Z_t^*)$. Hence, again by taking expected values, we obtain

$$\begin{aligned}
v(x,p) &= \mathbb{E}\left[e^{-r(\tau_0^*\wedge T)}(1-p\Gamma_{\tau_0^*\wedge T}^*)\tilde v(X_{\tau_0^*\wedge T}^*,Z_{\tau_0^*\wedge T}^*) + \int_0^{\tau_0^*\wedge T} e^{-rt}(1-p\Gamma_t^*)\,dD_t^*\right] \\
&\to \mathbb{E}\left[\int_0^{\tau_0^*} e^{-rt}(1-p\Gamma_t^*)\,dD_t^*\right]
\end{aligned}$$

as $T\to\infty$ by dominated convergence (the first term tends to 0 since $\tilde v(X_{\tau_0^*}^*,Z_{\tau_0^*}^*)=0$). Thus, we have proved that

$$J_1(x,p,D^*,\Gamma^*) = v(x,p) \ge \sup_{D\in\mathcal{A}_1} J_1(x,p,D,\Gamma^*), \quad \forall\,(x,p)\in\mathcal{O}.$$

Let us now consider $(x,p)\in([0,\infty)\times[0,1])\setminus\mathcal{O} =: \mathcal{O}^c$, i.e., $x>b(p)$ with $p\ne0$. Then,

$$v(x,p) = v(b(p),p) + x - b(p) = J_1(b(p),p,D^*,\Gamma^*) + x - b(p) = J_1(x,p,D^*,\Gamma^*).$$

Thus, we are left to prove that also in this case

$$J_1(x,p,D^*,\Gamma^*) \geq J_1(x,p,D,\Gamma^*), \quad \forall\, D \in \mathcal{A}_1.$$

For $(x,p) \in \mathcal{O}^c$, let the admissible strategy $D \in \mathcal{A}_1$ have an initial jump $\Delta D_0 = x - y$ with either $b(p) \leq y \leq x$ or $0 \leq y < b(p)$. In the former case, by definition (36) of Γ^*, we have that

$$J_1(x,p,D,\Gamma^*) = (1-\Gamma^*_0)J_1(b(q),q,D,\Gamma^*) + x - y = \frac{q(1-p)}{p}V(b(q)) + x - y,$$

where $q := c(y) \leq p$ (and hence $y = b(q)$). Since V is concave with $V'(b(p)) = 1/(1-p)$, then

$$\begin{aligned} J_1(x,p,D,\Gamma^*) &\leq \frac{q(1-p)}{p}\Big(V(b(p)) + \frac{y-b(p)}{1-p}\Big) + x - y \\ &= \frac{q}{p}\Big((1-p)V(b(p)) + y - b(p)\Big) + x - y \\ &\leq (1-p)V(b(p)) + x - b(p) = J_1(x,p,D^*,\Gamma^*). \end{aligned}$$

If instead $0 \leq y < b(p)$, then by a similar argument

$$\begin{aligned} J_1(x,p,D,\Gamma^*) &= J_1(y,p,D,\Gamma^*) + x - y = (1-p)V(y) + x - y \\ &\leq (1-p)V(b(p)) + x - b(p) = J_1(x,p,D^*,\Gamma^*). \end{aligned}$$

This concludes Step 1, i.e., shows that the strategy D^* is an optimal response to Γ^*.

Step 2. We now prove that Γ^* is an optimal response to D^*. Recall that

$$u(x,p) := \begin{cases} b(p)\frac{\psi(x)}{\psi(b(p))}, & x \leq b(p), \\ b(p), & x > b(p), \end{cases}$$

set $X^* := X^{D^*}$ with D^* defined in (33), $\tau^*_0 := \tau^{X^*}_0$, and let

$$Z^*_t := p \wedge c\Big(\sup_{0\leq s\leq t} X^*_s\Big), \quad t \geq 0, \qquad Z^*_{0-} := p,$$

as in (35) with $D = D^*$.

