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Proof of Thm (Long exact sequence of homotopy groups)

$$\boxed{\cdots \to \pi_{k+1}(B,*_B) \xrightarrow{\delta_{k+1}} \pi_k(F,*) \xrightarrow{\iota_*} \pi_k(E,*) \xrightarrow{p_*} \pi_k(B,*_B) \xrightarrow{\delta_k} \pi_{k-1}(F,*) \to \cdots}$$

Two consecutive maps in the sequence vanish more or less by construction. We are left with:

$\ker p_* \subseteq \operatorname{im} \iota_*$ $\quad \alpha \in C((S^k,N),(E,*)), \quad p_*([\alpha]) = 0$

$f_t : (S^k, N) \to (B, *_B)$ homotopy rel. basepoints from $p \circ \alpha$ to cst_{*_B}

equivalently: $f : I^k \times I_t \to (B, *_B)$ homotopy of maps from the "k-disc" I^k that send ∂I^{k+1} to $*_B$, $f_0 = p\circ\alpha$, $f_1 \equiv cst_{*_B}$

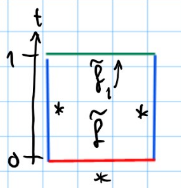
Homotopy lifting thm $\Rightarrow$ $\exists \tilde{f}_t$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I^{k+1} \arrow[ur, dashed, "\tilde{f}_t"] \arrow[r, "f_t"'] & B \end{tikzcd}$$

OBS: $\tilde{f}_0 = cst_*$ on $\tilde{f}_0|_{I^k\times\{0\}}$

$\tilde{f}_t \in p^{-1}(*_B)$ on $\partial I^k \times I_t$

$\tilde{f}_1 \in p^{-1}(*_B)$ on $I^k \times \{1\}$



$\Rightarrow [\alpha] = \iota_*[\tilde{f}_1]$, $[\tilde{f}_1] \in \pi_k(F)$.

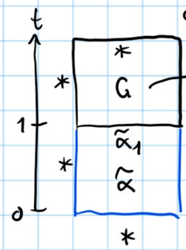
$\ker \delta \subseteq \operatorname{im} p_*$: $\quad \delta[\alpha] = 0$ for $\alpha : I^{k+1} \to B$, $\alpha|_{\partial I^{k+1}} = cst_{*_B}$

$$\Rightarrow \begin{tikzcd} & E \arrow[d, "p"] \\ I^k\times[0,1] \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & B \end{tikzcd}$$

$\tilde{\alpha} = cst_*$ on $\partial I^k \times [0,1]$, $I^k \times \{0\}$

but $\underbrace{\tilde{\alpha}|_{I^k\times[0,1]}}_{\delta[\alpha]} \in p^{-1}(*_B)$

If $\delta[\alpha] = 0$ by homotopy $G : I^i \times [0,1] \to p^{-1}(*_B)$



$\to p^{-1}(*_B) = F$

$\leftarrow$ htpic to α after post-composition w. p.

$\ker \iota_* \subseteq \operatorname{im} \delta$: Left to reader. $\square$

Exercise 14 Use the long exact sequence to compute the following homotopy groups of projective spaces:

1.) $\pi_k(S^1) = 0, \quad k > 1.$

Use principal bundle $\quad \mathbb{Z} \hookrightarrow \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$ (Similarly: $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n \twoheadrightarrow (S^1)^n$)

2.) $\pi_k(\mathbb{R}P^n) = \begin{cases} 0 & k=0 \\ \mathbb{Z}_2 & k=1 \\ \pi_k(S^n) & k>1 \end{cases} \quad n \geq 2 \qquad (\mathbb{R}P^0 = \{pt\}, \ \mathbb{R}P^1 = S^1)$

$\mathbb{Z}_2 \hookrightarrow S^n \twoheadrightarrow S^n/\mathbb{Z}_2 = \mathbb{R}P^n$

3.) $\pi_k(\mathbb{C}P^n) = \begin{cases} 0 & k=0,1 \\ \mathbb{Z} & k=2 \\ \pi_k(S^{2n+1}) & k>2 \end{cases} \qquad (\mathbb{C}P^1 = S^2)$

$S^1 \hookrightarrow S^{2n+1} \twoheadrightarrow S^{2n+1}/S^1 = \mathbb{C}P^n$

In particular, $\delta \neq 0$ in the above bundles $\Rightarrow$ these bundles are non-trivial.

