Holomorphic Curve Theories in Symplectic Geometry

Lecture VI

Georgios Dimitroglou Rizell

Uppsala University

UPPSALA UNIVERSITET

Goal of lecture

Today:

- Definition of Floer homology in the exact case.
- Application: Obstructs Hamiltonian displaceability.
- Moduli space of discs with boundary punctures (Associahedra).

UPPSALA UNIVERSITET

Take-home Message

If a closed Lagrangian can be displaced by a Hamiltonian isotopy, then it admits a non-constant pseudoholomorphic disc.

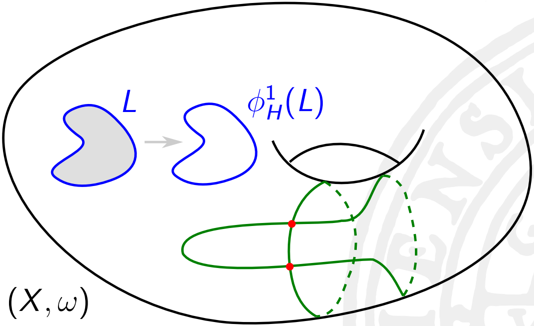


Figure: The blue Lagrangian L is displaceable by a Hamiltonian isotopy and bounds a holomorphic disc; the green Lagrangian which is homologically essential is not Hamiltonian displaceable.

UPPSALA UNIVERSITET

Plan

Displaceability

Definition

We say that a compact subset $C \subset (X,\omega)$ is *Hamiltonian displaceable* if there exists a Hamiltonian $H\colon X \times \mathbb{R} \to \mathbb{R}$ for which $C \cap \phi^1_H(C) = \emptyset$.

- Gromov showed in [Gro85] that a closed Lagrangian L which is Hamiltonian displaceable must admit a J-holomorphic disc with boundary on L for all tame J. Also see Chekanov's refinement [Che98].
- Floer later refined this to a chain complex in [Flo88], whose homology is a lower bound for

$$|L \pitchfork \phi^1_H(L)| \geq 0$$

This complex is typically impossible, or at least difficult, to define in the presence of pseudoholomorphic discs with boundary on L.

Displaceability

Example

- Any compact subset $C \subset (\mathbb{C}^n, \omega_0)$ is Hamiltonian displaceable; e.g. the translation $x_i \mapsto x_i + t$ is generated by the Hamiltonian $H = -y_i$.
- For curves in surfaces, the question of Hamiltonian displaceability can be solved completely by *area considerations*.

Remark

In order to deduce the existence of a J-holomorphic disc with boundary on a closed displaceable Lagrangian inside a non-compact symplectic manifold, one needs to control the behaviour of J outside of a compact subset

Displaceability

Why we need control at infinity:

- There are plenty of Hamiltonian isotopies, and they act on the space of tame almost complex structures:

$$(\phi^1_H)_* J := (D\phi^1_H)^{-1} \circ J \circ D\phi^1_H.$$

- We may use this action repeatedly to push some part of the interior of a J_0-holomorphic curve in $\mathbb{C}^n$, $n > 1$, with boundary on a fixed Lagrangian out to ∞ (make sure it leaves every compact subset).

Conditions on J

To gain control at infinity of a noncompact (X, ω) we will here assume that $J \in \mathcal{J}^{tame}(X, \omega)$ satisfies the following property: there exists a smooth proper function $f\colon X \to [-N, +\infty)$ such that:

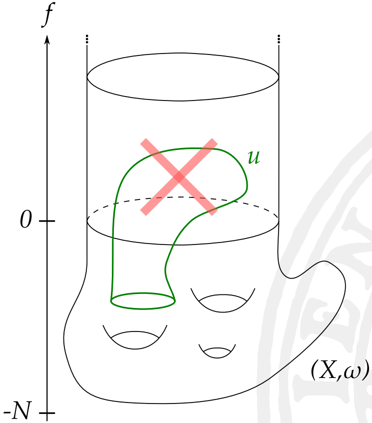
Non-negativity of the "Levi two-form" on J-complex lines, i.e.

$$-dd^c f(v, Jv) \geq 0, \;\; v \in T_p X,$$

holds for all $p \in f^{-1}[0, +\infty) \subset X$.

