Hiring and firing – a signaling game

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Abstract

We study a signaling game between an employer and a potential employee, where the employee has private information regarding his/her production capacity. At the initial stage, the employee communicates a salary claim, after which the true production capacity is gradually revealed to the employer as the unknown drift of a Brownian motion representing the revenues generated by the employee. Subsequently, the employer has the possibility to choose a time to fire the employee in case the estimated production capacity understeps the salary. In this set-up, we derive an equilibrium in which the employee provides a randomized salary claim, and the employer uses a threshold strategy in terms of the conditional probability for the high production capacity.

1 Introduction

Incomplete information is a key ingredient in many hiring processes, where full knowledge about the true capacity of a potential employee is rarely available to the employer at the hiring time. Instead, if the candidate is hired, such information will be gradually revealed to the employer with time. On the other hand, the potential employee would typically possess more accurate information, and would use this additional information when providing his/her salary claim. Naturally, a high salary is costly for the employer, and thus increases the risk for the employee of being fired. Therefore, there is a trade-off in the choice between a high salary claim to increase personal gains and a small salary claim to decrease the risk of being fired.

To model one possible instance of the strategic interaction between an employer and a potential employee, we set up and study a signaling game with asymmetric information between two players. The game is informally described as follows.

- The capacity μ of the employee (Player 1) is a random variable with a known two-point distribution.
- At time $t = 0$, Player 1 learns about the realization of the random variable μ, and presents to the employer (Player 2) a non-negotiable salary claim C; the salary can only take two values.
- At time 0, Player 2 observes the salary claim, and subsequently also noisy observations of μ, based upon which a choice is made of a stopping time τ to terminate the employment; here $\tau = 0$ corresponds to a case in which the salary claim is not accepted (no hiring), $0 < \tau < \infty$ corresponds to an accepted salary claim, but with firing in finite time, and $\tau = \infty$ to an accepted salary claim, with no firing taking place.
- Up to the termination time τ, Player 1 receives compensation at rate C per unit time. Player 2, on the other hand, earns a net payment stream consisting of increments of a stochastic process $\mu t + \sigma W_t - Ct$, where the noise is modeled by a Brownian motion W.

The above game is a *signaling game*, with two possible types of Player 1, corresponding to the two possible values of μ, and where Player 1 sends a signal by choosing the salary level C. As such, there is an incomplete and asymmetric information structure since the players have different knowledge about μ.

Variants of such games with asymmetric information have a long history within the literature on hiring of staff and salary formation. An early study of such a set-up is [14], where an example with a job seeker that can have two different types is studied. The job seeker knows his type and chooses an education level, where the cost of education depends on the type, thereby conveying information to the employer. In [1], an extension with a type-dependent continuation value for the job-seeker is studied, thus allowing for a future change in the salary level. In [4], a two-source learning mechanism is used, where in addition to the signal consisting of education level, the employer also observes a stochastic grade that is correlated with the type of the job-seeker. For another related study, allowing for more types of the job-seeker and competition between employers, see [5]. A signaling game outside of job market is explored in [3], where an owner of a company and a set of potential buyers are considered. The seller holds private information of the company type, and buyers learn gradually from noisy observations of the unknown type and from the actions (lack of actions) of the seller. For another continuous time game with asymmetric information with a similar two-source learning, see [6].

In line with the literature on signaling games as above, see also [8] and [12], we use the concept of *perfect Bayesian equilibrium* (PBE) as a solution concept. We show the existence of a semi-separating PBE, in which the strong type always chooses the high salary, whereas the weak type randomizes between the low and the high salary.

While our Bayesian game set-up is rather simplistic with only two possible types of the employee and two possible salary claims, it may serve as a benchmark for more involved problems. Such extensions could include, for example, effort controls, where the employee may control linearly the process X, but at a cost that depends on the type, continuous salary revisions, and competition between potential employers.

2 Set-up

To describe the game in further detail, let W be a standard Brownian motion, and let μ be a modified Bernoulli distributed random variable independent of W with

$$\mathbb{P}(\mu = \mu_1) = p = 1 - \mathbb{P}(\mu = \mu_0),$$

where μ_0, μ_1 and p are known constants with $\mu_0 < \mu_1$ and $p \in (0, 1)$. We assume that the employee (Player 1) generates to the employer (Player 2) a payment stream modeled as the increments of a process

$$X_t = \mu t + \sigma W_t,$$

where σ is a positive constant. The random variable μ will be referred to as *the capacity* of the employee.

