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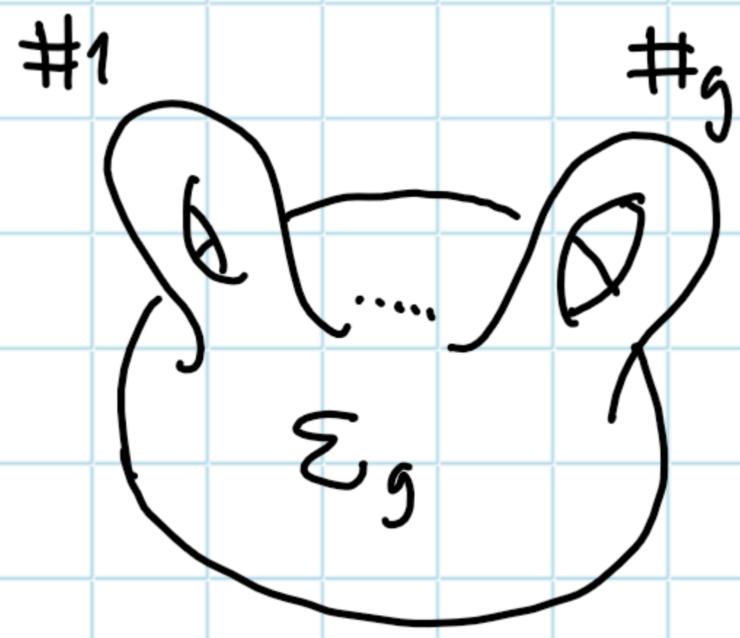
§ II. Morse Homology of Smooth Manifolds

The classification question for smooth manifolds up to diffeomorphism is not resolved (hard in dimensions 3 and 4) but progress has been made.

Surface of genus g

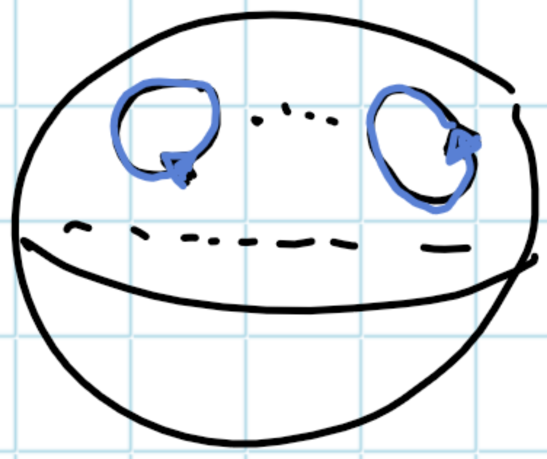


Dimension 1: S^1

Dimension 2: Closed orientable surfaces:

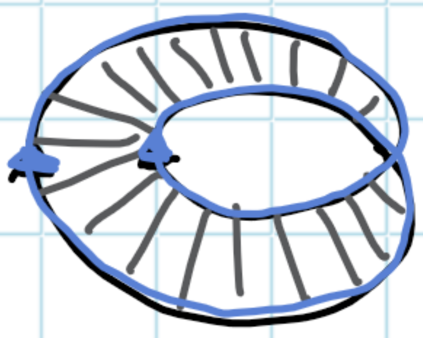
$$S^2 = \Sigma_0,\ \mathbb{T}^2 = S^1 \times S^1 = \Sigma_1,\ \Sigma_2, \ldots, \Sigma_g, \ldots$$

$$H_1(\Sigma_g) = \pi_1(\Sigma_g)/\text{comm} \cong \mathbb{Z}^{2g}$$



Non-orientable surfaces: $S^2 \setminus \#k$ disjoint balls

$\cup\ \#$ Möbius bands



$$\Big(\cong \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{k}\Big)$$

Fact $H_1 = \mathbb{Z}^{k-1} \times \mathbb{Z}_2 \Rightarrow H_1$ distinguishes all compact surfaces.

Dimension 3:

Thm [Perelman '03] A compact simply connected ($\pi_0 = \pi_1 = 0$) 3-dimensional manifold is diffeomorphic to S^3 (homeomorphism equally hard).

