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Morse Theory

$f: M \to \mathbb{R}$ **Morse**: in local coordinates $\varphi: U \to \mathbb{R}^n$, $U \subset M$

$\dim M = n$

$$\underbrace{d(f \circ \varphi^{-1})}_{\tilde{f}} = \left[\frac{\partial \tilde{f}}{\partial x_1}(\bar{x}), \ldots, \frac{\partial \tilde{f}}{\partial x_n}(\bar{x})\right] : \mathbb{R}^n \to \mathbb{R}^n$$

has 0 as a regular value, i.e.:

> The differential $D_{\bar{x}}(d(f \circ \varphi^{-1})) \in \mathrm{Mat}_{n,n}$ is surjective at all $\bar{x} \in \mathrm{Crit}\, \tilde{f}$ $(d_{\bar{x}}\tilde{f} = 0)$
>
> $$D_{\bar{x}}(d(f \circ \varphi^{-1})) = \left[\frac{\partial^2 \tilde{f}}{\partial x_i \partial x_j}(\bar{x})\right] =: H_{\bar{x}} \quad \text{Hessian}$$
>
> (OBS: independent of chart

Existence

$F \in C^\infty(M,N)$

critical points: $p \in \mathrm{Crit}\, F \subseteq M \iff T_pF : T_pM \to T_{F(p)}N$ not surjective

critical values: $F(\mathrm{Crit}\, F) \subseteq N$

regular values: $N \setminus F(\mathrm{Crit}\, F)$ OBS: all points in $N \setminus F(M)$ are regular!

Thm (Sard) For $F \in C^\infty(M,N)$ the regular values $N \setminus F(\mathrm{Crit}\, f)$ is a dense subset. (In fact: intersection of countably many open dense subsets; even finitely many if M is compact)

Thm Any $f \in C^\infty(M, \mathbb{R})$ can be perturbed to a Morse function $f + g$, where g is a suitable function with arbitrarily small C^2-norm.

Sketch of proof We perturb f in some chart. Then one can use bump functions to construct a global perturbation (left to the reader).

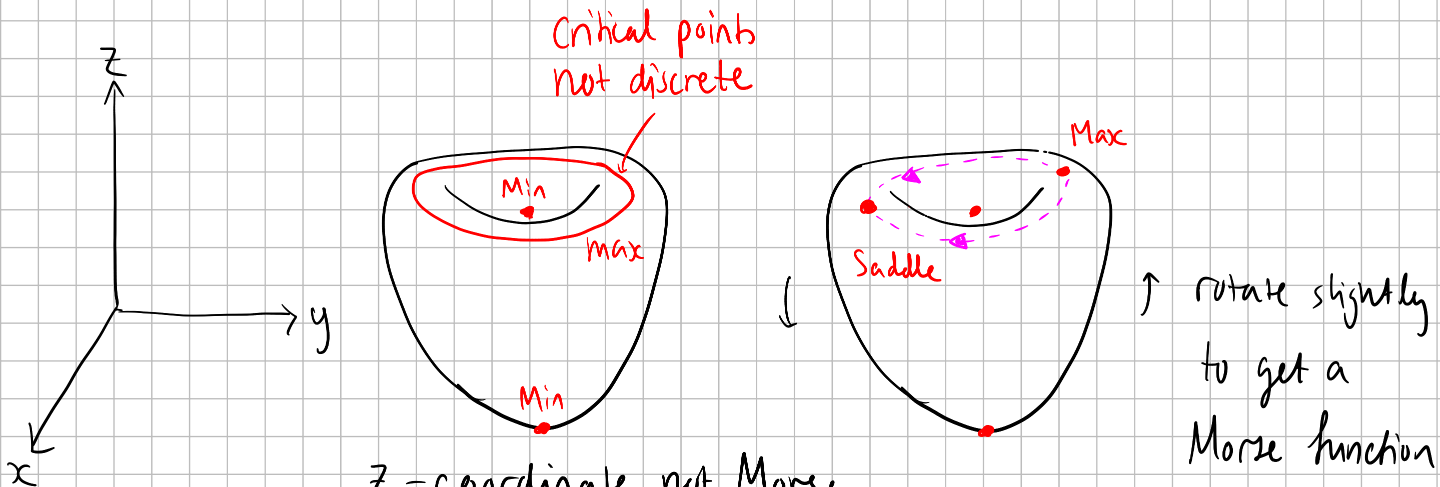
Sard's Theorem: $\exists$ a regular value of $d\tilde{f} = \left[\frac{\partial \tilde{f}}{\partial x_1} \cdots \frac{\partial \tilde{f}}{\partial x_n}\right]$ arbitrarily close to 0, say $\bar{\varepsilon} \in \mathbb{R}^n$ (this is the target space of $d\tilde{f}$)