Player 1 knows his capacity μ, and gives at the initial time ($t = 0$) a salary claim C in the set $\{c_0, c_1\}$, where $0 < c_0 < c_1$. Allowing for randomised strategies, a strategy of Player 1 consists of a pair $a = (a_0, a_1) \in \mathcal{P}$, where $\mathcal{P} = [0, 1]^2$ is the unit square. Here a_i represents the conditional probability of choosing $C = c_1$ given that Player 1 is of type i, $i = 0, 1$.

To describe the possible strategies of Player 2, denote by $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ the augmentation of the filtration generated by the process X, and by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the augmentation of the filtration generated by the process X and the random variable C. Also, let $\mathcal{T}^X$ be the collection of $\mathbb{F}^X$-stopping times, and $\mathcal{T}$ be the collection of $\mathbb{F}$-stopping times. Clearly, since C only takes two possible values, any stopping time $\tau \in \mathcal{T}$ can be decomposed as

$$\tau = \begin{cases} \tau_0 & \text{on } \{C = c_0\} \\ \tau_1 & \text{on } \{C = c_1\}, \end{cases} \tag{1}$$

where $(\tau_0, \tau_1) \in \mathcal{T}^X \times \mathcal{T}^X$. Conversely, defining τ by (1) for a given pair $(\tau_0, \tau_1) \in \mathcal{T}^X \times \mathcal{T}^X$ yields that $\tau \in \mathcal{T}$. Thus we may identify $\mathcal{T}$ with $\mathcal{T}^X \times \mathcal{T}^X$, and we therefore write $\tau = (\tau_0, \tau_1)$.

In addition to a pair $(a, \tau) \in \mathcal{P} \times \mathcal{T}$ of strategies, the definition of a perfect Bayesian equilibrium (given below) also requires the specification of a belief system $\Pi_0 = (\Pi_0^0, \Pi_0^1) \in \mathcal{P}$. Here Π_0^i represents the probability that Player 2 assigns to the event $\{\mu = \mu_1\}$ conditional on the signal $C = c_i$, $i = 0, 1$.

The payoff structure of the game is now described as follows. Up to the stopping time τ, Player 1 receives compensation for his/her work at rate C per unit of time. Player 2, on the other hand, receives increments of the net payment stream

$$X_t - Ct = (\mu - C)t + \sigma W_t \tag{2}$$

per unit of time. Both players seek to maximise the expected discounted future payoff. More precisely, for a given triple $(a, \tau, \Pi_0) \in \mathcal{P} \times \mathcal{T} \times \mathcal{P}$ with $a = (a_0, a_1)$, $\tau = (\tau_0, \tau_1)$ and $\Pi_0 = (\Pi_0^0, \Pi_0^1)$, and for a given discount rate $r > 0$, define

$$J_1^0(a, \tau) = (1 - a_0)\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} c_0 \, dt \Big| \mu = \mu_0\right] + a_0 \mathbb{E}\left[\int_0^{\tau_1} e^{-rt} c_1 \, dt \Big| \mu = \mu_0\right]$$

and

$$J_1^1(a, \tau) = (1 - a_1)\mathbb{E}\left[\int_0^{\tau_0} e^{-rt} c_0 \, dt \Big| \mu = \mu_1\right] + a_1 \mathbb{E}\left[\int_0^{\tau_1} e^{-rt} c_1 \, dt \Big| \mu = \mu_1\right].$$

Then J_1^i represents the expected payoff for Player 1 given the capacity $\mu = \mu_i$, $i = 0, 1$. Similarly, define

$$J_2^0(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^0}\left[\int_0^{\tau_0} e^{-rt}(\mu - c_0) \, dt\right]$$

and

$$J_2^1(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1} e^{-rt}(\mu - c_1) \, dt\right],$$

so that J_2^i is the expected payoff for Player 2 given that $C = c_i$, $i = 0, 1$. Here the subindex in the expected value indicates that the expected value is calculated using the belief system Π_0 as initial probability of the type $\mu = \mu_1$.

Definition 1. (Perfect Bayesian equilibrium.) *We call a triplet $(a^*, \tau^*, \Pi_0) \in \mathcal{P} \times \mathcal{T} \times \mathcal{P}$ a perfect Bayesian equilibrium (PBE) if the following conditions are satisfied.*

(A) Sequential rationality:

$$J_1^i(a^*, \tau^*) \geq J_1^i(a, \tau^*)$$

for $i = 0, 1$, and

$$J_2^i(\tau^*, \Pi_0) \geq J_2^i(\tau, \Pi_0),$$

$i = 0, 1$, for all pairs $(a, \tau) \in \mathcal{P} \times \mathcal{T}$.