Let $p \in [0,1]$ and assume $x \leq b(p)$. If $p \in [\hat{p},1]$, then $b(p) = 0$ and so $x = 0$ and the strategy $\Gamma \in \mathcal{A}_2$ is irrelevant since the game stops immediately. It hence suffices to check $p \in [0,\hat{p})$. For notational convenience we treat the case $p = 0$ separately at the end and assume first $p \in (0,\hat{p})$. Note that $X^*_t \leq b(Z^*_t)$ for every $t \geq 0$ and that Z^*_t, D^*_t and X^*_t are continuous for every $t \geq 0$. Define

$$\tilde{u}(x,p) := b(p)\frac{\psi(x)}{\psi(b(p))} \in C^2([0,\infty) \times (0,\hat{p})).$$

and let τ be any $\mathbb{F}^W$-stopping time s.t. $\tau \leq \tau^*_B$ *a.s.*, where $\tau^*_B = \inf\{t \geq 0 : X^*_t \geq B\}$. Define $\tau^* = \tau^*_{\varepsilon,T} := \tau^*_0 \wedge \tau^*_{B-\varepsilon} \wedge \tau \wedge T$ for $T, \varepsilon \geq 0$ arbitrary and note that $Z^*_t > 0$ for $t \in [0,\tau^*)$. By applying Ito's formula to $e^{-rt}\tilde{u}(X^*_t, Z^*_t)$ we obtain

$$\begin{aligned} e^{-r\tau^*}\tilde{u}(X^*_{\tau^*}, Z^*_{\tau^*}) &= \tilde{u}(x,p) + \int_0^{\tau^*} e^{-rs}\mathcal{L}\tilde{u}(X^*_s, Z^*_s)\,ds - \int_0^{\tau^*} e^{-rs}\tilde{u}_x(X^*_s, Z^*_t)\,dD^*_s \\ &\quad + \int_0^{\tau^*} \sigma e^{-rs}\tilde{u}_x(X^*_s, Z^*_s)\,dW_s + \int_0^{\tau^*} e^{-rs}\tilde{u}_p(X^*_s, Z^*_s)\,dZ^*_s. \end{aligned}$$

By definition of $\tilde{u}$, we have that $\mathcal{L}\tilde{u}(X^*_s, Z^*_s) = 0$ for every $0 \leq s \leq \tau^*$ and by construction of D^* and Z^* (recall (34)), we obtain

$$\begin{aligned} &\int_0^{\tau^*} e^{-rs}\tilde{u}_p(X^*_s, Z^*_s)\,dZ^*_s - \int_0^{\tau^*} e^{-rs}\tilde{u}_x(X^*_s, Z^*_s)\,dD^*_s \\ &\quad = \int_0^{\tau^*} e^{-rs}\Big(\tilde{u}_p(X^*_s, Z^*_s)c'(X^*_s) - \tilde{u}_x(X^*_s, Z^*_s)\lambda(X^*_s)\Big)\,d\bar{X}^*_s = 0 \end{aligned} \tag{42}$$

where the last equality holds by definition of λ in (25). Hence,

$$e^{-r\tau^*}\tilde{u}(X^*_{\tau^*}, Z^*_{\tau^*}) = \tilde{u}(x,p) + \int_0^{\tau^*} \sigma e^{-rs}\tilde{u}_x(X^*_s, Z^*_s)\, dW_s. \tag{43}$$

Since $\tilde{u}_x$ is bounded on $\{(x,p) : x \le b(p)\}$, the stochastic integral in (43) is a martingale. Since X^* and Z^* are continuous, applying the optional sampling theorem and using dominated convergence yields

$$\tilde{u}(x,p) = \mathbb{E}\Big[e^{-r\tau^*}\tilde{u}(X^*_{\tau^*}, Z^*_{\tau^*})\Big] \to \mathbb{E}\Big[e^{-r(\tau_0^*\wedge\tau)}\tilde{u}(X^*_{\tau_0^*\wedge\tau}, Z^*_{\tau_0^*\wedge\tau})\Big],$$

as $T \to \infty$ and $\varepsilon \to 0$, so

$$\tilde{u}(x,p) = \mathbb{E}\Big[e^{-r(\tau_0^*\wedge\tau)}\tilde{u}(X^*_{\tau_0^*\wedge\tau}, Z^*_{\tau_0^*\wedge\tau})\Big] \tag{44}$$