Consider the perturbation $\bar{x} \longmapsto \tilde{f}(\bar{x}) - \bar{\varepsilon} \cdot \bar{x}$ of $\tilde{f}$

$$d\left(\tilde{f} - \bar{\varepsilon} \cdot \_\right) = \left[\frac{\partial \tilde{f}_1}{\partial x_1} - \varepsilon_1, \ldots, \frac{\partial \tilde{f}_n}{\partial x_n} - \varepsilon_n\right]$$

$\mathrm{Crit}\left(\tilde{f} - \bar{\varepsilon} \cdot \_\right) = d\left(\tilde{f} - \bar{\varepsilon} \cdot \_\right)^{-1}(\bar{\varepsilon})$, and $\mathrm{Hessian}\left(\tilde{f} - \bar{\varepsilon} \cdot \_\right) = \mathrm{Hessian}\left(\tilde{f}\right)$

$\Rightarrow \quad \bar{x} \longmapsto \tilde{f}(\bar{x}) - \bar{\varepsilon} \cdot \bar{x}$ is Morse $\qquad \square$



z-coordinate not Morse

Now assume that M is compact

f Morse $\Rightarrow$ f has finitely many critical points.

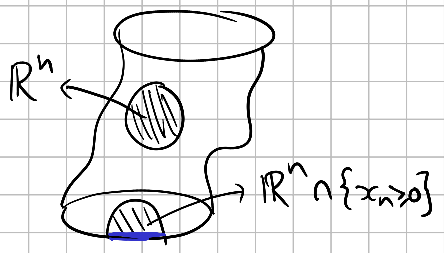
$f: M \longrightarrow f(M) = [\min f, \max f] \subseteq \mathbb{R}$

Consider $[a,b] \subseteq f(M) \setminus f(\mathrm{Crit}\, f)$

<u>Implicit function theorem:</u> $f^{-1}(a), f^{-1}(b) \subseteq M^n$ are submanifolds of $\dim = n-1$

Lem $f^{-1}[a,b] \cong_{C^\infty} f^{-1}(a) \times [a,b]$, and $f^{-1}(a) \cong f^{-1}(b)$

This is a manifold with <u>boundary</u>



$$\partial\left(f^{-1}[a,b]\right) = f^{-1}(a) \sqcup f^{-1}(b)$$

$f^{-1}[a,b]$ locally diffeomorphic to $\mathbb{R}^n$ <u>or</u> $\{x_n \geq 0\} \subseteq \mathbb{R}^n$

($\{x_n = 0\} \subseteq \mathbb{R}^n$ is a chart for the boundary)

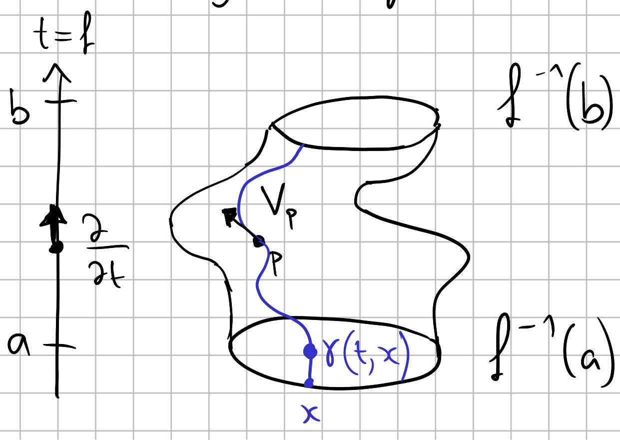
<u>Sketch of proof.</u> Lift the non-vanishing vector field ∂_t $[a,b]$ to a vector field V under $Tf: T(f^{-1}[a,b]) \to T\mathbb{R}$

