Thm. 25 1.) Connected sum is a well-defined operation on smooth isotopy classes of oriented knots.

2.) Conn. sum is associative & commutative

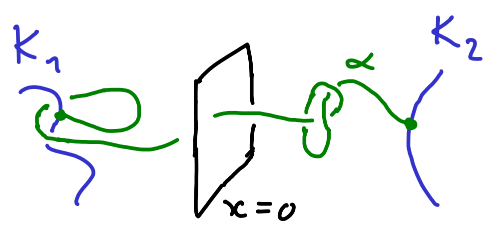
3.) $K \# \text{unknot} \overset{(\text{iso.})}{=} K$

4.) $g(K_1 \# K_2) = g(K_1) + g(K_2)$ ($\Rightarrow$ infinitely many iso. classes of knots!)

Def A knot K is prime if $K = K_1 \# K_2$

$\Rightarrow$ one of K_i, $i = 1, 2$, is the unknot.

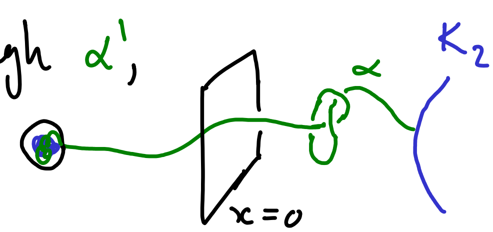
Ex The trefoil is prime by Thm. 25 (4.)



Proof of Thm. 25

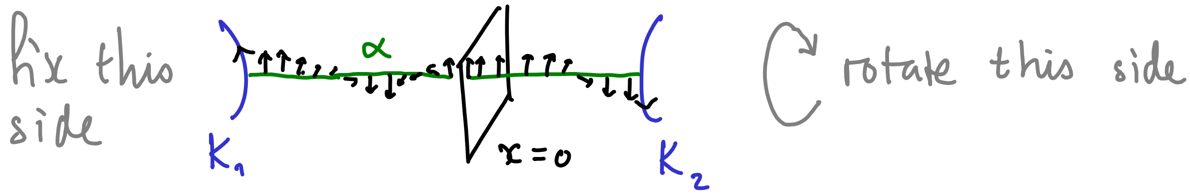
1.) **Claim** $K_1 \cup \alpha \cup K_2$ is ambient isotopic to $K_1 \cup \alpha' \cup K_2$ whenever α, α' are two paths in the construction.

Pf. Near endpoints: can be made to coincide "by hand".
To "undo the mess" in $\{x>0\}$: Shrink $\{x<0\}$ into a small ball and "thread" through α', and back through α.



Similarly in $\{x<0\}$.

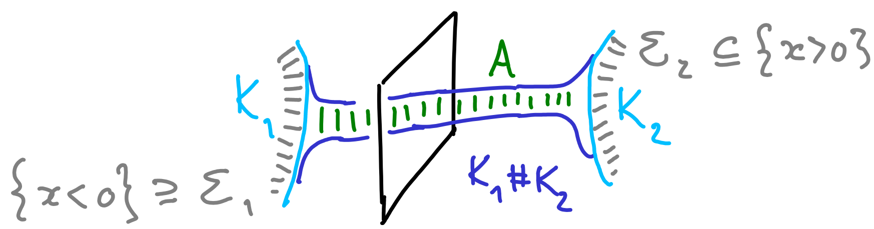
The additional choice of twisting along α in the construction can then be undone by an ambient rotation in $\{x>0\}$



2.) Exercise 27.)

3.) Obviously in view of (1.)

4.) Look near $\{x=0\}$. Using (1), we may assume

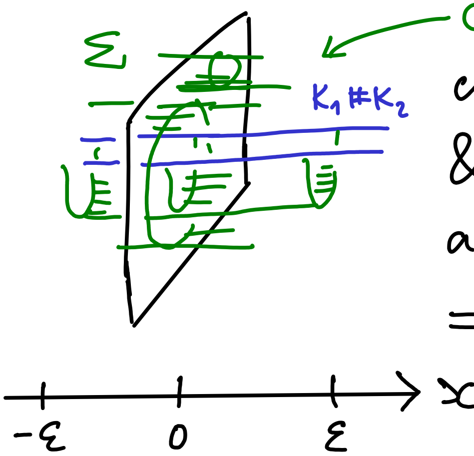


$g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ holds since we can glue Seifert surfaces for K_1 & K_2 by adding the band A. (Obs: $\mathbb{R}^3 \setminus (\Sigma_1 \cup \Sigma_2)$ is path connected, by (1.) we can assume $\alpha \subseteq \mathbb{R}^3 \setminus (\Sigma_1 \cup \Sigma_2 \setminus (K_1 \cup K_2))$.)

