Holomorphic Curve Theories in Symplectic Geometry

Lecture IX

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Goal of lecture

Today:

- More about the (closed exact) Fukaya Category.
- The wrapped Fukaya category: geometric setting and construction.

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Plan

1. Goal of lecture
2. *A*-infinity morphisms
3. The Wrapped Fukaya Category
4. The skeleton of a Liouville domain and a vanishing condition for Wrapped Floer homology.
5. References

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Section 2

A-infinity morphisms

The Fukaya Category

Recall that the Fukaya category consists of:

- **Objects:** exact closed Lagrangians (equipped with additional data);
- **Morphisms:** elements in the Floer complexes

  $$Hom(L_0, L_1) = CF(L_0, L_1)$$

  equipped with a differential derived from μ_1.
- **Composition:** the product derived from μ_2.
- **Higher compositions:** $\{\mu_d\}_{d \geq 3}$

Where all of $\{\mu_d\}$ satisfy the A_∞-*relations*.

The Fukaya Category

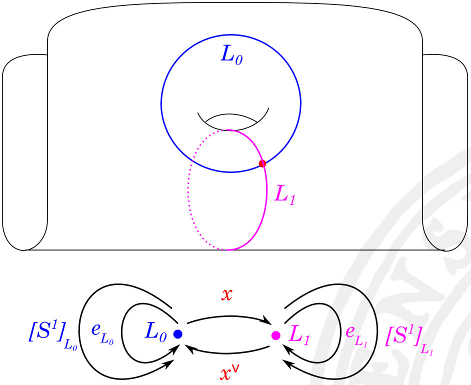


Figure: The generators of the endomorphisms and morphisms of two objects L_0 and L_1 in the Fukaya category. Here both L_i are exact embeddings of circles that intersect transversely in a unique point $x \in CF(L_0, L_1) = Hom(L_0, L_1)$.

The Fukaya Category

Note:

- *Very many* objects in general,

$$L,\ \phi_H^1(L),\ \phi_H^2(L),\ \phi_H^{2.7}(L),\ \phi_H^{2.71828}(L),\ \ldots$$

- Today and next lecture we will consider different tools for understanding thus huge category.
- In particular: It will be important to get a good notion of when objects (and categories) are *equivalent*.

The Fukaya Category

The *Donaldson category* is derived from the Fukaya category:

- **Objects:** exact closed Lagrangians (equipped with additional data);
- **Morphisms:** elements in the homology of the Floer complexes
$$Hom(L_0, L_1) = HF(L_0, L_1).$$
- **Composition:** homology level product induced by μ_2.

Since the product is associative on the homology level, the Donaldson category is a category in the ordinary sense!

Isomorphism in the Donaldson category

Proposition

Two Lagrangians L which are Hamiltonian isotopic are isomorphic objects in the Donaldson category.

Proof (1/4).

We just need the proof from [Lecture 7] of the invariance of the Floer complexes by multiplication with continuation elements.

- Let L and $\phi_H^1(L)$ be two objects in the Donaldson category induced by a Hamiltonian isotopy ϕ_H^t applied to a Lagrangian L;
- Consider the inverse Hamiltonian isotopy ϕ_G^t defined by

$$\phi_G^t(\phi_H^1(L)) = \phi_H^{1-t}(L).$$

Isomorphism in the Donaldson category

Proposition

Two Lagrangians L which are Hamiltonian isotopic are isomorphic objects in the Donaldson category.

Proof (2/4).

In [Lecture 8] invariance was established by showing that the continuation maps

$$CF(L,L) \xrightarrow{\Phi_H = l_{c_{L,H}}} CF(L,\phi^1_H(L)) \xrightarrow{\Phi_G = l_{c_{\phi^1_H(L),G}}} CF(L,L)$$

given by left multiplication with the continuation elements were each others inverses in homology.

Isomorphism in the Donaldson category

Proposition

Two Lagrangians L which are Hamiltonian isotopic are isomorphic objects in the Donaldson category.

Proof (3/4).

In other words:

$$[c_{\phi^1_H(L),G}]\cdot[c_{L,H}]=[e_L]$$

is the unit in homology.

$$\begin{tikzcd}
\varphi^1_G(\varphi^1_H(L))=L \bullet \arrow[r, bend left, "c_{L,H}"] & \bullet\ \varphi^1_H(L) \arrow[l, bend left, "c_{\varphi^1_H(L),G}"]
\end{tikzcd}$$

Isomorphism in the Donaldson category

Proposition

Two Lagrangians L which are Hamiltonian isotopic are isomorphic objects in the Donaldson category.