(This is a condition only on J outside of some compact subset.)
We will now derive a maximum-principle under these assumptions, that makes it impossible for J-holomorphic curves to partly escape to infinity.

Conditions on J



Conditions on J

Proposition

Let $u\colon (\Sigma, j) \to (X, J)$ be pseudoholomorphic, and Σ a connected (possibly open) Riemann surface whose boundary is contained inside $f^{-1}[-N, 0]$. If $f \circ u\colon \Sigma \to \mathbb{R}$ assumes a positive value, then $f \circ u$ is constant.

Proof (1/2).

It suffices to show that $f \circ u$ is sub-harmonic in local coordinates near a point which maps to a positive value under $f \circ u$.

Conditions on J

Proposition

Let $u\colon (\Sigma, j) \to (X, J)$ be pseudoholomorphic, and Σ a connected (possibly open) Riemann surface whose boundary is contained inside $f^{-1}[-N, 0]$. If $f \circ u\colon \Sigma \to \mathbb{R}$ assumes a positive value, then $f \circ u$ is constant.

Proof (2/2).

For a loc. def. J-holomorphic map $u\colon (B^2_\epsilon, j_0) \to (X, J)$ we have

$$4\Delta(f \circ u)dx \wedge dy = 2i\partial\overline{\partial}(f \circ u) = -dd^c(f \circ u)$$

(see [Lecture I]). The assumption on the Levi form gives

$$4\Delta(f \circ u)dx \wedge dy(\partial_x, j_0\partial_y) \geq 0$$

and thus $\Delta(f \circ u) \geq 0$

Conditions on J

- The above is clearly satisfied for $\mathbb{C}^n$: for $J = J_0$ and $f = \|\mathbf{z}\|^2/4$ we even have

$$-dd^c f = \omega_0.$$

- More generally: when (X, ω) is symplectomorphic to a "half symplectisation"

$$((0, +\infty)_t \times Y, d(e^t \alpha))$$

outside of a compact subset (where (Y, α) is a closed contact manifold) one can take $f = e^t$ and J to be *cylindrical* in the same subset. (See next slide.)

Example

The latter condition is satisfied for $(T^*M, d\theta_M)$ for M closed; take e.g. the complement of any fibre-wise convex smooth domain.

Cylindrial almost complex structures

Definition

An almost complex structure $J \in J^{comp}$ on the symplectisation

$$(\mathbb{R}_t \times Y, d(e^t\alpha))$$

is said to be *cylindrical* if

- J is t-invariant,
- $J(\ker\alpha \cap TY) = \ker\alpha \cap TY$ (i.e. J preserves the contact planes $\ker\alpha \subset TY$)
- $J\partial_t \in TY$ and satisfies

$$d\alpha(J\partial_t, \cdot) = 0 \text{ and } \alpha(J\partial_t) = 1$$

(i.e. $J\partial_t$ is the Reeb vector field on Y defined by α).

Cylindrial almost complex structures

An almost complex structure $J \in J^{comp}$ on the symplectisation

$$(\mathbb{R}_t \times Y, d(e^t\alpha))$$

which is cylindrical satisfies

$$-dd^c e^t = d(e^t\alpha) = \omega.$$

In other words, if (X, ω) can be equipped with an almost complex structure which is "cylindrical outside of a compact subset," then the aforementioned maximum principle is satisfied.