(B) Bayesian updating: *If $\min\{a_0^*, a_1^*\} < 1$, then*

$$\Pi_0^0 = \frac{p(1 - a_1^*)}{p(1 - a_1^*) + (1 - p)(1 - a_0^*)},$$

and if $\max\{a_0^*, a_1^*\} > 0$, then

$$\Pi_0^1 = \frac{p a_1^*}{p a_1^* + (1 - p) a_0^*}.$$

Remark 2. *While a key ingredient in our set-up is asymmetric information about the capacity μ, we point out that the set-up itself, including the numerical values of all parameters p, μ_0, μ_1, σ, r, c_1 and c_2, is common knowledge to both players.*

Remark 3. *Note that Player 1 is equipped with randomised strategies, whereas Player 2 is not. This is line with the asymmetric information structure, where Player 1 has more information and thus may benefit from hiding of information. Moreover, Player 1 acts at time 0 (revealing the realization of C), and once this is done, the game collapses to a single-player game of choosing a stopping time for Player 2, so randomisation of the stopping strategy is not needed.*

3 Filtering

From the perspective of Player 2, the problem is a two-source learning problem: at time $t = 0$, the salary claim is observed and the prior distribution of μ is updated in accordance with the specified belief system; at subsequent times $t > 0$, the posterior distribution is updated using observations of X.

Given $\pi \in [0, 1]$, define the process $\tilde{\Pi} := \tilde{\Pi}^\pi$ by

$$\tilde{\Pi}_t := \mathbb{P}_\pi(\mu = \mu_1 | \mathcal{F}_t^X),$$

where the index π indicates that the conditional probability is calculated using an initial estimate π for the event $\{\mu = \mu_1\}$. Thus $\tilde{\Pi}$ is the probability that $\mu = \mu_1$ conditioned merely on observations of X, and calculated with an initial belief π. It is well-known from filtering theory (see, e.g., [11]) that the conditional probability $\tilde{\Pi}$ satisfies

$$d\tilde{\Pi}_t = \omega \tilde{\Pi}_t (1 - \tilde{\Pi}_t)\, d\hat{W}_t,$$

where $\omega := (\mu_1 - \mu_0)/\sigma$ is the *signal-to-noise ratio* and the *innovations process*

$$\hat{W}_t := \frac{1}{\sigma} \left(X_t - \int_0^t (\mu_0 + (\mu_1 - \mu_0)\tilde{\Pi}_s)\, ds \right)$$

is an $\mathcal{F}^X$–Brownian motion.

Now, given a belief system $\Pi_0 = (\Pi_0^0, \Pi_0^1) \in \mathcal{P}$, we define the conditional probability process

$$\Pi_t := \begin{cases} \tilde{\Pi}_t^{\Pi_0^0} & \text{on } \{C = c_0\} \\ \tilde{\Pi}_t^{\Pi_0^1} & \text{on } \{C = c_1\}. \end{cases}$$

By the Bayesian updating property, Π_t coincides with

$$\mathbb{P}(\mu = \mu_1 | \mathcal{F}_t^{X,C})$$

on the event $\{C = c_i\}$ provided that $\mathbb{P}(C = c_i) > 0$.

Lemma 4. *Let $(\tau, \Pi_0) \in \mathcal{T} \times \mathcal{P}$. Then, for $i = 0, 1$, we have*

$$J_2^i(\tau, \Pi_0) = \mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt} (\mu_0 - c_i + (\mu_1 - \mu_0)\Pi_t) dt \right],$$

where the process Π satisfies

$$\begin{cases} d\Pi_t = \omega \Pi_t (1 - \Pi_t)\, d\hat{W}_t \\ \Pi_0 = \Pi_0^i. \end{cases} \tag{3}$$

Proof. By conditioning,

$$\mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt} \mu\, dt \right] = \mathbb{E}_{\Pi_0^i} \left[\mu \frac{1 - e^{-r\tau_i}}{r} \right] = \mathbb{E}_{\Pi_0^i} \left[(\mu_0 + (\mu_1 - \mu_0)\Pi_{\tau_i}) \frac{1 - e^{-r\tau_i}}{r} \right],$$

where $\Pi_t := \mathbb{P}_{\Pi_0^i} \left[\mu = \mu_1 | \mathcal{F}_t^X \right]$. By the above, Π_t satisfies (3). Moreover, by an application of Ito's formula and optional sampling,

$$\mathbb{E}_{\Pi_0^i} \left[(\mu_0 + (\mu_1 - \mu_0)\Pi_{\tau_i}) \frac{1 - e^{-r\tau_i}}{r} \right] = \mathbb{E}_{\Pi_0^i} \left[\int_0^{\tau_i} e^{-rt} (\mu_0 + (\mu_1 - \mu_0)\Pi_t)\, dt \right].$$

