III Fibre bundles (mainly principal bundles)

We have already seen an example: $\tilde{X}_{pt} \to X$ the universal cover (constructed out of X).

A different source: we want to construct an abstract family of objects. This is easier if we work with concrete representatives of such objects (e.g. maps are defined using features of the representatives). We then need to pass to a quotient to remove auxiliary choices.

Motivating example: The configuration space of m distinct points in $\mathbb{R}^n$.

Fix a representative $\underline{m} := \{1, 2, .., m\}$. When considered as unordered points, $\underline{m}$ satisfies an S_m-symmetry:

$\underline{m} \circlearrowleft S_m$ = group of symmetries (act by renumbering)

The embeddings of m ordered points in $\mathbb{R}^n$ form

$$\mathrm{Emb}(\underline{m}, \mathbb{R}^n) := \{(pt_1, .., pt_m) \in \mathbb{R}^n \mid pt_i = pt_j \Leftrightarrow i = j\}$$

$$\subseteq (\mathbb{R}^n)^m \quad (n \cdot m\text{-dim manifold!})$$

A family of m unordered points (in $\mathbb{R}^n$) is an element of

$$\mathrm{Conf}_m(\mathbb{R}^n) := \mathrm{Emb}(\underline{m}, \mathbb{R}^n)/S_m \subsetneq (\mathbb{R}^n)^{(m)} := (\mathbb{R}^n)^m/S_m \leftarrow \text{permuting coordinates}$$

which in this case is a non-compact manifold defined by the quotient proj. $\pi : \mathrm{Emb}(\underline{k}, \mathbb{R}^n) \twoheadrightarrow \mathrm{Conf}_k(\mathbb{R}^n)$

For concrete constructions (e.g. maps $\mathrm{Conf}_k(\mathbb{R}^n) \to \dots$) one again want to remove symmetry & obtain "concrete" embeddings.

A cont. choice of repr. $\underline{k} \hookrightarrow \mathbb{R}^n$ for points in Conf_k is called a <u>section</u>

$$\begin{tikzcd} \mathcal{U} \arrow[r, "\sigma"] \arrow[dr, "\mathrm{id}"'] & \mathrm{Emb}(\underline{k}, \mathbb{R}^n) \arrow[d, "\pi"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathcal{U} \end{tikzcd}$$

It is an important point that sections might only exist <u>locally</u>.

<u>Ex.</u> $\mathrm{Emb}(\underline{m}, \mathbb{R}^1) \cong \mathrm{Conf}_m(\mathbb{R}^1) \times S_m \leftarrow$ "trivial product"

$\mathrm{Conf}_m(\mathbb{R}^1) \cong \mathbb{R}^m$

⚠ not natural

Exercise 8. Show that $\mathrm{Emb}(\underline{2}, \mathbb{R}^2) \overset{\text{htpy. eq.}}{\sim} \partial(\text{Möbius band})$

$$\mathrm{Conf}_2(\mathbb{R}^2) \sim S^1$$

By cont. deforming any two distinct pts to become antipodal on $S^1 \subseteq \mathbb{R}^2$.

Next we will see that $\mathrm{Emb}(\underline{m}, \mathbb{R}^n) \to \mathrm{Conf}_m(\mathbb{R}^n)$ is an example of a <u>principal S_m-bundle</u>, which is <u>trivial</u> only when $n=1$.

Principal bundles

Let G be a (metric) top. group, and E a (metric) top. space equipped with a free (no fixpoints) right G-action, i.e.

$$r: E \times G \xrightarrow{\text{cont.}} E \qquad \bullet\ x \cdot e = x$$

$$(x,g) \mapsto x \cdot g \in E \qquad \bullet\ (x \cdot g) \cdot h = x \cdot (gh)$$

Further requirement: quotient top. on E/G induced by

$$\pi : E \twoheadrightarrow B := E/G = \{x \cdot G \mid x \in E\}$$

sets of G-orbits

is a metric topology.