The Floer complex: The vector space

For two closed Lagrangians $L_0, L_1 \subset (X, \omega)$ that intersects transversely $L_0 \pitchfork L_1$ (the intersection is a finite number of points) we define:

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{F} \cdot x$$

- The above canonical basis is graded under the presence of suitable additional data. (To be dealt with later.)
- The coefficients are taken to be in a suitable field $\mathbb{F}$; in general one needs the “Novikov field”, i.e. power series of the form

$$\Lambda^R := \left\{ \sum_{i=1}^{\infty} a_i T^{\lambda_i}, \;\; a_i \in R, \lambda_i \in \mathbb{R}, \lim_{i \to +\infty} \lambda_i = +\infty \right\}.$$

The Floer complex: The vector space

For two closed Lagrangians $L_0, L_1 \subset (X, \omega)$ that intersects transversely $L_0 \pitchfork L_1$ (the intersection is a finite number of points) we define:

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \mathbb{F} \cdot x$$

- We can always take $\mathbb{F} = \mathbb{Z}_2$ when both L_i are *exact* Lagrangians.
- If, in addition, L_i are equipped with *spin structures* we we can take $\mathbb{F} = \mathbb{Q}$ (or even $\mathbb{Z}$).

The Floer complex: The vector space

From now on we will assume that L_i are closed, connected, *exact* Lagrangians and will take $\mathbb{F} = \mathbb{Z}_2$.
In particular $(X, \omega) = (X, d\lambda)$ and $\lambda|_{TL_i} = df_i$ for primitives

$$f_i \colon L_i \to \mathbb{R}.$$

Definition

The *action* of an intersection $x \in L_0 \cap L_1$ is

$$\mathfrak{a}(x) := f_0(x) - f_1(x) \in \mathbb{R}$$

- The primitives f_i of $d\lambda|_{TL_i}$ are only determined by the embedding $L_i \subset (X, d\lambda)$ *up to an unspecified constant.*
- Likewise, only *difference of action* between two generators is uniquely determined by $L_i \subset X$.

The Floer complex: The differential

The differential

$$d\colon CF(L_0, L_1) \to CF(L_0, L_1)$$

is defined as follows.

- The differential d depends only on the choice J_t of a generic one-parameter family of tame almost complex structures on (X, ω)
- For $x \in L_0 \cap L_1$ a basis element of $CF(L_0, L_1)$ we define

$$d(x) = \sum_{y \in L_0 \cap L_1} \sum_{\substack{M \in \pi_0(\mathcal{M}_{J_t}(x,y)) \\ \mathrm{index} M = 1}} y$$

where we proceed to describe the moduli space $\mathcal{M}_{J_t}(x, y)$.

The Floer complex: The moduli space

The "moduli space of Floer strips from x to y"

$$\mathcal{M}_{J_t}(x, y)$$

consists of those smooth maps

$$u\colon (\{s + it; t \in [0, 1]\}, \{t = 0\}, \{t = 1\}) \to (X, L_0, L_1)$$

which

- are pseudoholomorphic for the domain-dependent complex structure J_t on X, i.e.

  $$du(\partial_t) = du(j_0 \cdot \partial_s) = J_t \cdot du(\partial_s)$$

  is satisfied.
- have finite energy $0 \leq \int_u \omega < \infty$.
- $t \mapsto u_s(t) = u(s + it)$ converge uniformly to the constant map $t \mapsto x \in L_0 \cap L_1$ (resp. $t \mapsto y$) as $s \to +\infty$ (resp. $s \to -\infty$).

The Floer complex: The moduli space

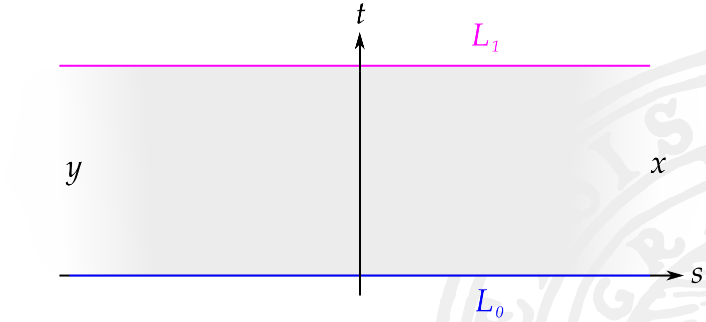


Figure: A Floer strip used in the differential. The input is $x \in L_0 \cap L_1$ and the output is $y \in L_0 \cap L_1$.