Consequently,

$$\begin{aligned} J_2^i(\tau, \Pi_0) &= \mathbb{E}_{\Pi_0^i}\left[\int_0^{\tau_i} e^{-rt}(\mu - c_i)\,dt\right] \\ &= \mathbb{E}_{\Pi_0^i}\left[\int_0^{\tau_i} e^{-rt}(\mu_0 - c_i + (\mu_1-\mu_0)\Pi_t)\,dt\right]. \end{aligned}$$

□

4 A semi-separating equilibrium

Note that if $c_1 \geq \mu_1$, then the net drift $\mu - C$ in (2) is non-positive, and Player 2 would always choose immediate firing ($\tau = 0$). Similarly, if $\mu_0 \geq c_1$, then $\tau = \infty$ would always be optimal. Thus, to rule out degenerate cases, a minimal assumption is that $\mu_0 < c_1 < \mu_1$. Moreover, we will make the additional assumption that $c_0 < \mu_0$ so that the net drift $\mu - C$ in (2) is positive on the event $\{C = c_0\}$.

The aim of the current section is thus to derive a perfect Bayesian equilibrium under the assumption that

$$0 < c_0 < \mu_0 < c_1 < \mu_1. \tag{4}$$

In subsections 4.1-4.2 we use intuitive arguments to obtain a candidate equilibrium, which is then verified in Subsection 4.3.

4.1 The employer's perspective

Under the assumption (4), the lower salary level c_0 is smaller than the capacity μ with probability one. Thus, if (4) holds, then it is clear that if Player 1 chooses $C = c_0$, then an optimal response for Player 2 should be to choose $\tau_0 = \infty$.

On the other hand, on the event $\{C = c_1\}$, Player 2 would stop if there is sufficient evidence that $\mu = \mu_0$. More precisely, we expect a boundary level b such that

$$\tau_1 := \inf\{t \geq 0 : \Pi_t \leq b\} \tag{5}$$

is an optimal response for Player 2. To determine b, standard optimal stopping theory based on the dynamic programming principle (see, e.g., [14]) suggests that the pair (V, b), where

$$V(\pi) := \sup_\tau \mathbb{E}_\pi\left[\int_0^\tau e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\tilde{\Pi}_t)\,dt\right],$$

solves the free-boundary problem

$$\begin{cases} \mathcal{L}V + \mu_0 - c_1 + (\mu_1 - \mu_0)\pi = 0 & \pi \in (b,1) \\ V(b) = 0 \\ V_\pi(b) = 0 \\ V(1-) = (\mu_1 - c_1)/r, \end{cases} \tag{6}$$

where

$$\mathcal{L} = \frac{1}{2}\omega^2\pi^2(1-\pi)^2\frac{d^2}{d\pi^2} - r.$$

Here the two boundary conditions at b constitute the so-called condition of smooth fit, and the boundary condition at $\pi = 1$ corresponds to receiving payments at rate $\mu_1 - c_1$ until time $\tau = \infty$.

To solve the free-boundary problem (6), one readily verifies that the general solution of the ODE is given by

$$V(\pi) = A_1(1-\pi)\left(\frac{\pi}{1-\pi}\right)^{\gamma_1} + A_2(1-\pi)\left(\frac{\pi}{1-\pi}\right)^{\gamma_2} + \frac{\mu_0 - c_1 + (\mu_1-\mu_0)\pi}{r},$$

where $\gamma_1 < 0$ and $\gamma_2 > 1$ are the solutions of the quadratic equation

$$\gamma^2 - \gamma - \frac{2r}{\omega^2} = 0, \tag{7}$$

and A_1 and A_2 are arbitrary constants. Imposing the boundary condition at $\pi = 1$, we must have $A_2 = 0$, and so the two remaining boundary conditions yield

$$\begin{cases} A_1(1-b)\left(\frac{b}{1-b}\right)^{\gamma_1} + \frac{\mu_0 - c_1 + (\mu_1-\mu_0)b}{r} = 0 \\ A_1(\gamma_1 - b)\left(\frac{b}{1-b}\right)^{\gamma_1} + \frac{(\mu_1-\mu_0)b}{r} = 0. \end{cases}$$

Eliminating A_1, we find that

$$b = \frac{-(c_1-\mu_0)\gamma_1}{\mu_1 - c_1 - (\mu_1-\mu_0)\gamma_1}. \tag{8}$$

For a graphical illustration of the function V and the threshold b, see Figure 1.