The above resolves the Poincaré Conjecture in dimension 3.

Poincaré originally believed that $H_1 = 0$ would be sufficient, however, he found the following counter-example... but first

Quaternions The quarternions is the 4-dim$_{\mathbb{R}}$ non-commutative unital $\mathbb{R}$-algebra $\mathbb{H} := \left\{ \begin{bmatrix} a+i\cdot b & c+i\cdot d \\ -c+i\cdot d & a-i\cdot b \end{bmatrix} ; a,b,c,d \in \mathbb{R} \right\} \cong \mathbb{R}^4$

$SU(2) = \mathbb{H} \cap \{ a^2+b^2+c^2+d^2=1 \} \cong S^3$ the unit quarternions

(SU(n) acts transitively on S^{2n-1}, hence $S^3 \cong SU(2)/SU(1) \cong SU(2)$)

SU(2) acts on $\mathrm{Im}\,\mathbb{H} := \left\{ \begin{bmatrix} ib & c+id \\ -c+id & -ib \end{bmatrix} \right\} \cong \mathbb{R}^3$ by conjugation

$B \in \mathrm{Im}\,\mathbb{H} \qquad A \cdot B \cdot A^{-1} \in \mathrm{Im}\,\mathbb{H}, \quad A \in SU(2)$

Preserves norm $\Rightarrow p : SU(2) \xrightarrow{\text{group morphism}} SO(3)$, $\ker = \{ \pm \mathrm{Id} \} \triangleleft SU(2)$

Exercise 19 Show that p is surjective with $\ker = \mathbb{Z}_2$.

Hence it follows that

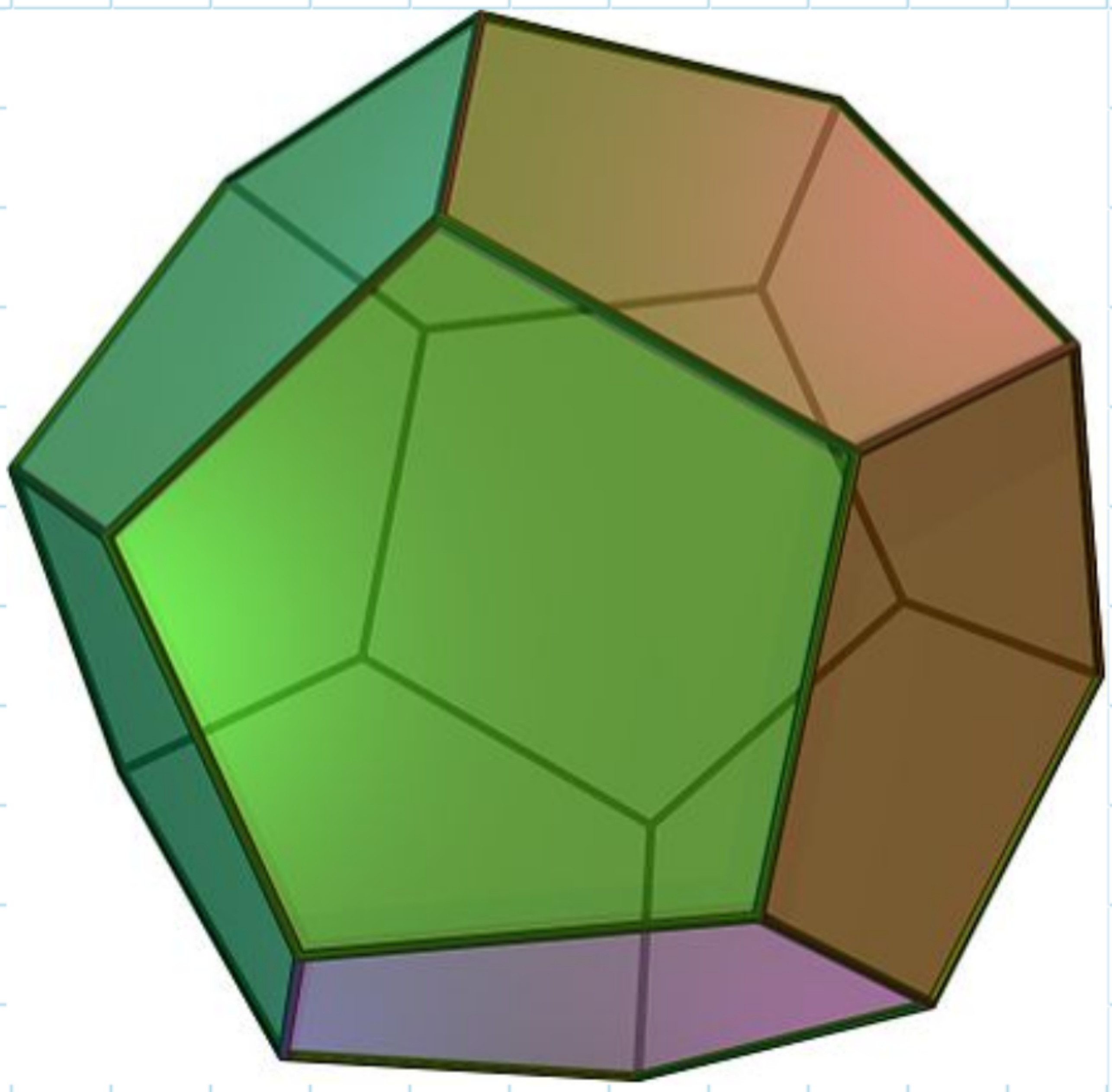
$$\begin{array}{ccccc} \mathbb{Z}_2 & \hookrightarrow & SU(2) & \xrightarrow{p} & SO(3) \\ & & \text{\rotatebox{90}{$\cong$}} & & \\ & & S^3 & & \end{array}$$