Def. E as above is called a (loc. trivial) principal G-bundle if $\forall x \in B\ \exists\ \Phi_U : \pi^{-1}(U) \xrightarrow{\cong} U \times G$ for some open $U \ni x$ for which $\boxed{\Phi_U(y \cdot g) = (\pi(y), g_y \cdot g)}$ is satisfied. (*)

A morphism of G-bundles is a cont. $\Psi : E_1 \to E_2$ for which $\boxed{\Psi(\underset{r_2}{y \cdot g}) = \underset{r_1}{\Psi(y) \cdot g}}$ (G-equivariant)

Ex/Def. The trivial G-bundle over B is $E = B \times G$

$\pi : E \to B$ can. proj., $r((x,g),h) = (x, gh)$

Rmk A princ. G-bundle is "loc. isomorphic" to the trivial bundle

We call B the <u>base</u>, E the <u>total space</u>, and G the <u>fibre</u>

Φ_U <u>local trivialisation</u>

Facts • $(*) \Rightarrow \Phi_V \circ \Phi_U^{-1} : (U \cap V) \times G \to (U \cap V) \times G$

$$(x, g) \longmapsto (u, \varphi_{VU}(x) \cdot g)$$

can. proj. to G

where $\varphi_{VU} = pr_G \circ \Phi_V \circ \Phi_U^{-1}(\_\_, e) : U \cap V \longrightarrow G$ satisfies:

(i) $\varphi_{UV} \cdot \varphi_{VU} \equiv e$.

(ii) $\varphi_{WV} \cdot \varphi_{VU} = \varphi_{WU}$ (the <u>cocycle condition</u>)

- We can construct a principal bundle $E \to B$ from the choice of an open cover of B & assoc. maps $\{\varphi_{UV}\}$ satisfying (i) & (ii) $\leftarrow$ highly non-trivial!

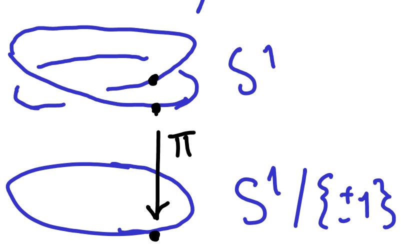
- Ψ induces $\psi : E_1/G \to E_2/G$, i.e.:

$$\begin{tikzcd} E_1 \arrow[r, "\Psi"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\circlearrowleft"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "\psi"'] & B_2 \end{tikzcd}$$

- $\Phi_{U_2} \circ \Psi \circ \Phi_{U_1}(x, g) = (\psi(x), \xi_{U_2 U_1}(x) \cdot g)$

subgp. of $U(1) \cong S^1$

Ex. $S^1 \xrightarrow{\pi} S^1/\{\pm 1\} \cong S^1$ is a non-trivial $\mathbb{Z}_2 = \{\pm 1\}$-bundle over the base S^1.



<u>Reason</u>: $\pi_0(E) = \{0\} \neq \pi_0(S^1 \times \mathbb{Z}_2)$

Long exact sequence of htpy. groups

Recall:

A sequence (of morphisms in some alg. category)

$$\cdots \to G_{i+1} \xrightarrow{f_{i+1}} G_i \xrightarrow{f_i} G_{i-1} \to \cdots$$

is <u>exact</u> if $\ker f_i = \operatorname{im} f_{i+1}$

Thm. 8. Fix $pt \in E$, let $\iota : G \xrightarrow{\cong} \pi^{-1}(\pi(pt)) \subseteq E$ be $\iota(g) = x \cdot g$, then there exists a long exact sequence of htpy. groups

$$\cdots \to \pi_i(G) \xrightarrow{\iota_*} \pi_i(E, pt) \xrightarrow{\pi_*} \pi_i(B, \pi(pt)) \xrightarrow{\partial_*} \pi_{i-1}(G) \to \cdots$$

"pt" not needed by Exc. 6.c.

Rmk ∂_* is the restriction to $\partial D^i = S^{i-1}$ of a <u>lift</u>

$\tilde{\alpha} : D^i \to E$ of $\alpha : D^i \to B \in \pi^i(B, \pi(pt))$ $\left(\alpha(\partial D^i) = \pi(pt)\right)$

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ D^i \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & B \end{tikzcd}$$

Cor. 9. The universal cover $\tilde{X}_{pt}$ of a conn. space X is <u>connected.</u>

Proof. $(\pi_*)_0$ is obviously surj. Check that $(\partial_*)_1$ is an iso! (almost an tautology) $\square$