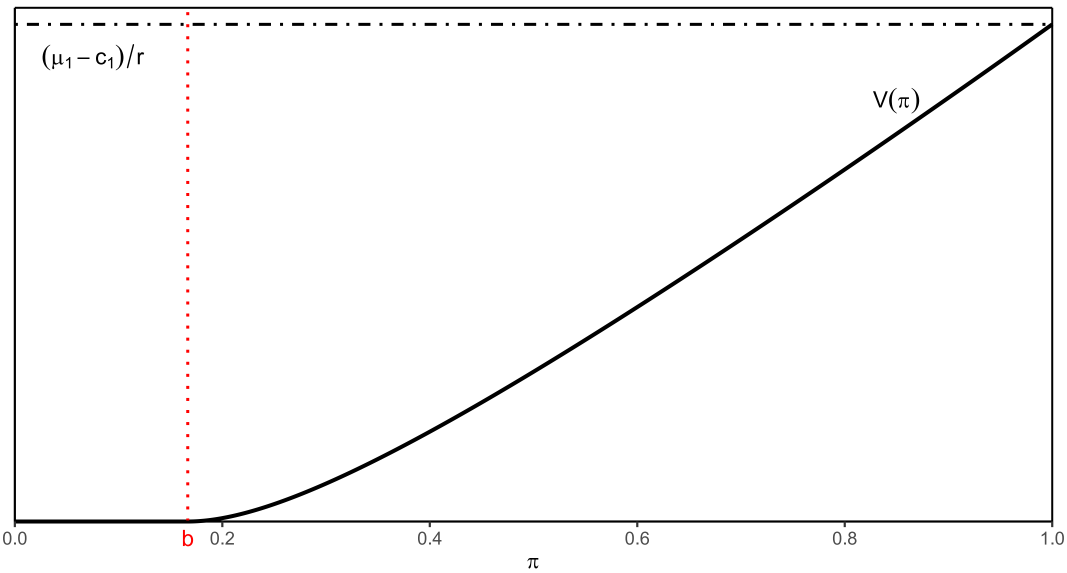

Figure 1: The value function $V(\pi)$ of the employer in the case when $C = c_1$ is chosen. The parameter values chosen for this example figure are $c_1 = 1.5$, $\mu_0 = 1.4$, $\mu_1 = 1.7$, $r = 0.05$ and $\sigma = 1$. The value function attains positive values only after the boundary level $b \approx 0.167$, and it approaches its maximum value $(\mu_1 - c_1)/r$ for π close to 1.

4.2 The employee's perspective

We now take the perspective of Player 1. We will construct an equilibrium in which Player 1 always chooses $C = c_1$ on the event $\{\mu = \mu_1\}$, and on the event $\{\mu = \mu_0\}$ uses a strategy such that $\mathbb{P}(C = c_1 | \mu = \mu_1) = a_0 = 1 - \mathbb{P}(C = c_0 | \mu = \mu_1)$ for some $a_0 \in [0,1]$ to be determined. Thus, in the notation of Section 2, we consider the strategy $a = (a_0, 1) \in \mathcal{P}$.

As note above, on the event $\{C = c_0\}$, Player 2 would use $\tau_0 = \infty$. By the indifference principle in game theory (see, e.g., [7] or [10]), to have an equilibrium with a strategy pair (a^*, τ^*) in which Player 1 uses a mixed strategy $a^* = (a_0, 1)$ with $a_0 \in (0,1)$ and Player 2 uses $\tau^* = (\infty, \tau_1)$ with τ_1 as in (5), we need that the expected payoffs $J_1^0((0,1), \tau^*)$ and $J_1^0((1,1), \tau^*)$ coincide. Clearly, choosing $C = c_0$ gives the expected payoff

$$J_1^0((0,1), (\infty, \tau_1)) = c_0/r$$

for Player 1.