Proof (4/4).

The equality

$$[c_{L,H}] \cdot [c_{\phi^1_H(L),G}] = [e_{\phi^1_H(L)}]$$

is proven verbatim.

In conclusion: we have shown that the morphism $[c_{L,H}] \in CF(L, \phi^1_H(L))$ is invertible with inverse

$$[c_{L,H}]^{-1} = [c_{\phi^1_H(L),G}] \in CF(\phi^1_H(L), L).$$

□

A_∞-morphisms

The natural relation between A_∞-algebras (e.g. $CF(L, L)$) is that of A_∞*-morphisms.*

Definition ((1/2))

If $(A, \{\mu_d^A\})$, $(B, \{\mu_d^B\})$ are two A_∞-algebras, then a A_∞*-morphism* is a sequence of maps $\{f_d\}_{d=1,2,3\ldots}$

$$f_d \colon A^{\otimes d} \to B$$

of degree $d-1$ that satisfy the following infinite sequence of equations...

A_∞-morphisms

The natural relation between A_∞-algebras (e.g. $CF(L, L)$) is that of A_∞*-morphisms.*

Definition ((Continued... (2/2))

For each fixed $d = 1, 2, 3, \ldots$ we have

$$\sum_{i_1+\ldots+i_j=d} \mu_j^B(f_{i_j}(x_d, \ldots, x_{d-i_j+1}), \ldots, f_{i_1}(x_{i_1}, \ldots, x_1)) =$$

$$\sum_{s\leq d} (-1)^{\maltese_t} f_{d-s+1}(x_d, \ldots, x_{t+s+1}, \mu_s^A(x_{t+s}, \ldots, x_{t+1}), x_t, \ldots, x_1)$$

where the sign is

$$\maltese_t = \sum_{i=1}^{t} |x_i|.$$

A_∞-morphisms

- **For** $d = 1$**:** the relation simply says that f_1 is a chain map:

$$\mu_1^B \circ f_1 = f_1 \circ \mu_1^A$$

- **For** $d = 2$**:** the relation *almost* says that f preserves the product:

$$\mu_2^B(f_1(x_2), f_1(x_1)) = f_1 \circ \mu_2^A(x_2, x_1) \pm f_2(\mu_1^A(x_1), x_2) \pm f_2(x_2, \mu_1^A(x_1))$$

(in particular f_1 is a map of homology algebras).

An important feature of A_∞-morphisms

Unlike DGA quasi-isomorphisms (DGA morphisms that induce isomorphism on the level of homology), A_∞ quasi-isomorphisms admit inverses in the following sense:

Theorem (Prouté; see [Pro11])

For an A_∞-morphism

$$\{f_d\}\colon (A, \{\mu_d^A\}) \to (B, \{\mu_d^B\})$$

for which $[f_1]\colon H(A, \mu_1^A) \to H(B, \mu_1^B)$ is an isomorphism of vector spaces, there exists an A_∞-morphism

$$\{g_d\}\colon (B, \{\mu_d^B\}) \to (A, \{\mu_d^A\})$$

which is an inverse in homology, i.e. $[g_1] = [f_1]^{-1}$.

A_∞-morphisms

Remark

The previous theorem exhibits an important feature of A_∞-algebras which is <u>not true</u> for morphisms of differential graded algebras (DGAs). (Reason: it is difficult to construct a map <u>to</u> a free DGA from a non-free DGA)

We will see: The analogous property holds also for A_∞-modules (more about them the next lecture); In other words, there is no need to add formal inverses to the quasi-isomorphisms when considering the derived category of A_∞-modules over an A_∞-algebra.

A_∞-quasi-iso. from continuation

We just saw that the continuation elements induced by a Hamiltonian isotopy are isomorphisms in the Donaldson category (the homology of the Fukaya category). We will now extend these morphisms to A_∞-quasi isomorphisms of the Fukaya category.

A_∞-quasi-iso. from continuation

- For L' is a small Hamiltonian pert. of L there is an A_∞-algebra
$$CF(L,L) := CF(L,L').$$
- For a ϕ^1_H and its inverse ϕ^1_G i.e. $\phi^t_G(\phi^1_H(L)) = \phi^{1-t}_H(L)$ and the associated continuation elements
$$c_{L',H} \in CF(L',\phi^1_H(L'))$$
$$c_{\phi^1_H(L),G} \in CF(\phi^1_H(L),\phi^1_G(\phi^1_H(L))) = CF(\phi^1_H(L),L)$$

Theorem

The "two-sided" continuation map
$$f_1^{H,G} := r_{c_{\phi^1_H(L),G}} \circ l_{c_{L',H}} \colon CF(L,L') \to CF(\phi^1_H(L),\phi^1_H(L'))$$
extends to an A_∞ quasi-isomorphism $\{f_d\}$.