Recall: $Gl_n(\mathbb{R}) \sim O(n) \qquad \& \qquad Gl_n(\mathbb{C}) \sim U(n) \qquad$ (Exercise 3)

$$O(n) \cong \underset{\det=1}{SO(n)} \amalg \underset{\det=-1}{SO(n)}$$

Exercise 15 Compute the homotopy groups of the following Lie groups

1.) $SO(n-1) \hookrightarrow SO(n) \twoheadrightarrow S^{n-1}$

$\Rightarrow \pi_k(SO(n)) \cong \pi_k(SO(n-1)) \quad k < n-2, \quad \pi_{n-2}(SO(n-1)) \twoheadrightarrow \pi_{n-2}(SO(n))$

$SO(1) = pt, \quad SO(2) = S^1.$

Conclude: $\pi_1(SO(n)) = \mathbb{Z}/\mathbb{Z}\cdot m \qquad$ (generated by a single element)

2.) $U(n-1) \hookrightarrow U(n) \twoheadrightarrow S^{2n-1}$

$\Rightarrow \pi_k(U(n)) \cong \pi_k(U(n-1)) \quad k < 2n-2, \quad \pi_{2n-2}(U(n-1)) \twoheadrightarrow \pi_{2n-2}(U(n))$

$U(1) = S^1$

Conclude $\pi_1(U(n)) = \mathbb{Z}$ for all $n \geq 1$.

Exercise 16 $SO(2) \hookrightarrow SO(3) \twoheadrightarrow S^2 \quad \delta_2 : \underset{\mathbb{Z}}{\underset{\parallel}{\pi_2(S^2)}} \xrightarrow{2\cdot} \underset{\mathbb{Z}}{\underset{\parallel}{\pi_1(SO(2))}}$ Conclude

$\Rightarrow \pi_1(SO(3)) = \mathbb{Z}_2$

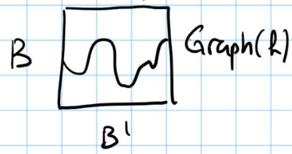
(in fact $SO(3) \cong \mathbb{R}P^3$)

Useful constructions

Pull-back If • $p: E \twoheadrightarrow B$ is a principal G-bundle (or fibre bundle), and

• $f: B' \to B$ in C^∞,

$f^*E = E|_{\mathrm{Graph}(f)}$



then $f^*E := \{(b',x) \in B' \times E \mid f(b') = p(x)\} \xrightarrow{p_f} B'$

is a smooth manifold for which the G-action inherited from $B' \times E$ induces the structure of a principal G-bundle $f^*E \twoheadrightarrow B'$

Thm Homotopic maps $f, g: B' \to B$ induce pullback bundles that are isomorphic as bundles over B'. $\boxed{f^*E \cong g^*E}$

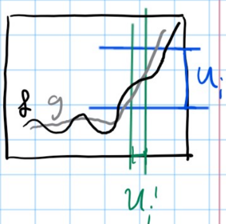
needs smooth bump functions (not true for holomorphic bundles)

Proof $B' \times E \xrightarrow{(\mathrm{id},p)} B' \times B$ is a principal G-bundle. Note that any local trivialisation $\Phi: p^{-1}(U) \to U \times G$ above $U \overset{\text{open}}{\subseteq} B$ induces a local trivialisation $\Phi': \underbrace{B' \times p^{-1}(U)}_{\subseteq B' \times E} \to (B' \times U) \times G$ above the open set $B' \times U \subseteq B' \times B$.

This local trivialisation restricts to a local trivialisation on f^*E, g^*E.