for any $\mathbb{F}^W$-stopping time $\tau \le \tau_B$ *a.s.* Now, for any $\Gamma \in \mathcal{A}_2$, define the $\mathbb{F}^W$-stopping times

$$\gamma(\rho) := \inf\{t \ge 0 : \Gamma_t(X^*) > \rho\}, \qquad \rho \in [0,1),$$

and let $\gamma_B(\rho) := \gamma(\rho) \wedge \tau_B^* \le \tau_B^*$. Since $\tilde{u} = u$ on $\{(x,p) : x \le b(p)\}$, equality (44) for $\tau = \gamma_B(\rho)$ reads

$$u(x,p) = \mathbb{E}\Big[e^{-r(\tau_0^*\wedge\gamma_B(\rho))}u(X^*_{\tau_0^*\wedge\gamma_B(\rho)}, Z^*_{\tau_0^*\wedge\gamma_B(\rho)})\Big], \qquad \rho \in [0,1).$$

Thus,

$$\begin{aligned} u(x,p) &= \int_0^1 \mathbb{E}\Big[e^{-r(\tau_0^*\wedge\gamma_B(\rho))}u(X^*_{\tau_0^*\wedge\gamma_B(\rho)}, Z^*_{\tau_0^*\wedge\gamma_B(\rho)})\Big]\, d\rho \\ &\ge \int_0^1 \mathbb{E}\Big[e^{-r(\tau_0^*\wedge\gamma_B(\rho))}X^*_{\tau_0^*\wedge\gamma_B(\rho)}\Big]\, d\rho \end{aligned} \tag{45}$$

where the inequality holds because $\psi(x)$ is concave for $x \le B$ with $\psi(0) = 0$.

Last, we note that

$$e^{-r(\tau_0^*\wedge\gamma_B(\rho))}X^*_{\tau_0^*\wedge\gamma_B(\rho)} \ge e^{-r(\tau_0^*\wedge\gamma(\rho))}X^*_{\tau_0^*\wedge\gamma(\rho)} \qquad a.s. \tag{46}$$

since $X_t^* \le B$ for all $t > 0$ and $r > 0$ and thus

$$u(x,p) \ge \int_0^1 \mathbb{E}\Big[e^{-r(\tau_0^*\wedge\gamma(\rho))}X^*_{\tau_0^*\wedge\gamma(\rho)}\Big] d\rho = J_2(x,p,D^*,\Gamma).$$

If $\Gamma = \Gamma^*$, then by (36) we have that $\gamma^*(\rho) \le \tau_B^*$ for every $\rho \in [0,1)$, where

$$\gamma^*(\rho) := \inf\{t \ge 0 : \Gamma_t^*(X^*) > \rho\}, \qquad \rho \in [0,1)$$

and thus the inequality in (46) is an equality in this case. Moreover, Γ_t^* only increases when Z_t^* increases and $Z^* = Z_t^* := p \wedge c\Big(\sup_{0\le s\le t} X_s\Big)$ so

$$u(X^*_{\tau_0^*\wedge\gamma^*(\rho)}, Z^*_{\tau_0^*\wedge\gamma^*(\rho)}) = b(c(\bar{X}_{\tau_0^*\wedge\gamma^*(\rho)})) = X^*_{\tau_0^*\wedge\gamma^*(\rho)}$$

in (45). Thus all the inequalities above become equalities and

$$u(x,p) = J_2(x,p,D^*,\Gamma^*). \tag{47}$$

If $p = 0$, we have $u(x,0) = \tilde{u}(x,0) = b(0)\frac{\psi(x)}{\psi(b(0))} = B\frac{\psi(x)}{\psi(B)}$ and $Z_t^* = 0$ for all $t \geq 0$. Applying Ito's formula to $e^{-rt}u(X_t^*, 0)$ between 0 and $\tau_0 \wedge \tau \leq \tau_B^*$ and using the properties of $\psi(x)$ gives

$$e^{-r(\tau_0\wedge\tau)}\tilde{u}(X_{\tau_0\wedge\tau}, 0) = \tilde{u}(x,0) - \int_0^{\tau_0\wedge\tau} e^{-rs}\tilde{u}_x(X_s^*, 0)dD_s^* + \int_0^{\tau_0\wedge\tau} e^{-rs}\sigma\tilde{u}_x(X_s^*, 0)dW_s.$$