Construction of A_∞-q.is.

Construction of A_∞-morphism: f_1.

- The map f_1 counts two-sided continuation strips (two punctured discs) of index zero.
- The chain-map property of f_1 and invertibility on homology level follows similarly to the proof of invariance of the Floer complex.
- Similarly, the component f_d will be defined by counting solutions in $\mathcal{R}_d$ of $d+1$-punctured discs for a *suitable* boundary condition which we defined inductively. (In any case, one must make sure inputs and outputs live in the apprioriate Floer complexes.)

Construction of A_∞-q.is.

Construction of A_∞-morphism: f_d, $d \geq 2$.

The boundary condition is defined inductively on $\mathcal{R}_d$ as follows. Assume that we have defined the boundary condition for all $d' \leq d$.

- Glue the discs that correspond to every term in the equation for the A_∞-morphism (one disc that defines μ_j, and possibly several that define f_i's). This defines the boundary condition near $\partial\mathcal{R}_d$.
- Choose an extension of this boundary condition over all of $\mathcal{R}_d$.
- The component f_d of the A_∞-morphism is defined by counting index zero solutions of this moduli space. (They constitute a finite number of solutions in the interior of $\mathcal{R}_d$.)

$\square$

A_∞-morphisms

In fact, the following useful lemma shows that algebra alone can be used to derive the above quasi-isomorphism of endomorphism A_∞-algebras in an A_∞-category:

Lemma

If multiplication (i.e. composition) from the left and right with a μ_1-closed element (i.e. morphism) $x \in Hom(L_0, L_1)$ induces quasi-isomorphisms of complexes

$$l_x \colon Hom(L_0, L_0) \xrightarrow{\simeq} Hom(L_0, L_1) \text{ and } r_x \colon Hom(L_1, L_1) \xrightarrow{\simeq} Hom(L_0, L_1)$$

then there is an A_∞ quasi-isomorphism $Hom(L_0, L_0) \to Hom(L_1, L_1)$.

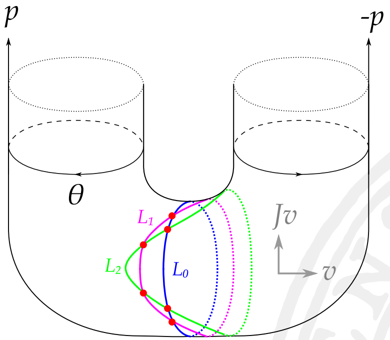
Case of closed exact Lagrangians

Theorem (Abouzaid [Abo11])

The A_∞-algebra $CF(L, L)$ for a closed exact Lagrangian L with $\mathbb{F}$-coefficients is quasi-isomorphic (as an A_∞-algebra) to the unital differential graded algebra $C^*(L, \mathbb{F})$ of singular cochains (this is an A_∞-algebra with $\mu_d = 0$ for all $d \geq 3$).

A sketch of proof will be given next lecture (based upon a computation of the wrapped Fukaya category together with classical algebraic topology of the based loop space).

A_∞-structure of the zero-section.



Exercise: L_1 and L_2 are Hamiltonian perturbations of the zero-section $L_0 \subset (T^*S^1, d(p\,d\theta))$. Compute the Floer complex $CF(L_0, L_1)$, all products $CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$, and then identify the continuation element $c_{L_1,H} \in CF(L_1, L_2)$.

Generation of the Fukaya category

The problem of understanding the Fuakaya category needs more input.

- The reason is that, almost in no case do we understand all objects (Lagrangians), even up to the equivalence of Hamiltonian isotopy.
- Next lecture we will see that, in some favourable situations, a finite set of Lagrangians "generate" the entire category.

Section 3

The Wrapped Fukaya Category

The Wrapped Fukaya Category

- The **wrapped Fukaya category** $\mathcal{W}(X, \eta)$ is a generalisation of the exact Fukaya category to also include *Lagrangians with Legendrian boundary* inside a Liouville domain $(X, d\eta)$ that are cylindrical near ∂X.
- This turns out to be a very rich and powerful invariant of the Liouville domain (with the choice of primitive of symplectic form).
- In Tobias' lecture, you will also learn that it sometimes can be computed.

First we need to define the necessary geometric background.