The Floer complex: The moduli space

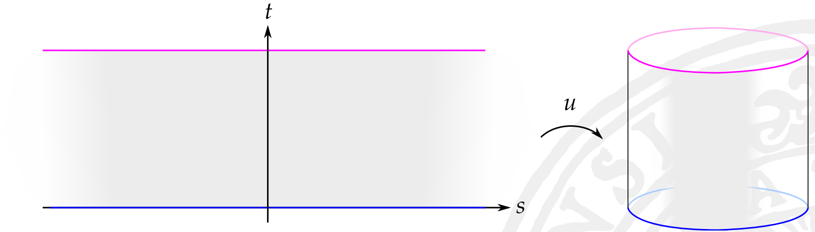


Figure: A strip whose symplectic area is infinite; it is given as the universal cover of a holomorphic annullus with boundary on two disjoint Lagrangians.

The Floer complex: The moduli space

- The reason why we need a t-dependence is to achieve transversality, so that the moduli spaces of Floer strips becomes a smooth manifold of the expected dimension i.e. the Fredholm index.
- In [EES07] Ekholm–Etnyre–Sullivan managed to get rid of this assumption in the exact case (for a carefully chosen almost complex structure).

The Floer complex: The moduli space

- The requirement of finite symplectic area (energy) together with holomorphicity gives an a priori uniform convergence of the functions $t \mapsto u_s(t) = u(s+it)$ to constants as $s \to \pm\infty$.
- One can get rid of the domain-dependence of the Cauchy–Riemann equation by considering instead $\mathbf{J}$-holomorphic sections over

$$X \times \{s+it;\ \ t \in [0,1]\} \to \{s+it;\ \ t \in [0,1]\}$$

with $\mathbf{J}_{(\mathrm{pt},s,t)} = J_t(\mathrm{pt}) \oplus j_0$.

The Floer complex: The moduli space

By Stokes' theorem, any $u \in \mathcal{M}_{J_t}(x,y)$ satisfies

$$\int_u \omega = \int_u d\lambda = \mathfrak{a}(x) - \mathfrak{a}(y)$$

(Exactness is of course crucial here!) On the other hand, recall that pseudoholomorphic maps satisfy

$$\int_u \omega \geq 0$$

with equality if and only if u is constant.

The Floer complex: The moduli space

- The moduli space of Floer strips has a linearisation which is elliptic, and hence there is a well-defined Fredholm index.
- The index of a contant strip in $\mathcal{M}_{J_t}(x,x)$ which maps into a double point $x \in L_0 \cap L_1$ is equal to zero.
- In the exact case, only the constant strip lives in the moduli space $\mathcal{M}_{J_t}(x,x)$; the formula for its symplectic area in terms of the asymptotics yields $\mathfrak{a}(x) - \mathfrak{a}(x) = 0$;
- The moduli space has a natural $\mathbb{R}$-action by reparametrisation $s \to s + s_0$ which is free unless the strip is constant (by its asymptotic properties).

The Floer complex: The differential

The differential

$$d\colon CF(L_0, L_1) \to CF(L_0, L_1)$$

is defined on a basis element $x \in L_0 \cap L_1$ by

$$d(x) = \sum_{y \in L_0 \cap L_1} \sum_{\substack{M \in \pi_0(\mathcal{M}_{J_t}(x,y)) \\ \mathrm{index} M = 1}} y$$

where the area formula (Stoke's theorem) implies that $d(x)$ is a sum of generators action strictly lower than $\mathfrak{a}(x)$.

The Floer complex

Theorem

Floer [Flo88] When $L_i \subset (X, d\lambda)$, $i = 0, 1$, are closed exact Lagrangian submanifolds and J_t is cylindrical outside of a compact subset then

1. *d is well-defined;*
2. *$d^2(x) = 0$;*
3. *A compactly supported Hamiltonian isotopy ϕ_H^t of $(X, d\lambda)$, and choice of two-parameter family of almost complex structures $J_{s,t}$, induces a chain map*

$$\Phi_{H,J_{s,t}} \colon CF(L_0, L_1; J_{-1,t}) \to CF(L_0, \phi_H^1(L_1); J_{1,t})$$

which induces isomorphism in homology; and

The Floer complex

Theorem

Floer [Flo88] When $L_i \subset (X, d\lambda)$, $i = 0, 1$, are closed exact Lagrangian submanifolds and J_t is cylindrical outside of a compact subset then