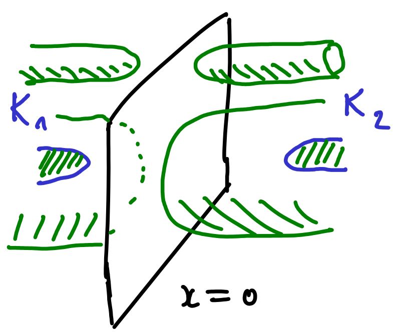
The opposite inequality is seen by splitting any Seifert surface Σ of $K_1 \# K_2$ along $\{x=0\}$.

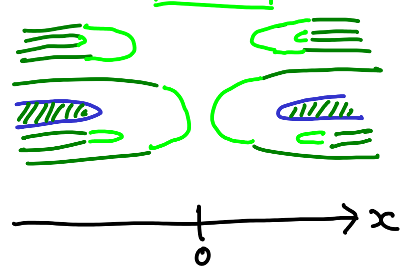
After a generic perturbation of Σ we may assume that it takes the following form in $[-\varepsilon,\varepsilon]_x \times \mathbb{R} \times \mathbb{R} \subseteq \mathbb{R}^3$:

$\Sigma \cap \{|x| \leq \varepsilon\} =$ cylinders over closed curves in $\{x=0\}$ & a strip over a single path w. bdy $= K_1 \# K_2 \cap \{x=0\}$



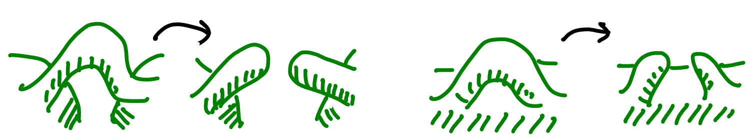
Cut open cylinders along $\{x=0\}$ and add discs





Resulting surface does not intersect $\{x=0\}$ & has bdy $= K_1 \cup K_2$.

Throw away closed components.



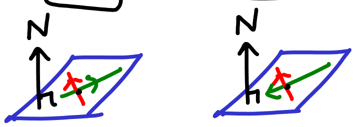
Compute χ (c.f. Exc. (2.5.)) to show that total genus cannot increase! $\square$

6. The Alexander polynomial

Intersection form

For any oriented surface Σ we have an "intersection form" defined on curves in the following manner.
Let $x, y \subseteq \Sigma$ be two closed <u>oriented</u> curves (here: immersed)

$$\boxed{x \bullet y \overset{\text{def}}{=} \sum 1 - \sum 1 \in \mathbb{Z}}$$



after a small generic perturbation (well-def & indep. of perturbation)

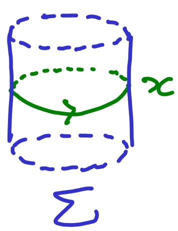
$\Rightarrow$ "$\bullet$" is "linear" $\quad (-1)^i x \bullet (-1)^j y = (-1)^{i+j} x \bullet y$

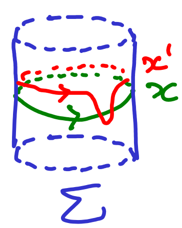
"skew symmetric" $\quad x \bullet y = - y \bullet x$

Change of orientation

$\Rightarrow \quad x \bullet x = 0.$

Geometric computation (x embedded)





$x \bullet x \overset{\text{def}}{=} x \bullet x' = 0$

nbhd of $x \cong$ annulus
(two-sided since Σ is orientable, so not a Klein bottle)

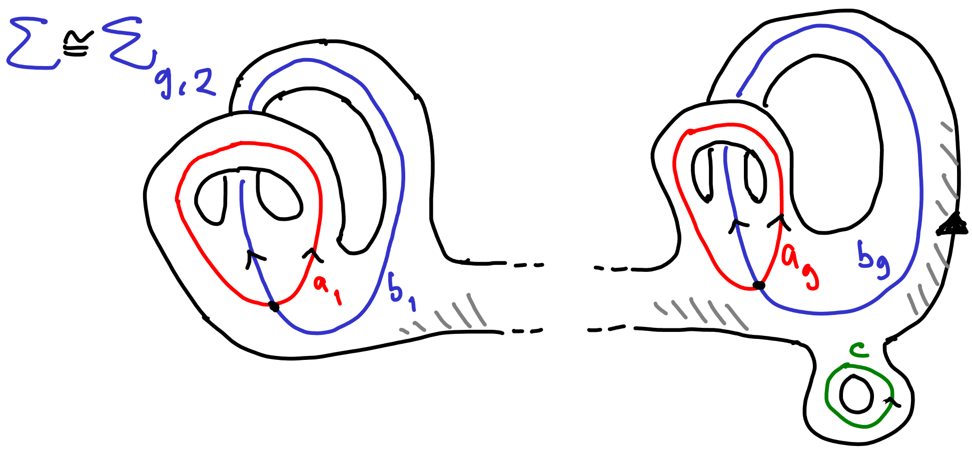
<u>Fact</u> $\_\bullet\_ : H_1(\Sigma) \times H_1(\Sigma) \to \mathbb{Z}$ is well-def, $\mathbb{Z}$-bilinear, & <u>skew symmetric.</u>

The Seifert matrix

Let $\Sigma \subseteq \mathbb{R}^3$ be a Seifert surface for $\partial\Sigma = L$.