Taking expected value and arguing as above thus gives

$$u(x,0) = \mathbb{E}\left[e^{-r(\tau_0\wedge\tau_B^*)}u(X^*_{\tau_0\wedge\tau_B^*}, 0)\right] = e^{-r(\tau_0\wedge\tau_B^*)}X_{\tau_0\wedge\tau_B^*} = J_2(x, 0, D^*, \Gamma^*)$$

and

$$\begin{aligned} u(x,0) =& \int_0^1 \mathbb{E}\left[e^{-r(\tau_0\wedge\gamma_B(\rho))}u(X_{\tau_0\wedge\gamma_B(\rho)}, 0)\right]d\rho \geq \int_0^1 \mathbb{E}\left[e^{-r(\tau_0\wedge\gamma_B(\rho))}X_{\tau_0\wedge\gamma_B(\rho)}\right]d\rho \\ \geq& J_2(x, p, D^*, \Gamma) \end{aligned}$$

where we again have used convexity of ψ and the fact that any stopping time $\gamma(\rho) > \tau_B^*$ yields a lower payoff that τ_B^*.

The above treats the case $x \leq b(p)$ so let us finalize the proof by considering $x > b(p)$. We have, for every $\Gamma \in \mathcal{A}_2$, that

$$u(x,p) = u(b(p),p) \geq J_2(b(p), p, D^*, \Gamma) = J_2(x, p, D^*, \Gamma),$$

where the last equality holds by the precedence of Player 1 over Player 2 and since $D_0^* = x - b(p)$ for $x > b(p)$. Similarly, we obtain

$$u(x,p) = u(b(p),p) = J_2(b(p), p, D^*, \Gamma^*) = J_2(x, p, D^*, \Gamma^*).$$

Hence, Γ^* is an optimal response to D^*. Together with Step 1, this implies that (D^*, Γ^*) is a NE and that the equilibrium values are v and u, respectively. This concludes the proof. □

Remark 12. *It is a remarkable feature of the equilibrium stratgey (D^*, Γ^*) that it allows the process Π^* to reach 0 in finite time, thereby completely ruling out the possibility that a competitor exists if he did not stop the game yet. Indeed, let $x \leq b(p)$, then we have*

$$X_t^* = Y_t - \bar{Y}_t + f(\bar{Y}_t)$$

and thus $\bar{X}_t^* = f(\bar{Y}_t)$ where f is an increasing bounded function such that $f(x) = B$ for all $x \geq \Lambda(B) + B$. Consequently, $\Pi_t^* = p \wedge c(\bar{X}_t^*) = p \wedge c(f(\bar{Y}_t)) = c(B) = 0$ for all

$$t \geq \tau_B = \inf\{s \geq 0 : Y_s \geq \Lambda(B) + B\},$$

the first time the unrestricted Brownian motion (with drift) Y reaches $\Lambda(B) + B$ (which is finite a.s.).

6 A numerical example

To provide the reader with further intuition, we conclude by looking at some numerical experiments. Throughout the section, we consider parameters $\mu = 0.03$, $\sigma = 0.12$, and $r = 0.01$. The optimal strategy $\tilde{D}$ in the single-player de Finetti problem given by (13) then amounts to reflection at $B \approx 1.12$.

Note that whereas the qualitative form of the single-player strategy de Finetti problem is fixed, the nature of the NE strategy for Player 1 varies depending on the value of $p \in [0,1]$. To be more precise, if Player 1 is certain that no competitor exists, i.e., if $p = 0$, then the problem degenerates into the standard single-player de Finetti problem and the optimal strategy is $\tilde{D}$