4. *When L_1 is obtained by perturbing L_0 to the graph of $dg \in \Omega^1(L)$ inside a Weinstein neighbourhood $U \subset T^*L_0$ of L_0, then suitable choices yields an identification*

$$(CF(L_0, L_1), d) = (C^{Morse}(-g), \partial^{Morse}),$$

of complexes with action filtration (R.H.S. is the Morse homology complex of $-g \colon L_0 \to \mathbb{R}$ generated by $\mathrm{crit}(g)$).

The Floer complex: An example

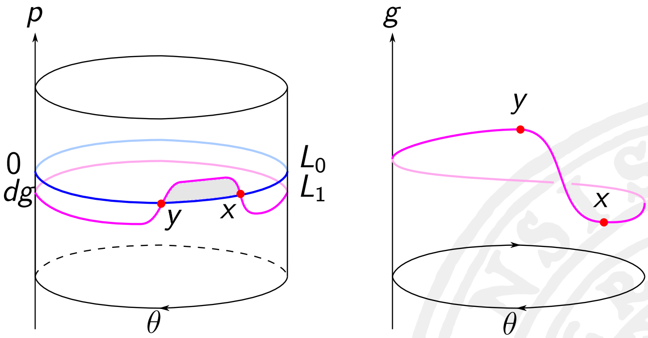


Figure: The Floer homology complex $CF(L_0, L_1)$ for $L_0 = 0_M$ the zero section in $(T^*S^1, dp \wedge d\theta)$ and $L_1 = dg$ the exterior derivative of a Morse function $g\colon S^1 \to \mathbb{R}$ with precisely two critical points. The two holomorphic strips contribute $d(x) = y - y = 0$.

The Floer complex

Corollary

- *When the homology $HF(L_0, L_1)$ is nonzero, then L_0 intersects any image of L_1 under any compactly supported Hamiltonian isotopy.*
- *Since the Morse homology always is non-zero, it follows that a closed exact Lagrangian is not Hamiltonian displaceable.*

- Instead of proving isomorphism with Morse homology, the next lecture we will mimic the proof of the fact that "Morse homology is nonvanishing" to give a condition for when the Floer homology is nonvanishing.

The Floer complex

Proof that d is well-def.

This follows from a version of Gromov's compactness theorem that we will formulate later:

- Since L_i are exact, the components of the moduli spaces $\mathcal{M}_{J_t}(x, y)$ which consist of solutions of $\mathrm{index} = 1$ become compact zero-dimensional manifold after taking quotients by automorphisms (translations).
- For compactness, the fact that the energy of solutions in $\mathcal{M}_{J_t}(x, y)$ are automatically bounded, is crucial. (Gromov's compactness needs an assumption of energy bound!)

□

The Floer complex

Proof that $d^2 = 0$.

This follows from a compactness argument together with a gluing argument, that we will postpone until next time.
Roughly:

- Two strips u, v can be glued to a new solution $u\sharp v$ if their asymptotics match;
- The Fredholm index is *additive* under this operation i.e. $\mathrm{index}(u\sharp v) = \mathrm{index}(u) + \mathrm{index}(v)$;
- After taking a quotient by reparam. we obtain a compact $1 + 1 - 1 = 1$-dimensional manifold; A compact one-dimensional manifold has an *even* number of boundary points!

□

The Floer complex

Proof of invariance (1/3).

Today we define the chain map

$$\Phi_{H,J_{s,t}} \colon CF(L_0, L_1; J_{-1,t}) \to CF(L_0, \phi_H^t(L_1); J_{1,t}).$$

The chain-map property, as well as the property of being invertible in homology, will be postponed until next time.
We assume that

- $J_{s,t}$ is constant inside $\{|s| \geq 1\}$;
- $\phi_H^s = \mathrm{Id}_X$ for $s \leq 0$, and $\phi_H^s = \phi_H^1$ for $s \geq 1$.