Wrapped Fukaya Category: geometric setup

Definition

A *Liouville domain* is compact exact symplectic manifold $(X, d\eta)$ with smooth boundary, for which the Liouville vector field ζ defined by $\iota_\zeta d\eta = \eta$ is <u>outwards</u> transverse to the boundary.

Remark

- By Stokes' theorem ζ cannot point inwards along the entire bdy.
- The boundary $(Y, \alpha) := (\partial X, \eta|_{T\partial X})$ is a contact manifold; the negative Liouville flow gives a collar neighbourhood of ∂X of the form of a subset of the symplectisation

$$((-\epsilon, 0] \times \partial X, d(e^t\alpha)),$$

  where ζ and η are identified with ∂_t and $e^t\alpha$, resp.

Wrapped Fukaya Category: geometric setup

- In order to make the maximum principle hold, we will use a tame almost complex structure on $(X, d\eta)$ which is *cylindrical* on the above collar
$$((-\epsilon, 0] \times \partial X, d(e^t\alpha))$$
as defined in [Lecture 6]. (The collar is a subset of the symplectisation of (Y, α).)
- One can also complete the compact domain $(X, d\eta)$ by attaching a semi-infinite cylindrical end
$$((0, +\infty) \times \partial X, d(e^t\alpha));$$
using the fact that a cylindrical almost compelex structure is invariant under translation of the t-coordinate gives an extension to the entire end.

Wrapped Fukaya Category: geometric setup

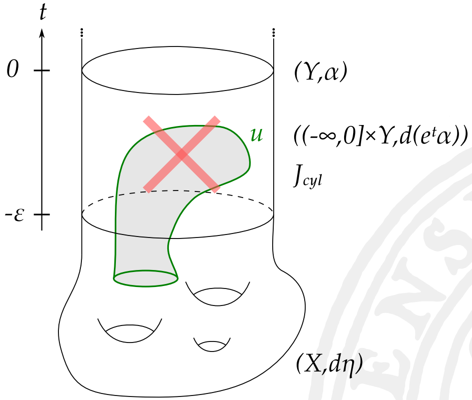


Figure: A Liouville domain with a cylindrical almost complex structure J_{cyl} in the collar of the boundary satisfies the maximum principle for pseudoholomorphic curves; see [Lecture 6].

Wrapped Fukaya Category: geometric setup

The objects of the Fukaya category $\mathcal{W}(X, \eta)$ are Lagrangians $L \subset (X, d\eta)$ which are

- Exact: the restriction $\eta|_{TL} = df$ is exact for some smooth function $f \colon L \to \mathbb{R}$;
- Cylindrical near ∂X: inside some collar neighbourhood, L is of the form
$$(-\epsilon, 0] \times \Lambda \subset ((-\epsilon, 0] \times \partial X, d(e^t\alpha))$$
The Lagrangian condition together with tangency to ∂_t gives that $\alpha|_{T\Lambda} \equiv 0$, i.e.
    - f is locally constant near ∂X;
    - the boundary $\Lambda = \partial L$ is a *Legendrian* submanifold of the contact manifold (Y, α).

(We also allow the case $\Lambda = \emptyset$, i.e. L closed and exact.)

Wrapped Fukaya Category: geometric setup

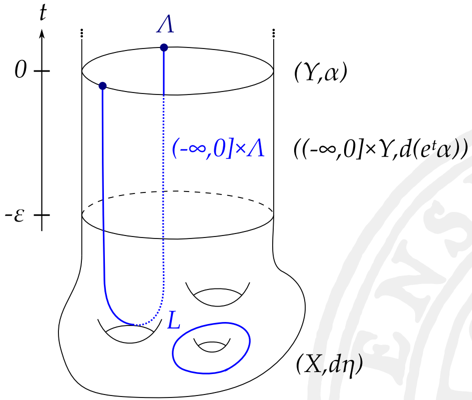


Figure: A Liouville domain with an exact Lagrangian with Legendrian boundary.

Wrapped Fukaya Category: geometric setup

In fact, the following crucial property holds:

Lemma (Lemma 7.2 in [AS10])

The maximum principle also holds for discs with boundary on Lagrangians with Legendrian boundary when J is cylindrical near the contact boundary $\partial X \subset X$.

Remark

As we will see, compactness is much more subtle. Even the moduli space of constant discs is not compact!