is a principal bundle, and $SO(3) \cong \mathbb{RP}^3$

Unlike for surfaces, three-dim. manifolds cannot be distinguished by H_1.

Ex (Poincaré Homology sphere)

The symmetry group $< SO(3)$ of the regular dodecahedron $\subseteq \mathbb{R}^3$ is the alternating group of 5 elements



Dodecahedron. Source: Wikipedia

$A_5 \quad \lhd \quad S_5$ symmetric group of five elements

A_5: products of an even nr of permutations

S_5: set-theoretic bijections of $\{1,2,3,4,5\}$

$|A_5| = 60 \qquad |S_5| = 5! = 120$

The Poincaré homology sphere is the homogeneous space:

$$SO(3)/A_5 \cong SU(2)/\tilde{A}_5 \qquad \tilde{A}_5 := p^{-1}(A_5)$$

LES

$$\Rightarrow \underbrace{\pi_1(SU(2))}_{0} \to \pi_1(SO(3)/A_5) \xrightarrow{\delta} \underset{\tilde{A}_5}{\underset{\parallel}{\pi_0(\tilde{A}_5)}} \to \underbrace{\pi_0(SU(2))}_{0}$$

$\Rightarrow \delta$ is a group isomorphism $\quad \pi_1(SO(3)/A_5) \cong \tilde{A}_5 \overset{2:1}{\twoheadrightarrow} A_5$

Fact $\quad H_1(SO(3)/A_5) = \tilde{A}_5/\text{comm} = 0 \quad (A_5 \,\&\, A_5 \text{ perfect})$

In dimensions 2 & 3, there are strong constraints on π_1.

On the other hand:

<u>Fact</u> Any finitely presented group is π_1 of some compact 4-dimensional smooth manifold. <u>Hence:</u> general classification is hopeless.

<u>Poincaré Conjecture in dimension ≥ 4:</u>

<u>Note</u> $\pi_1(\mathbb{C}P^2) = 0 = \pi_1(S^4)$, but $\pi_2(\mathbb{C}P^2) = \mathbb{Z} \neq \pi_2(S^4)$

But, it makes sense to ask: if M^n a closed n-dim manifold that is <u>homotopy equivalent</u> to S^n, is it the case that M is <u>diffeomorphic</u> to S^n?