The Floer complex

Proof of invariance (2/3).

Consider a moduli space $\mathcal{M}_{J_{s,t}}(x, y)$ is defined similarly as before; It consists of smooth maps

$$u\colon (\{s + it; t \in [0, 1]\}, \{t = 0\}) \to (X, L_0)$$

which

- satisfy the boundary condition $u(s + i) \in \phi_H^{-s}(L_1)$
- satisfies the Cauchy–Riemann equation

$$du(\partial_t) = J_{-s,t} du(\partial_s)$$

- The asymptotic at $s = +\infty$ (resp. $s = -\infty$) is $x \in L_0 \cap L_1$ (resp. $y \in L_0 \cap \phi_H^1(L_1)$).

The Floer complex: The moduli space

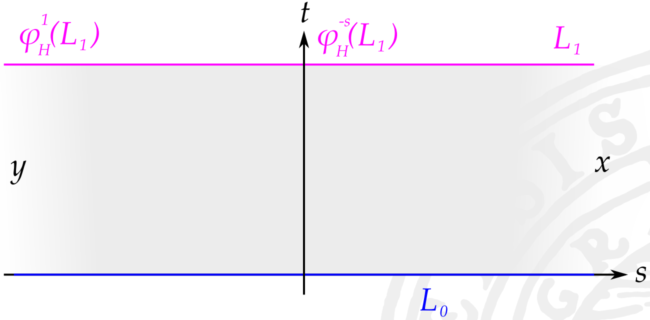


Figure: A strip used in the continuation map, the input is $x \in L_0 \cap L_1$ while the output is $y \in L_0 \cap \phi_H^1(L_1)$.

The Floer complex

Proof of invariance (3/3).

We finally define

$$\Phi_{H,J_{s,t}}(x) = \sum_{y \in L_0 \cap \phi_H^1(L_1)} \sum_{\substack{M \in \pi_0(\mathcal{M}_{J_{s,t}}(x,y)) \\ \mathrm{index} M = 0}} y$$

on any basis element $x \in L_0 \cap L_1$, where $y \in L_0 \cap \phi_H^1(L_1)$. Note that the components of the above moduli spaces that are counted all have expected dimension zero. (In the definition of the differential, the components had expected dimension one.) □

The Floer complex

Definition

The map

$$\Phi_{H,J_{s,t}} \colon CF(L_0, L_1; J_{-1}) \to CF(L_0, \phi_H^1(L_1); J_1)$$

between Floer complexes induced by the Hamiltonian isotopy ϕ_H^t and path of almost complex structures $J_{s,t}$ is called a *continuation map*.

Exercise

The continuation map induced by $H \equiv 0$ and $J_{s,t} \equiv J_t$ is the *identity map*.

Associahedra

For more operations in Floer homology, we need to introduce the configurations space of boundary punctures on D^2.

Recall:

- There is a unique simply connected Riemann surface with boundary by the *uniformisation theorem*: (D^2, j_0).
- The real Möbius transformations $\mathrm{Aut}(D^2)$ act transitively on triples of distinct cyclically ordered points in ∂D^2. (Any element in $\mathrm{Aut}(D^2)$ is determined uniquely by its image of such a triple.)

Associahedra

Set $p_0 = -1 \in \partial D^2$. The space of configurations of $d \geq 2$ additional distinct points

$$\iota\colon \{p_1, \ldots, p_d\} \hookrightarrow \partial D^2 \setminus \{p_0\},$$

called *boundary punctures*, which are required to

- respect the order on $(-\pi, \pi) = \partial D^2 \setminus \{p_0\}$, i.e.

$$\iota(p_1) < \ldots < \iota(p_d),$$

  and where

- we identify two such configurations that differ by an element in $\mathrm{Aut}(D^2)$ (which thus fixes p_0),

will be denoted by

$$\mathcal{R}_d = \mathrm{Emb}^{ord}(\{p_1, \ldots, p_d\}, \partial D^2 \setminus \{p_0\})/\sim$$

Associahedra

Since $\mathrm{Aut}(D^2)$ acts transitively on three cyclically ordered distinct points, one deduces that

$$\mathcal{R}_d \cong \mathbb{R}^{d-2}, \ \ d \geq 2.$$

We can e.g. pick the unique representatives which satisfy

$$p_1 \mapsto 1 \text{ and } p_2 \mapsto \sqrt{-1}.$$

BUT, there are of course many other choices: $\mathrm{Aut}(D^2)$ is a non-compact group.