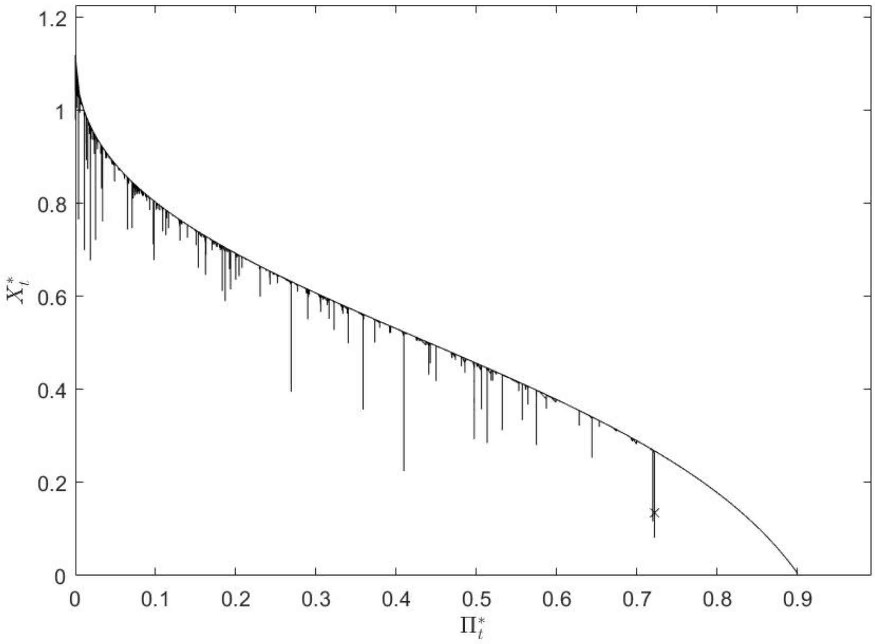

Figure 1: A simulated path of (Π^*, X^*) reflected along the boundary $p \mapsto b(p)$. The starting point $(p, x) \approx (0.72, 0.13)$ is illustrated by the cross $\times$.

(and Player 2 would stop as soon as X hits B). On the other hand, if Player 1 has sufficient evidence of the existence of a competitor, i.e., if $p \in [\hat{p}, 1]$ where $\hat{p} := (V'(0) - 1)/V'(0)$, then the agent extracts the whole resource immediately and the game terminates at $t = 0$. The most interesting scenario is when $p \in (0, \hat{p})$. In this case, the NE described in Theorem 11 amounts to a (possible) initial lump sum extraction of size $(x - b(p))^+$, and then continuous extraction as to reflect the two-dimensional process (X^*, Π^*) along the boundary b, with reflection in the prescribed direction $(u_p, -u_x)$. Figures 1 and 2 are derived with initial values $p = 0.8 \cdot \hat{p} \approx 0.72$ and $x = \frac{b(p)}{2} \approx 0.13$, putting us in the last of the three cases above.

Figure 3 shows the boundary $p \mapsto b(p)$ (or equivalently $x \mapsto c(x)$) together with the direction of reflection of the equilibrium process (X^*, Π^*). Note that $b(0) = B$ and $b(\hat{p}) = 0$. Figures 1 and 2 show a simulated path of the equilibrium process (X^*, Π^*) and the corresponding processes Π^*, Γ^*, and D^*, respectively. Flat portions of Γ^*, Π^*, and D^* correspond to X^* being strictly below the boundary $b(\Pi^*)$. Note also that in Figure 1, the process Π^* reaches 0 in finite time, ruling out the existence of a competitor playing the equilibrium strategy if he did not stop yet, see Remark 12.