Wrapped Fukaya Category: geometric setup

Example

The main example of Lagrangians that satisfy the condition are the *cotangent fibres* $T^*_{\mathrm{pt}}M$ in the cotangent bundle $(T^*M, d\theta_M)$ of a closed smooth manifold; in fact θ_M vanishes on $T^*_{\mathrm{pt}}M$. (Strictly speaking, we must take intersection with to some fibre-wise convex subdomain $X \subset T^*M$ to get a compact Liouville domain.)
More generally: conormal lifts of closed submanifolds of M are exact Lagrangians with Legendrian boundary (the fibre $T^*_{\mathrm{pt}}M$ is the conormal lift of the point $\mathrm{pt} \in M$).

Wrapped Fukaya Category: geometric setup

Remark

The Floer complex of a Lagrangian $L \subset (X, d\eta)$ with Legendrian boundary is *not* invariant under non-compactly supported Hamiltonian isotopies;

- This is still the case, even if we require that the Hamiltonian isotopy preserves the cylindrical condition outside of a compact subset. (I.e. one which induces a *Legendrian isotopy* of its Legendrian boundary $\partial L \subset \partial X$.)

Wrapped Fukaya Category: geometric setup

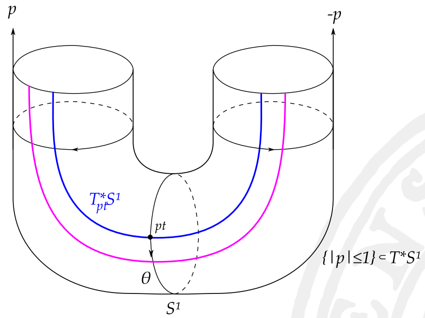


Figure: The cotangent fibre $T^*_{pt}S^1 \subset (T^*S^1, d(p\,d\theta))$ and a Hamiltonian isotopic copy which is disjoint.

Wrapped Fukaya Category: geometric setup

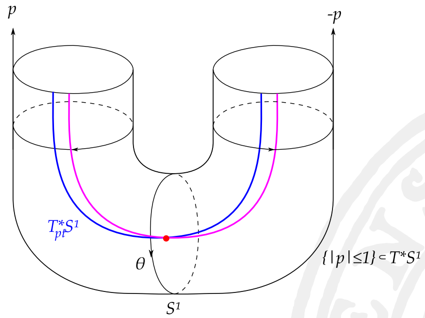


Figure: The cotangent fibre $T^*_{pt}S^1 \subset (T^*S^1, d(p\,d\theta))$ and a Hamiltonian isotopic copy which intersects the fibre transversely in a single point.

Wrapped Fukaya Category: geometric setup

We solve the problem of the invariance of $CF(L_0, L_1)$ by *wrapping* the Legendrian boundary of L_0 by the *positive Reeb flow*.
More precisely, we consider the Hamiltonian isotopy

$$\phi^1_{H^{wr}_\lambda} \colon (X, d\eta) \to (X, d\eta)$$

where the Hamiltonian is supported near the collar of the contact boundary $\partial X \subset X$, where it can be expressed as follows in terms of the coordinate t on the collar induced by the primitive η:

$$H^{wr}_\lambda(t) = \lambda \rho_\lambda(e^t) - C_\lambda, \ \ \lambda > 0.$$

Wrapped Fukaya Category: geometric setup

The wrapping Hamiltonian:

$$H_\lambda^{wr}(t) = \lambda\rho_\lambda(e^t) - C_\lambda, \ \ \lambda > 0.$$

Here $C_\lambda \in \mathbb{R}$ and ρ_λ is a bump function chosen so that:

- $\frac{d}{dt}H_\lambda^{wr}(t) = \lambda e^t$ near $t = 0$;
- $\frac{d^2}{dt^2}H_\lambda^{wr}(t) \geq 0$ (convexity); and
- $H_\lambda^{wr}(t) \geq 0$ and its support is contained inside the collar.

Wrapped Fukaya Category: geometric setup

The *Reeb vector field* of (Y, α) is the vector field $R_\alpha \in \Gamma(TY)$ defined by

$$d\alpha(R_\alpha, \cdot) = 0 \text{ and } \alpha(R_\alpha) = 1.$$

The Hamiltonian vector field can thus be expressed as:

$$X_{H_\lambda^{wr}} = e^{-t} \frac{d}{dt} H_\lambda^{wr}(t) \cdot R_\alpha$$

where

$$\frac{d}{dt} H_\lambda^{wr}(t) = \lambda e^t$$

near the contact boundary $\{t = 0\} = \partial X$, and hence

$$X_{H_\lambda^{wr}} = \lambda R_\alpha$$

is a constant multiple of the Reeb vector field there.