To determine the expected payoff $J_1^0((1,1),\tau^*)$ for Player 1 when $C=c_1$ is chosen, note that on the event $\{C=c_1\}$, Player 2 would first re-evaluate the probability that Player 1 has the larger capacity $\mu=\mu_1$ according to the specified belief system Π_0. Moreover, by the Bayesian updating requirement of the belief system, we have

$$\Pi_0^1 = \mathbb{P}(\mu=\mu_1|C=c_1) = \frac{\mathbb{P}(\mu=\mu_1, C=c_1)}{\mathbb{P}(C=c_1)} = \frac{p}{p+(1-p)a_0}.$$

Thus Π_t makes an initial jump from $\Pi_{0-}=p$ up to $\Pi_0^1 = \frac{p}{p+(1-p)a_0} \geq p$, and then it diffuses with dynamics

$$d\Pi_t = \omega\Pi_t(1-\Pi_t)\,d\hat{W}_t.$$

From the perspective of Player 1, however, $\hat{W}$ is not a Brownian motion since Player 1 knows the true drift μ. Instead,

$$d\Pi_t = \omega\Pi_t(1-\Pi_t)\,d\hat{W}_t = -\omega^2\Pi_t^2(1-\Pi_t)\,dt + \omega\Pi_t(1-\Pi_t)\,dW_t.$$

Consequently, the value

$$U(\pi) := J_1^0((1,1),\tau^*) = \mathbb{E}_\pi\left[\int_0^{\tau_1} e^{-rt}c_1\,dt \Big| \mu=\mu_0\right]$$

for Player 1 solves

$$\begin{cases} \frac{\omega^2\pi^2(1-\pi)^2}{2}U_{\pi\pi} - \omega^2\pi^2(1-\pi)U_\pi - rU + c_1 = 0 & \pi\in(b,1) \\ U(b)=0 \\ U(1-) = c_1/r. \end{cases} \tag{9}$$

The ODE in (9) has general solution

$$U(\pi) = B_1\left(\frac{\pi}{1-\pi}\right)^{\gamma_1} + B_2\left(\frac{\pi}{1-\pi}\right)^{\gamma_2} + c_1/r,$$

where $\gamma_1<0$ and $\gamma_2>1$ are the solutions of (7) as before, and B_1 and B_2 are arbitrary constants. Similarly as above, the boundary condition at $\pi=1$ yields $B_2=0$, and then the boundary condition at $\pi=b$ gives

$$B_1 = \frac{-c_1}{r}\left(\frac{b}{1-b}\right)^{-\gamma_1},$$

so

$$U(\pi) = \begin{cases} \frac{c_1}{r}\left(1-\left(\frac{\pi(1-b)}{(1-\pi)b}\right)^{\gamma_1}\right) & \pi>b \\ 0 & \pi\leq b. \end{cases}$$

Now recall that we are looking for $a_0\in(0,1)$ so that

$$\frac{c_0}{r} = U\left(\frac{p}{p+(1-p)a_0}\right).$$

This is possible only if $U(p)<c_0/r$, i.e. if

$$\frac{p}{1-p} < \frac{b}{1-b}\left(1-\frac{c_0}{c_1}\right)^{1/\gamma_1}.$$

Equivalently, we need to have

$$p < \hat{p} := \frac{b(1-\frac{c_0}{c_1})^{1/\gamma_1}}{1-b+b(1-\frac{c_0}{c_1})^{1/\gamma_1}}. \tag{10}$$

Moreover, in that case, a_0 should be chosen so that

$$\frac{p}{p+(1-p)a_0} = \frac{b(1-\frac{c_0}{c_1})^{1/\gamma_1}}{1-b+b(1-\frac{c_0}{c_1})^{1/\gamma_1}},$$

i.e.

$$a_0 = \frac{p(1-b)}{(1-p)b(1-\frac{c_0}{c_1})^{1/\gamma_1}}.$$

For a graphical illustration of the value function U and the indifference point $\hat{p}$, see Figure 2.