Consider a homotopy f_t from $f_0 = f$ to $f_1 = g$. For $N \gg 0$, the maps $f_{i/N}$ & $f_{i+1/N}$ are arbitrarily close. In particular, we can cover B' by open subsets U'_j for which $f_{i/N}(U'_j)$ & $f_{(i+1)/N}(U'_j) \subseteq B$ both are contained in some open subset $U_j \subseteq B$ over which E has a trivialisation

$$\Phi_j: p^{-1}(U_j) \to U_j \times G.$$



W.l.o.g.: Consider the case when f & g are close in C^∞-distance

We get induced trivialisations $B' \times p^{-1}(U_j) \to U_j \times G$ and thus also

$$\Phi_{f,j}: p_f^{-1}(U'_j) \to U'_j \times G, \quad \Phi_{g,j}: p_g^{-1}(U'_j) \to U'_j \times G$$

Note: The transition functions $\Phi_{f,j} \circ \Phi_{f,i}^{-1}$ & $\Phi_{g,j} \circ \Phi_{g,i}^{-1}: U'_i \cap U'_j \times G \circlearrowleft$ are close in C^∞-distance.

After a deformation $\hat{\Phi}_{g,i} := (\mathrm{id}_{U'_i}, \underbrace{\ell_{g(n)}}_{U'_i \to G \ C^\infty\text{-close to } cst_e}) \circ \Phi_{g,i}$ they can be assumed to be equal (left to the reader) $\Rightarrow f^*E \cong g^*E$ as G-bundles over B' □

Induced bundles

If G acts smoothly on V from the left $\begin{aligned} G \times V &\to V \\ (g,v) &\mapsto g \cdot v \end{aligned}$

then any principal G-bundle $p: E \twoheadrightarrow B$ induces a fibre-bundle

- $E_V := (P \times V)/G$ where $\begin{aligned} (P \times V) \times G &\longrightarrow P \times V \\ ((p,v), g) &\longmapsto (p \cdot g, g^{-1} \cdot v) \end{aligned}$ is the "diagonal G-action" (need to first make G act from the right)
- Base is again B
- $p_V: E_V \twoheadrightarrow B$ is induced by $p: P \times V \longrightarrow B$ which is G-invariant.
- $F := V$
- Structure group $=$ Image of G inside $\mathrm{Diff}^\infty(V)$ (⚠ G-action need not be faithful)
- $H_b := \left\{ \begin{aligned} \varphi_{b,p}: V &\hookrightarrow (\{p\} \times V)/G \subseteq \underbrace{(P \times V)/G}_{= E_V} \\ v &\mapsto [(p,v)] \end{aligned} \right\}$ $q(p) = b$ $\left(\varphi_{b,p \cdot g}^{-1} \circ \varphi_{b,p} = v \mapsto g^{-1} \cdot v \right)$

Conversely: From a fibre bundle $p: E \twoheadrightarrow B$ with structure group G we can perform the "inverse" of the above operation, i.e. construct a principal G-bundle

$$P_E := \{H_b\} \subseteq C^\infty(V, E)$$

$\downarrow$ B, $\varphi \mapsto p(\varphi(pt))$ for some choice of $pt \in V$, endowed with the right G-action

Rmk When the G-action on V is faithful ($g \cdot v = v \ \forall v \Rightarrow g = e$), then these operations are indeed inverse. In particular: E_V trivial $\Leftrightarrow$ P_E trivial

Exercise 17 Verify that 1.) a constant map $cst: B \to \{pt\} \subseteq B'$ satisfies $cst^* P \cong B \times G$, i.e. is globally trivial.

2.) A trivial G-bundle induces a trivial fibre bundle E_V.

Cor All bundles on a contractible space are trivial

Proof $id: B \to B$ is homotopic to $cst: B \to \{pt\} \subseteq B$

Hence $id^* E \cong cst^* E = B \times F$. $\square$

Exercise 18 $G = Gl_2(\mathbb{R})$ acts faithfully on $\mathbb{R}^2$. Consider the principal "frame bundle" $G \hookrightarrow E := \{A \in Gl_3(\mathbb{R}); \|Ae_1\|=1, \langle Ae_1, Ae_i\rangle = 0 \; i=2,3\} \twoheadrightarrow E/G \cong S^2$, and show that the induced bundle is $E_{\mathbb{R}^2} \cong TS^2$. Conclude using Exercise 16 that TS^2 is non-trivial.

Classifying spaces

Classifying spaces are "universal families" of a group G, and we can use them to replace G-bundles over B by maps from B into the classifying space.