Wrapped Fukaya Category: geometric setup

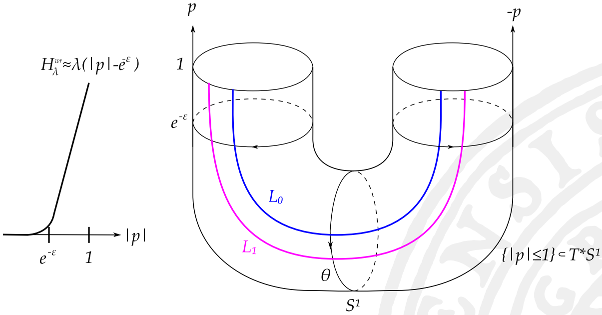


Figure: The cotangent fibre $T^*_{pt}S^1 \subset (T^*S^1, d(p\,d\theta))$ and a Hamiltonian isotopic copy which is disjoint.

Wrapped Fukaya Category: geometric setup

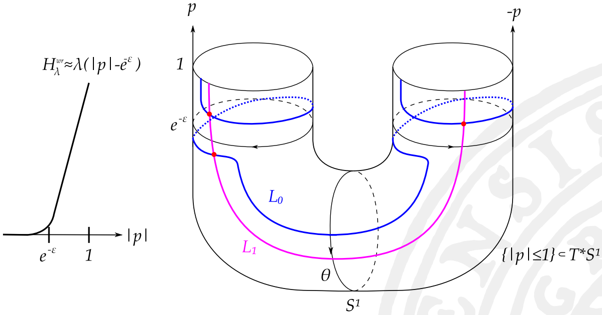


Figure: The cotangent fibre $T^*_{pt}S^1 \subset (T^*S^1, d(p\,d\theta))$ and a Hamiltonian isotopic copy which are disjoint.

Wrapped Fukaya Category: directed system

The morphism space in the *Wrapped Fukaya Category* is given by the *Wrapped Floer complex*, i.e.

$$Hom(L_0, L_1) = CW(L_0, L_1) = \lim_{\lambda \to +\infty} CF(\phi^1_{H^{wr}_\lambda}(L_0), L_1)$$

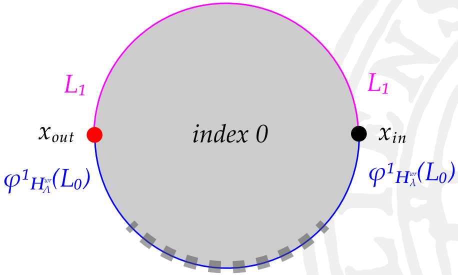
where the directed system

$$CF(\phi^1_{H^{wr}_\lambda}(L_0), L_1) \to CF(\phi^1_{H^{wr}_\Lambda}(L_0), L_1)$$

for $\Lambda > \lambda$ is given by *continuation*.

Wrapped Fukaya Category: directed system

$$CF(\phi^1_{H^{wr}_\lambda}(L_0), L_1) \to CF(\phi^1_{H^{wr}_\Lambda}(L_0), L_1),$$
$$\Lambda > \lambda > 0,$$



Wrapped Fukaya Category: directed system

For a suitable system of wrapping Hamiltonians H^{wr}_λ, the above directed system is actually a *sequence of canonical inclusions*

$$CF(\phi^1_{H^{wr}_\lambda}(L_0), L_1) \hookrightarrow CF(\phi^1_{H^{wr}_\Lambda}(L_0), L_1), \ \ \Lambda > \lambda > 0.$$

One needs the following property:

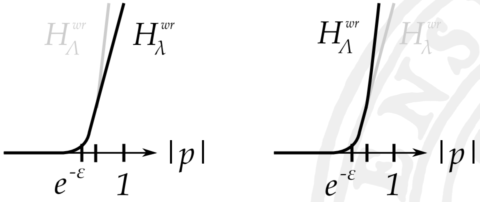


Figure: The Hamiltonian H^{wr}_Λ coincides with H^{wr}_λ except in some small neighbourhood of the boundary. It satisfies $H^{wr}_\Lambda \geq H^{wr}_\lambda$ everywhere, while its slope near the boundary $\Lambda > \lambda$ is strictly greater.

Wrapped Fukaya Category: perturbations

Analytic subtlety:

- In the case of closed exact Lagrangians we had freedom to choose arbitrary Hamiltonian deformations when defining moving boundary conditions.
- When the Lagrangian has a non-compact cylindrical ends, this is still true in the compact part;
- However, in the cylindrical ends we must use the *wrapping Hamiltonian* $\phi^{\rho(\theta)}_{H^{wr}_\lambda}$;

Wrapped Fukaya Category: perturbations

More importantly: For compactness, when traversing the boundary of the disc in the *positive* direction, the total wrapping near ∂X needs to *non-positive*.