(By Whitehead's Thm + Rel. Hurewicz: M^n homotopy eq. to S^n $\Leftrightarrow$ M is a <u>simply connected</u> homology sphere)

<u>Homeomorphism</u> true in all dimensions. $n=3$: Perelman '03
$n=4$: Freedman '82
$n>4$: Smale '60

<u>Diffeomorphism</u> • Not known in dimension $n=4$

• $n>4$ the problem was solved by Cerf, Milnor, Kervaire, Smale, ..
finitely many smooth structures / Diff^∞ for each $n>4$
(the answer depends on n)

Ex (Milnor's exotic 7-spheres '56)

Consider the S^3-bundle $M_{\ell,m} \twoheadrightarrow S^4$ with clutching function

$$\phi_{\ell,m}\ \partial D^4 = S^3 \longrightarrow SO(4) < \mathrm{Diff}^\infty(S^3) \quad \text{(not a group morphism!)}$$
$$A \in SU(2) \longmapsto \{ SU(2) \ni B \mapsto A^m \cdot B \cdot A^\ell \in SU(2) \}$$

Fact $M_{\ell,m}$ homeomorphic to $S^7 \iff m+\ell = \pm 1$

diffeomorphic to $S^7 \iff (m-\ell)^2 \equiv 1 \pmod 7$ & $m+\ell = \pm 1$

Exercise 20

Show that the so called Hopf fibration

$$\begin{array}{ccccc} S^1 & \hookrightarrow & S^3 & \twoheadrightarrow & \mathbb{CP}^1 \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \right\} & < & SU(2) & \twoheadrightarrow & S^2 \end{array}$$

has clutching function $\phi : \partial D^2 = S^1 \to S^1 < \mathrm{Diff}^\infty(S^1)$ acts on itself by mult.

$$\phi = \mathrm{Id}_{S^1}$$

Hint: Use the connection between the clutching function and the LES from the end of L.7.

Exercise 21

Show that $M_{1,0} \cong S^7$

$$\begin{array}{ccccc} S^3 & \hookrightarrow & S^7 & \twoheadrightarrow & S^4 \\ \text{\rotatebox{90}{$\cong$}} & & \cap & & \text{\rotatebox{90}{$\cong$}} \\ SU(2) & & \mathbb{H} \oplus \mathbb{H} & & \mathbb{HP}^1 \end{array}$$

"quaternionic projective line"

Morse theory

The classification question for smooth manifolds can be approached by Morse theory.

If $f: M \to \mathbb{R}$, then $Tf: TM \to T\mathbb{R}$ is the tangent map
$$T\mathbb{R} = \mathbb{R} \times \mathbb{R}_{\text{fibre}}$$

The fibre component is a fibre-wise linear functional
$df: TM \to \mathbb{R}$, i.e. $df_x : T_x\mathbb{R}^n \to \mathbb{R}$ is linear for all $x \in M$

i.e. for $M \subseteq \mathbb{R}^N$, the classical derivative along M.

Def $f \in C^\infty(M, \mathbb{R})$ is Morse if the differential

$d(f \circ \varphi_i^{-1}) = df \circ T\varphi_i^{-1} : T\mathbb{R}^n \to \mathbb{R}$ in local coordinates $M \supseteq U_i \xrightarrow[\varphi_i]{\cong} \mathbb{R}^n$ satisfies:
$$T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$$

$$\boxed{0 \text{ is a regular value of } \begin{array}{c}\mathbb{R}^n \longrightarrow \mathbb{R}^n \\ d(f\circ\varphi_i^{-1})(\partial_{x_1}, \dots, \partial_{x_n})\end{array} \left(\begin{array}{l}\text{matrix expression} \\ \text{of } \quad d(f\circ\varphi_i^{-1})\end{array}\right)}$$

i.e. the Hessian $\dfrac{\partial^2 (f\circ\varphi_i^{-1})}{\partial x_\alpha \partial x_\beta}$ is non-degenerate at all $\operatorname{Crit} f := \{x \in M ; df_x = 0\}$

Consequence The critical points $\operatorname{Crit} f \subseteq M^n$ are of codimension $= n = \dim M$. Hence they are isolated (thus finite when M is compact)