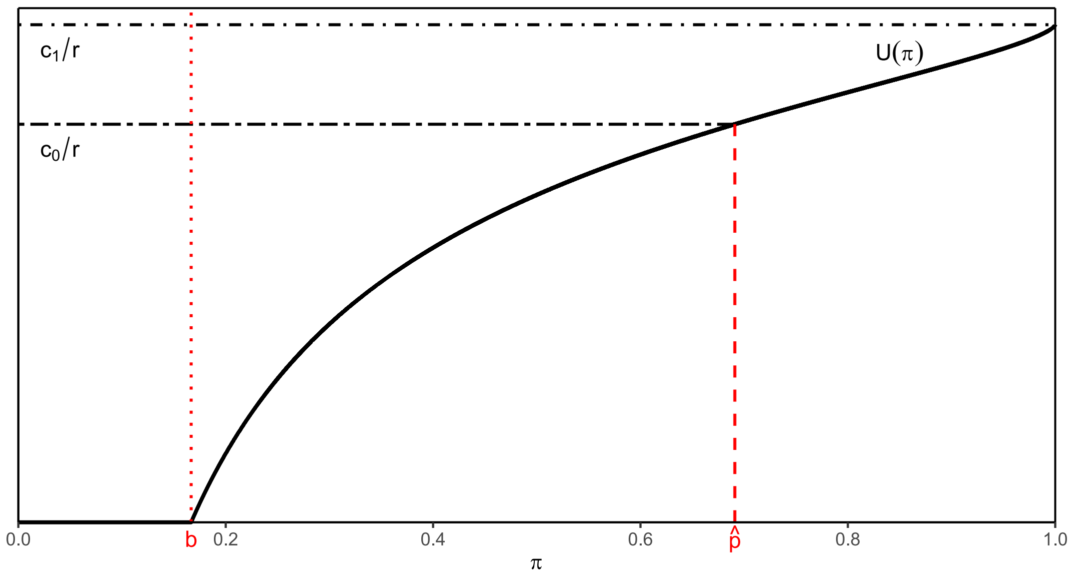

Figure 2: The value function $U(\pi)$ for the weak type employee when the high salary $C = c_1$ is chosen. The parameter values of c_1, μ_0, μ_1, r and σ are the same as in Figure 1, and $c_0 = 1.2$. Here $\hat{p}$ is the unique value so that $U(\hat{p}) = c_0/r$.

4.3 Verification of equilibrium

We now summarize the strategies described above; in Theorem 5 we then verify that these strategies together constitute a perfect Bayesian equilibrium.

Let

$$b = \frac{-(c_1-\mu_0)\gamma_1}{\mu_1-c_1-(\mu_1-\mu_0)\gamma_1}$$

as in (8) above, and define the strategy $a^* = (a_0^*, a_1^*)$ of Player 1 by

$$a_0^* = \begin{cases} \frac{p(1-b)}{(1-p)b(1-\frac{c_0}{c_1})^{1/\gamma_1}} & p < \hat{p} \\ 1 & p \geq \hat{p} \end{cases}$$

and $a_1^* = 1$, where $\hat{p}$ is as in (10). Moreover, let $\tau^* = (\tau_0^*, \tau_1^*)$ be defined by

$$\tau_0^* := \infty$$

and

$$\tau_1^* := \inf\{t \geq 0 : \tilde{\Pi}_t^{\Pi_0^1} \leq b\},$$

where $\Pi_0^1 := \hat{p} \vee p$. Also, let $\Pi_0 := (\Pi_0^0, \Pi_0^1) = (0, \hat{p} \vee p)$.

Theorem 5. *Assume that (4) holds. Then the triplet (a^*, τ^*, Π_0) specified above is a perfect Bayesian equilibrium. Moreover, if $p < \hat{p}$, the equilibrium is semi-separating; if $p \geq \hat{p}$, then the equilibrium is of pooling type.*

Proof. We first note that, by construction, the belief system Π_0 satisfies the Bayesian updating property. The proof of sequential rationality is divided into two parts.

Optimality of τ^*. First note that $\Pi_0^0 = 0$ yields that

$$J_2^0(\tau, \Pi_0) = \mathbb{E}\left[\int_0^{\tau_0} e^{-rt}(\mu_0 - c_0)\, dt \Big| \mu = \mu_0\right] \le \frac{\mu_0 - c_0}{r} = J_2^0(\tau^*, \Pi_0)$$

for any $\tau \in \mathcal{T}$, so $\tau_0^* = \infty$ is a rational response to $C = c_0$.

Next, if the employer observes the event $\{C = c_1\}$, then the stopping time

$$\tau_1^* := \inf\{t \ge 0 : \Pi_t \le b\}$$

is used, where

$$\Pi_t := \mathbb{P}_{\Pi_0^1}(\mu = \mu_1 | \mathcal{F}_t^X).$$

By Section 3,

$$d\Pi_t = \omega \Pi_t (1 - \Pi_t)\, d\hat{W}_t,$$

so an application of Ito's formula together with (6) shows that

$$Y_t := e^{-rt} V(\Pi_t) + \int_0^t e^{-rs}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_s) ds$$

is a bounded supermartingale. For any stopping time $\tau' = (\tau_0', \tau_1') \in \mathcal{T}$, optional sampling therefore gives that