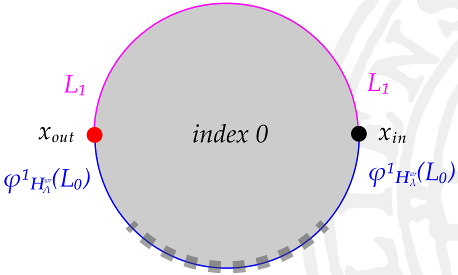
- Continuation strips from

$$CF(\phi^1_{H^{wr}_\lambda}(L_0), L_1) \to CF(\phi^1_{H^{wr}_\Lambda}(L_0), L_1)$$

  are admissible if and only if $\Lambda \geq \lambda > 0$.
- Continuation strips going the other way are not admissible: wrapping can create intersection points and is thus typically *not* an isomorphism in homology.
- (In the closed case, i.e. when L has empty boundary, one can of course still choose arbitrary Hamiltonians.)

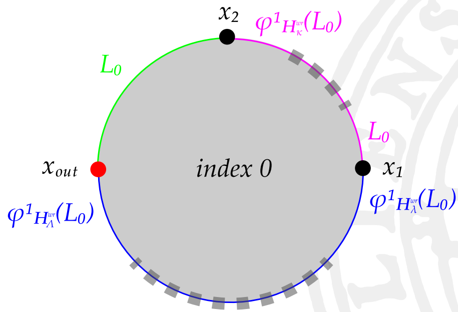
Admissible continuation strip

$$CF(\phi^1_{H^{wr}_\lambda}(L_0), L_1) \to CF(\phi^1_{H^{wr}_\Lambda}(L_0), L_1),$$
$$\Lambda \geq \lambda > 0,$$



Admissible three-punctured disc

$$CF(\phi^1_{H^{wr}_\kappa}(L_0), L_0) \otimes CF(\phi^1_{H^{wr}_\lambda}(L_0), L_0) \to CF(\phi^1_{H^{wr}_\Lambda}(L_0), L_0),$$
$$\Lambda, lambda, \kappa > 0, \;\; \Lambda \geq \lambda + \kappa.$$



Inadmissible discs

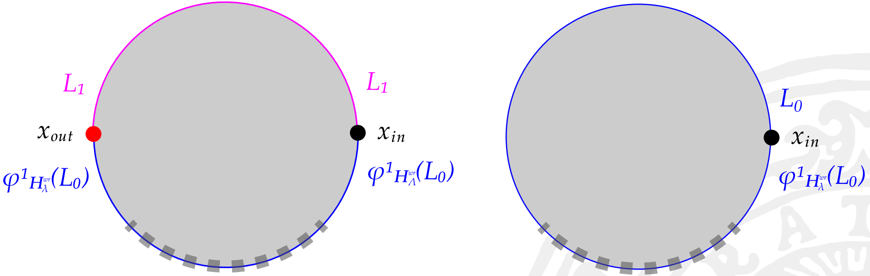


Figure: Here $\Lambda > \lambda > 0$. Compactness for the type of disc shown on the left would allow us to show that wrapping is an isomorphism in homology. The compactness of the discs to the right would show that wrapped Floer has a counit, and thus does not vanish.

Wrapped Fukaya Category: perturbations

- We refer to [AS10] for a set-up which solves the problem of defining the A_∞-endomorphism;
- A more modern approach due to Seidel performs a localisation of an A_∞-precategory under continuation morphisms [GPS20];

Remark

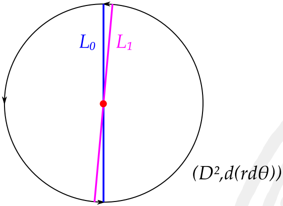
In the literature the moving boundary condition is often replaced by a Hamiltonian perturbation term in the Cauchy–Riemann equation; these two perspectives are in fact equivalent, as they differ by a change of coordinates.

Non co-unitality and vanishing

Remark

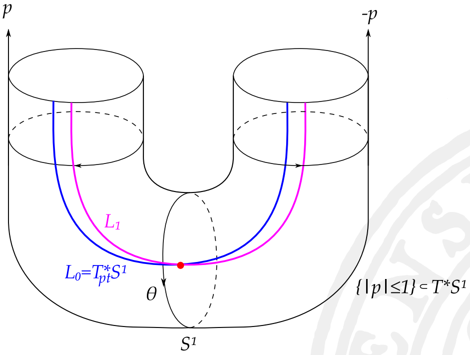
The wrapped Floer complex complex of an exact Lagrangian is always, unital but it is co-unital only when the Lagrangian is closed. Indeed, as we will see, the Floer complex $CW(L, L)$ is sometimes acyclic when L is exact with a non-empty Legendrian boundary.