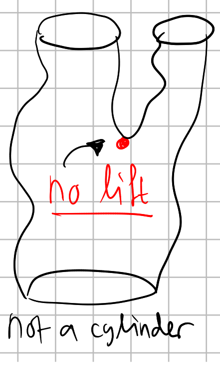
(possible since Tf is surjective there: no crit points)

The cylinder is parametrised by solving the ODE

$$\begin{cases} \dot{\gamma}(t,x) = V_{\gamma(t,x)} \\ \gamma(0,x) = x \in f^{-1}(a) \end{cases}$$

$\square$

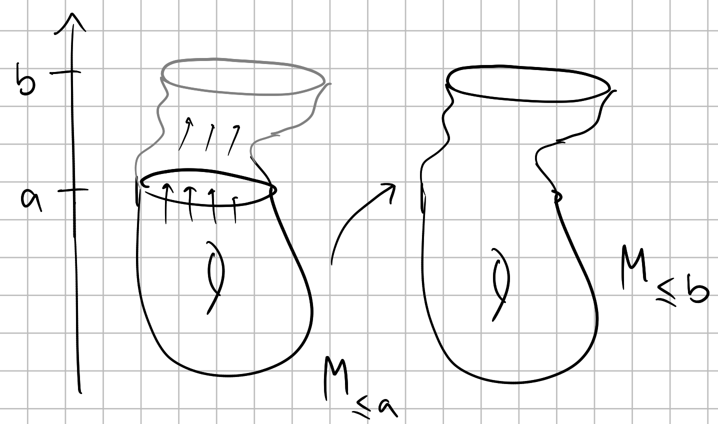




Similarly,

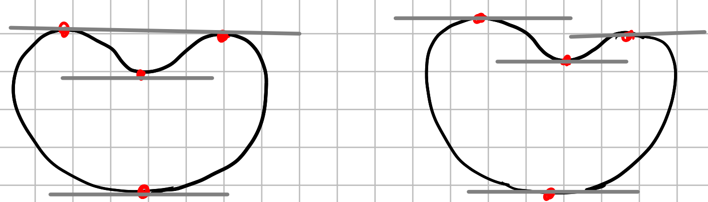
Lem The manifold $M_{\leq a} := f^{-1}(-\infty, a]$ with boundary $\partial(M_{\leq a}) = f^{-1}(a)$ is diffeomorphic to $M_{\leq b}$ if there are no critical values in $[a,b]$

$\Big($ Use the construction of a diffeomorphism $g : (a-\varepsilon, a] \xrightarrow{\cong} (a-\varepsilon, b]$ s.t $g|_{(a-\varepsilon, a-\varepsilon/2]} = id$ $\Big)$



In order to describe M, it suffices to understand how $M_{\leq t}$ is changed when t passes one of the finitely many critical values.

After a perturbation by bump functions near each critical points: <u>different critical points have different critical values</u>