$$\begin{aligned} V(\Pi_0^1) &\ge \mathbb{E}\left[e^{-r(T\wedge\tau_1')} V(\Pi_{T\wedge\tau_1'}) + \int_0^{T\wedge\tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_t) dt\right] \\ &\ge \mathbb{E}\left[\int_0^{T\wedge\tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_t) dt\right] \\ &\to \mathbb{E}\left[\int_0^{\tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_t) dt\right] \end{aligned}$$

as $T \to \infty$ by bounded convergence. Since

$$\mathbb{E}\left[\int_0^{\tau_1'} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_t) dt\right] = J_2^1(\tau', \Pi_0)$$

by Lemma 4, we find that

$$J_2^1(\tau', \Pi_0) \le V(\Pi_0^1) \tag{11}$$

for all $\tau' \in \mathcal{T}$.

Furthermore, for τ^*, the stopped process $Y_{t\wedge\tau_1^*}$ is a martingale, so optional sampling and bounded convergence give

$$\begin{aligned} V(\Pi_0^1) &= \mathbb{E}\left[e^{-r(T\wedge\tau_1^*)} V(\Pi_{T\wedge\tau_1^*}) + \int_0^{T\wedge\tau_1^*} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_t) dt\right] \\ &\to \mathbb{E}\left[\int_0^{\tau_1^*} e^{-rt}(\mu_0 - c_1 + (\mu_1 - \mu_0)\Pi_t) dt\right] = J_2^1(\tau^*, \Pi_0) \end{aligned}$$

as $T \to \infty$, which together with (11) implies that τ_1^* is an optimal response to $C = c_1$.

Optimality of a^*. We have that

$$\begin{aligned}
J_1^0(a,\tau^*) &= (1-a_0)\frac{c_0}{r} + a_0 \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1\,dt \Big| \mu=\mu_0\right] \\
&= (1-a_0)\frac{c_0}{r} + a_0 U(\hat p \vee p) \\
&\le (1-a_0^*)\frac{c_0}{r} + a_0^* U(\hat p \vee p) = J_1^0(a^*,\tau^*),
\end{aligned}$$

where the inequality holds since if $p > \hat p$ then we have $U(\hat p \vee p) = U(p) \ge c_0/r$ and $a_0^* = 1 \ge a_0$, and if $p \le \hat p$ then we have $U(\hat p \vee p) = U(\hat p) = c_0/r$.

Similarly,

$$\begin{aligned}
J_1^1(a,\tau^*) &= (1-a_1)\frac{c_0}{r} + a_1 \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1\,dt \Big| \mu=\mu_1\right] \\
&\le \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1\,dt \Big| \mu=\mu_1\right] = J_1^1(a^*,\tau^*),
\end{aligned}$$

where the inequality follows from the inequalities

$$\mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1\,dt \Big| \mu=\mu_1\right] \ge \mathbb{E}_{\Pi_0^1}\left[\int_0^{\tau_1^*} e^{-rt} c_1\,dt \Big| \mu=\mu_0\right] = U(\Pi_0^1) \ge c_0/r.$$

□

Remark 6. *Consider the strategy pair (a,τ), where $a=(0,0)$ and $\tau=(\infty,0)$; in words, Player 1 always chooses $C=c_0$ (regardless of his type) and Player 2 never stops if $C=c_0$ and stops immediately if $C=c_1$. Then also (a,τ,Π_0) with $\Pi_0=(p,\Pi_0^1)$ is a perfect Bayesian equilibrium (of pooling type) provided the belief Π_0^1 is chosen small enough (e.g., $\Pi_0^1 \le b$). However, in this equilibrium, both types have the same equilibrium value c_0/r, and the strong type is more likely to deviate to the off-equilibrium signal $C=c_1$ than the weak type; as a consequence, it can be checked that this equilibrium is not* divine*, see e.g. [2].*

Remark 7. *We have analyzed the game under the assumption (4) that $c_0 < \mu_0 < c_1 < \mu_1$. However, the game can also be set up in a non-degenerate way by altering the ordering to $\mu_0 < c_0 < c_1 < \mu_1$. In that case, also the smaller salary c_0 provides a negative running reward for the employer for one of the possible types of the employee, so a semi-separating equilibrium is no longer feasible. In addition to the pooling equilibrium with $a=(0,0)$ described in Remark 6, one also obtains a pooling equilibrium with $a=(1,1)$, which is supported by a sufficiently small belief Π_0^0.*

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