- Non-compactness of the space is a result of the fact that, in a sequence $\{r_i \in \mathcal{R}_n\}_i$, two or more points can collide.

Associahedra

- Assume that $\{r_i\}$ is a sequence of representatives of elements in $\mathcal{R}_d$ which diverge.
- After acting by $\phi_i \in \mathrm{Aut}(D^2)$ with $\phi_i(-1) = -1$, we obtain a possibly different divergent sequence.
- For a suitable choice of sequence $\phi_i \circ r_i$ of representatives, we may assume that a subsequence converges to an element in $\mathcal{R}_{d'}$ for $2 \leq d' \leq d$. (Roughly speaking: the automorphisms ϕ_i can be used to separate the limiting clusters of points, making sure that *at least* three clusters form.)

Associahedra

There are many *different* choices of reparametrisations which can be used to extract a limit configuration. Here is one example:

Exercise

For any $j_0 \neq 0$ (resp. $j_0 = 0$), there is a sequence $\{\phi_i\}$, where $\phi_i(-1) = -1$, under which

- $\phi_i \circ r_i(p_{j_0}) = 1$ (resp. $\phi_i \circ r_i(p_{j_0}) = -1$),
- no sequence $\{\phi_i \circ r_i(p_j)\}$ for $j \neq j_0$ has 1 (resp. -1) as a limit point,
- there are at least three distinct limit points.

I.e. we can "zoom in" on the j_0:th boundary puncture in the limit, and extract an element in $\mathcal{R}_{d'+1}$ in which p_{j_0} is not forming a cluster of colliding points.

Associahedra

Theorem (Devadoss [Dev99])

For a suitable metric on $\mathcal{R}_d$ there is a natural compactification

$$\overline{\mathcal{R}}_d \cong K_d$$

by adding "nodal configurations", where K_d denotes the $d-2$-dimensional associahedron (a.k.a. Stasheff polyhedron). Moreover, the boundary faces of the polyhedron $K_d = \overline{\mathcal{R}}_d$ of dimension $\dim \overline{\mathcal{R}}_d - 1 = d - 3$ is given by the products

$$K_{d'} \times K_{d''} = \overline{\mathcal{R}}_{d'} \times \overline{\mathcal{R}}_{d''}$$

with $d' + d'' = d + 1$, where these products naturally correspond to nodal configurations.

Associahedra

The metric on K_d gives the same notion of convergence as in Gromov's compactness theorem:

- There exists a nodal disc in the "Gromov sense", whose every disc component has at least three boundary points which are either *nodes* or *boundary punctures*.
- All nodes and boundary punctures are distinct.
- There exists a sequence of diffeomorphisms ϕ_i of D^2 which identifies (D^2, j_0) with (D^2, Γ, j_i), and where (D^2, j_i) together and its boundary punctures converge in C^∞_{loc} to the nodal disc away from the curves Γ, and which respects the position of the boundary punctures.

The Floer complex: The moduli space

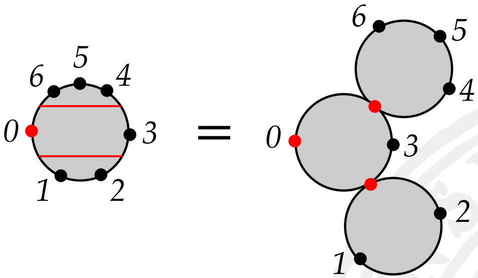


Figure: A nodal disc with boundary punctures which lives in $\mathcal{R}_3 \times \mathcal{R}_3 \times \mathcal{R}_2$. Note that each component has at least three boundary points which are either nodes or boundary punctures. In addition, nodes and boundary punctures are disjoint.