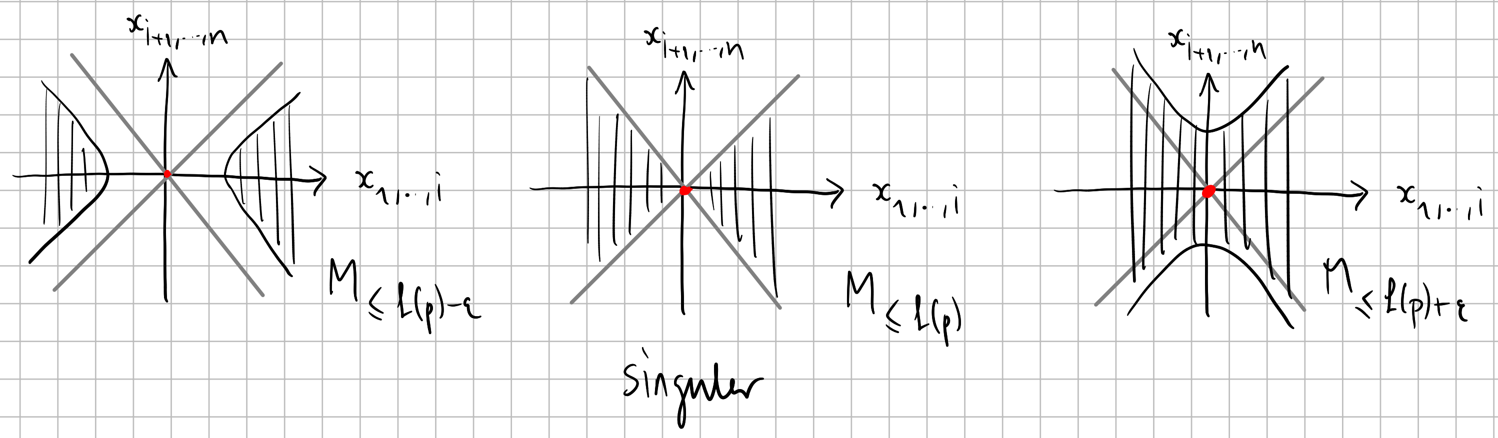


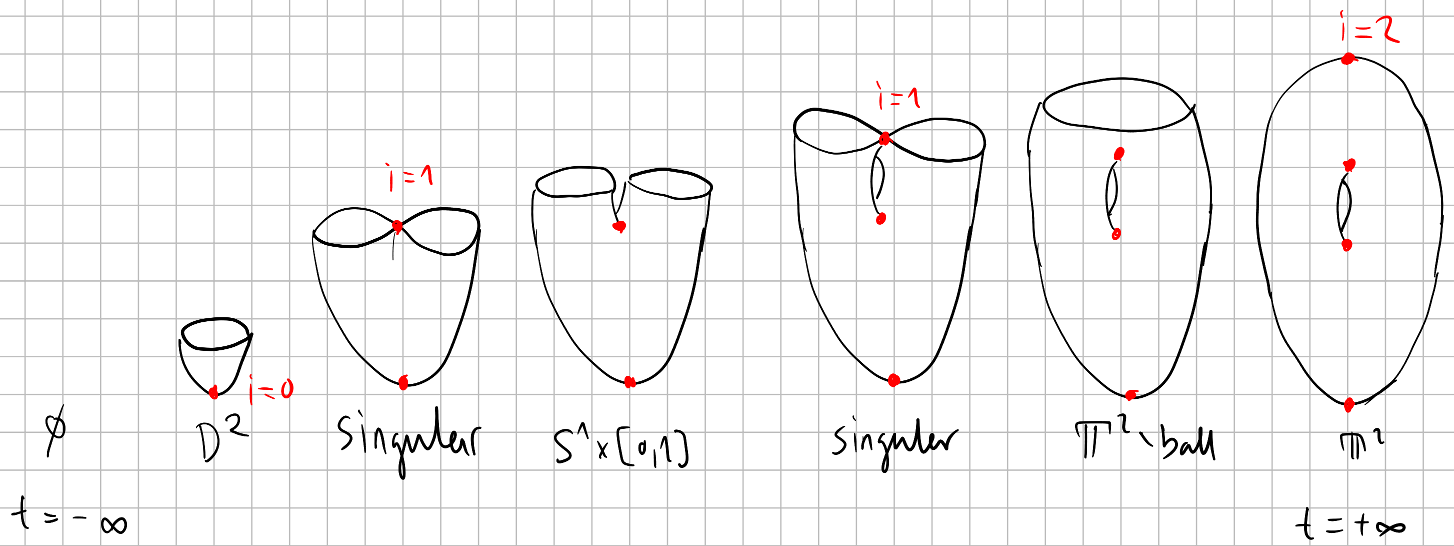
The crucial result is the Morse lemma:

Lem (The Morse Lemma) Near each critical point $p \in M$ of a Morse function $f: M \to \mathbb{R}$, there exists a local chart $\varphi: \begin{matrix} U & \xrightarrow{\cong} & \mathbb{R}^n \\ p & \longmapsto & 0 \end{matrix}$ in which

$$\boxed{f \circ \varphi^{-1}(\bar{x}) = f(p) - (x_1^2 + \dots + x_i^2) + (x_{i+1}^2 + \dots + x_n^2)}$$

$\Rightarrow$ Hessian is $H = \begin{bmatrix} -1 & & & & & \\ & \ddots & & & 0 & \\ & & -1 & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} \text{signature} = i \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n-i \end{matrix}$





Exercise 22

Show that if M^n is compact without boundary & admits a Morse function with precisely two critical points, then

- M^n is homeomorphic to S^n
- Show diffeomorphism when $n = 1, 2$
  (In fact: true for $n \leq 6$)