Def The classifying space of a Lie group G is a topological space (CW-complex) BG which admits a contractible principal G-bundle $EG \twoheadrightarrow BG$.

By Whitehead's theorem: contractibility $\Leftrightarrow$ $\pi_k(EG) = 0$ for all k.

$\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \mathbb{R}^3 \subseteq \dots \subseteq \mathbb{R}^\infty$

$S^1 \subseteq S^2 \subseteq S^3 \subseteq \dots \subseteq S^\infty$

$\mathbb{R}P^1 \subseteq \mathbb{R}P^2 \subseteq \mathbb{R}P^3 \subseteq \dots \subseteq \mathbb{R}P^\infty$

$\mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq \mathbb{C}P^3 \subseteq \dots \subseteq \mathbb{C}P^\infty$

$Gl_1 \subseteq Gl_2 \subseteq Gl_3 \subseteq \dots \subseteq Gl_\infty$

Where $X^\infty := \bigcup_{i=1}^{\infty} X^i = \lim_{i\to\infty} X^i$

these spaces are endowed with the limit topology:

$$\boxed{U \subseteq X^\infty \text{ open} \Leftrightarrow \text{open in all } X^i}$$

they are CW-complexes.

⚠ Each point in X^∞ lives in some X^i $|i| < \infty$, e.g. $\mathbb{R}^\infty \subsetneq \ell^\infty$

Ex

1.) $\mathbb{Z}^n \hookrightarrow \underbrace{\mathbb{R}^n}_{\text{contractible}} \twoheadrightarrow \mathbb{T}^n \cong (S^1)^n \Rightarrow B\mathbb{Z}^n \simeq (S^1)^n$

2.) $\pi_k(S^\infty) = 0$, since every map $\gamma: S^k \to S^\infty$ has image contained inside some $S^N \subseteq S^\infty$ w. $0 << N < \infty$. $\Rightarrow S^\infty$ contractible

$\mathbb{Z}_2$ acts on S^n and hence on S^∞:

$\mathbb{Z}_2 \hookrightarrow S^\infty \twoheadrightarrow \mathbb{R}P^\infty \Rightarrow B\mathbb{Z}_2 = \mathbb{R}P^\infty$

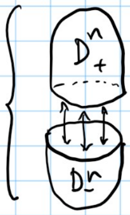
3.) S^1 acts on S^{2n+1} and hence on $S^\infty = \lim_{i\to+\infty} S^{2i+1}$

$S^1 \hookrightarrow S^\infty \twoheadrightarrow \mathbb{C}P^\infty \Rightarrow BS^1 = \mathbb{C}P^\infty$

Thm. Each principal G-bundle $E \twoheadrightarrow B$ is isomorphic to f^*EG for some uniquely determined homotopy class of maps $[f] \in [B, BG]$.

We illustrate this for G-bundles over spheres $E \twoheadrightarrow S^n$.

By Cor. above: E is isomorphic to the "clutching construction"



$$S^n \left\{ \begin{array}{c} D^n_+ \times G \\ \uparrow \\ D^n_- \times G \end{array} \right. \qquad (x_-, g) \sim (x_+, h(x_+)\cdot g) \qquad x \in S^{n-1}$$

the bundle is trivial on any disc by Cor above

$h: S^{n-1} \to G$
"clutching function"

for some $h: S^{n-1} \to G$

Fact: 1.) $\pi_n(S^n) \xrightarrow{\delta_n^{EG}} \pi_{n-1}(G)$ (c.f. Exercise 16)
$[id_{S^n}] \mapsto [h]$

The LES for $EG \twoheadrightarrow BG$ is

$$\cdots 0 \to \pi_{n+1}(BG) \underset{\cong}{\xrightarrow{\delta_{n+1}^{EG}}} \pi_n(G) \to 0 \to \pi_n(BG) \underset{\cong}{\xrightarrow{\delta_n^{EG}}} \pi_{n-1}(G) \to 0 \to$$

$EG \sim pt$, $[h] \in \pi_{n-1}(G)$

2.) The sought class of maps $S^n \to BG$ is given by $[f] = \delta_n^{-1}[h]$.
i.e. $f^*EG \cong E$

Rmk $H^*(BG)$ computes the group cohomology of G