Associahedra

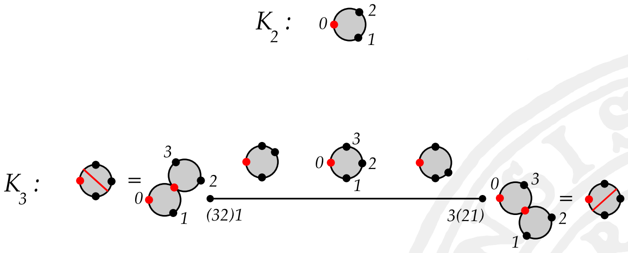


Figure: The assoceahedra $K_2 = \overline{\mathcal{R}}_2 = \{\star\}$ and $K_3 = \overline{\mathcal{R}}_3 = I$. The boundary vertices correspond to possible decompositions of the d-ary multiplication $d \cdot (d-1) \cdot \ldots \cdot 1$ into sequences of binary operations.

Associahedra

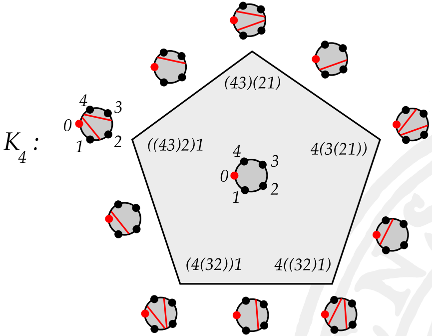


Figure: The associahedron $K_4 = \overline{\mathcal{R}}_4$ is the pentagon. The boundary vertices corresponds to possible decompositions of the 4-ary multiplication $4 \cdot 3 \cdot 2 \cdot 1$ into sequences of binary operations.

Associahedra

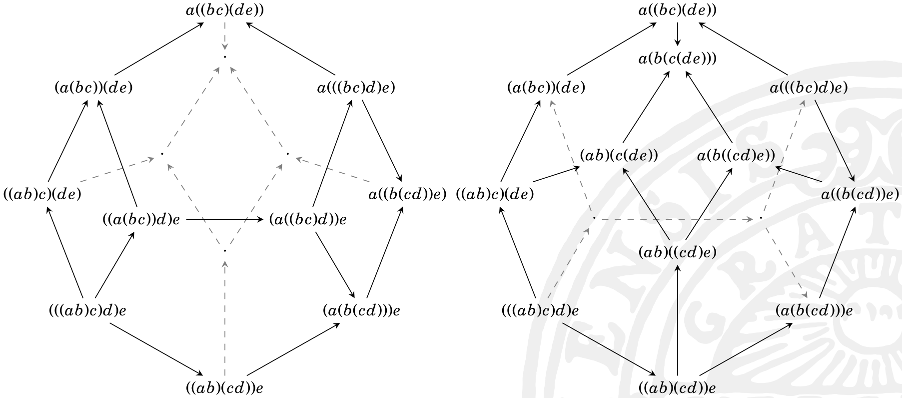


Figure: The space K_5. Source: Wikipedia

UPPSALA UNIVERSITET

References

Yu. V. Chekanov.
Lagrangian intersections, symplectic energy, and areas of holomorphic curves.
Duke Math. J., 95(1):213–226, 1998.

S. L. Devadoss.
Tessellations of moduli spaces and the mosaic operad.
In *Homotopy invariant algebraic structures (Baltimore, MD, 1998)*, volume 239 of *Contemp. Math.*, pages 91–114. Amer. Math. Soc., Providence, RI, 1999.

T. Ekholm, John Etnyre, and Michael Sullivan.
Legendrian contact homology in $P \times \mathbf{R}$.
Trans. Amer. Math. Soc., 359(7):3301–3335 (electronic), 2007.

A. Floer.
Morse theory for Lagrangian intersections.
J. Differential Geom., 28(3):513–547, 1988.

M. Gromov.
Pseudoholomorphic curves in symplectic manifolds.
Invent. Math., 82(2):307–347, 1985.