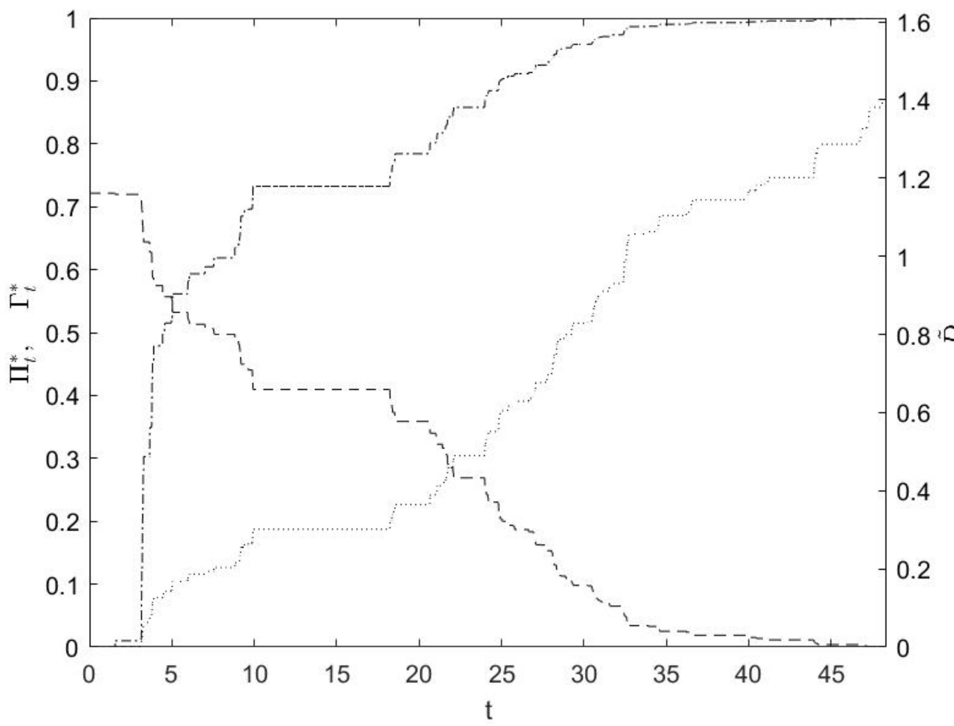

Figure 2: Auxiliary processes Π^* (dashed), Γ^* (dash-dot), and D^* (dotted).

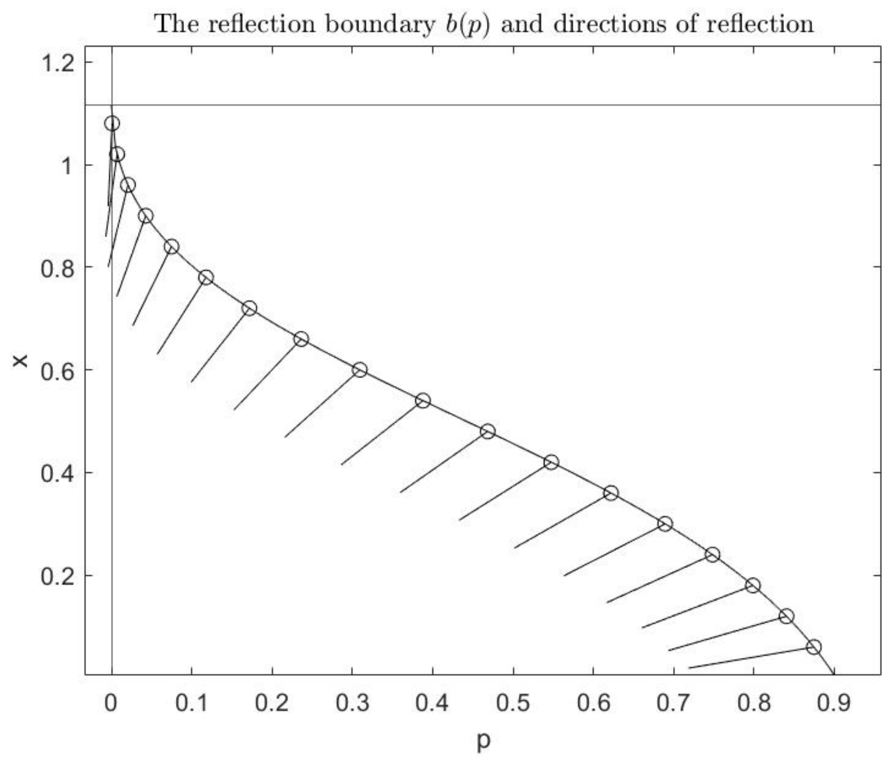

Figure 3: The boundary $p \mapsto b(p)$ and the direction of reflection for the equilibrium process (X^*, Π^*). The top horizontal line is $x = B \approx 1.1$ and represents the level at which to exert the control in the single-player de Finetti problem.

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