(Recall: connected, orientable compact)

oriented $\Rightarrow$ orientation of Σ



orientation of K

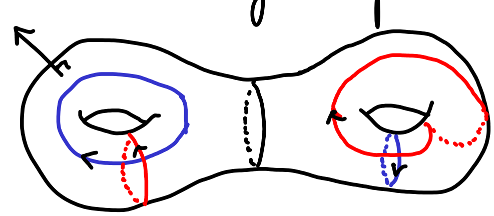
orient Σ with upward pointing normal

The oriented embedded curves a_i, b_i, $i = 1, .., g$ satisfy the intersection properties

$$\boxed{a_i \bullet b_j = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}} \qquad (a_i \bullet a_j = b_i \bullet b_j = 0 \text{ always true})$$

Fact: Any collection of closed curves a_i, b_i satisfying the above is a basis of $H_1(\Sigma_{g,k}) \cong \mathbb{Z}^{2g}$, $k = 0$ or 1.

In general $H_1(\Sigma_{g,k}) \cong \mathbb{Z}^{2g+k-1}$, where the additional $k-1$ curves can be taken among the k bdy. components (on which "$\bullet$" is trivial)



Pick a basis of curves as above and write $e_1 = a_1,\ e_2 = b_1,\ e_3 = a_3,\ \dots,\ e_{2g-1} = a_g,\ e_{2g} = b_g,\ \overbrace{e_{2g+1} = c_1, \dots}^{k-1 \text{ bdy comp.}}$

The Seifert matrix induced by Σ is

$$V \in \mathrm{Mat}_{2g,2g}(\mathbb{Z}) \text{ with entries}$$

$$v^i_j = \mathrm{lk}(e_i, e_j^{\#})$$

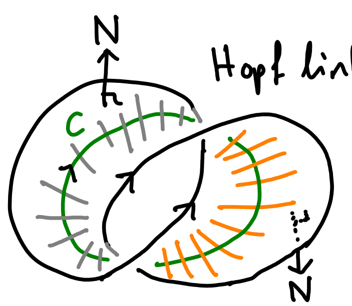
row

col.

pushoff of e_j along normal of Σ.

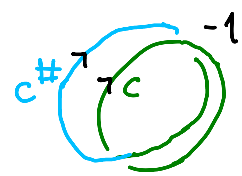
V is invariant / change of basis & certain stabilisations, but we will not focus V itself here.

Ex

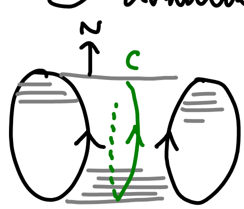


Hopf link

$$V = [\mathrm{lk}(c, c^{\#})] = [-1]$$

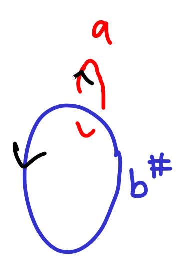


Σ = annulus (twisted)

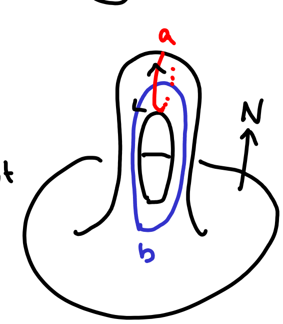


unlink $\quad V = [0]$

Σ = annulus (untwisted)



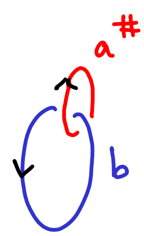
$\mathrm{lk}(a, b^{\#})$



Unknot

$$V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$\mathrm{lk}(b, a^{\#})$



The **Alexander polynomial** is the Laurent polynomial

$$\Delta_L(t) := t^N \cdot \det(tV - V^t) \in \mathbb{Z}[t, t^{-1}]$$

$N \in \mathbb{Z}$ s.t.
- honest polynomial
- not divisible by t (unless zero)

(⚠ there are different conventions)

We compute $\Delta_{\circ\circ}(t) = 0$, $\Delta(t) = t-1$,

$$\Delta_{\circ}(t) = t^{-1} \cdot \det \begin{bmatrix} 0 & -1 \\ t & 0 \end{bmatrix} = 1.$$

Clearly $\boxed{\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_2}(t)}$

Exercise 28.) Show that (1.) $\pm t^{\deg \Delta} \Delta(t^{-1}) = \Delta(t)$

(2.) $\Delta_{\text{Knot}}(1) = \pm 1$

Exercise 29.) Calculate $\Delta(t) = 1 - t + t^2$

Thm. 26. (Alexander, '28) The Alexander polynomial is an isotopy invariant of links.