A Lagrangian in the disc.



Exercise: Find the complex $CW(L_0, L_1)$ for $L_0 = \{0\} \times [0,1] \subset D^2$ and its Hamiltonian perturbation L_1.

A cotangent fibre



Exercise: Find the complex $CW(L_0, L_1)$ (Next time we will see: There is a quasi-isomorphism of this A_∞-algebra $CW(L_0, L_0)$ and the Laurent polynomial ring $\mathbb{F}[a, a^{-1}]$, equipped with a vanishing differential.)

The Skeleton of a Liouville domain

Recall: By definition the Liouville vector field $\zeta \in TX$ which is the symplectic of the primitive η of the symplectic form on $(X, d\eta)$ is outwards transverse along the entire boundary ∂X.
In particular,

$$\mathrm{Skel}(X, \eta) := \bigcap_{N=1}^{\infty} \phi_{\zeta}^{-N}(X)$$

is a closed subset.
Using the Liouville flow one can construct:

Lemma

There is a symplectomorphism to the half symplectisation

$$(X \setminus \mathrm{Skel}(X, \eta), d\eta) \cong ((-\infty, 0] \times \partial X, d(e^t \alpha)), \;\; \alpha = \eta|_{T\partial X}$$

that preserves the Liouville vector fields.

Section 4

The skeleton of a Liouville domain and a vanishing condition for Wrapped Floer homology.

The Skeleton of a Liouville domain

Example

1. The skeleton of $(D^{2n}, -dd^c\|\mathbf{z}\|^2/4)$ is the origin;
2. The skeleton of $(T^*M, d\theta_M)$ is the zero section;
3. McDuff has constructed examples where the skeleton is a hypersurface [McD91].
4. The skeleton cannot have full measure, since the backwards Liouville flow shrinks the volume:

$$\phi_\zeta^t d\eta = e^t d\eta.$$

Examples of skeleta

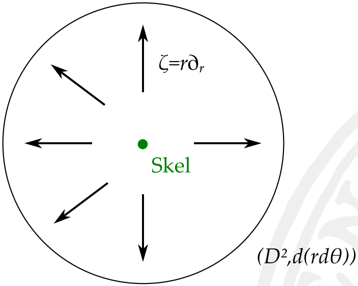


Figure: Skeleton of $(D^2, d(r^2/2d\theta))$.

Examples of skeleta

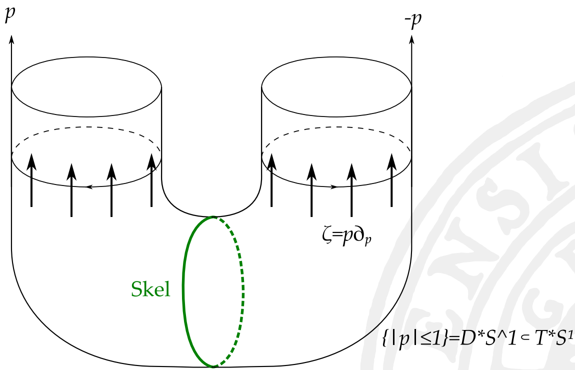


Figure: Skeleton of $(D^*S^1, d(pd\theta))$.

A vanishing condition

Theorem ([CRGG19])

If $L \subset (X \setminus \mathrm{Skel}(X, \eta), d\eta)$ is an exact Lagrangian with (possibly empty) Legendrian boundary, then $CW(L, L) = 0$.

Corollary

If $\mathrm{Skel}(X, \eta) \subset X^{2n}$ is of dimension at most $n - 1$ (strictly less than half!), then any $L \subset X$ satisfies $CW(L, L) = 0$.

Proof of Corollary.

Since

$$\dim L + \dim \mathrm{Skel}(X, \eta) < 2n = \dim X,$$

a small generic Hamiltonian perturbation disjoins L from the skeleton. $\square$

A vanishing condition

Theorem ([CRGG19])

If $L \subset (X \setminus \mathrm{Skel}(X, \eta), d\eta)$ is an exact Lagrangian with (possibly empty) Legendrian boundary, then $CW(L, L) = 0$.

Idea of proof.

Consider a continuation map for a "vertical displacement""

$$\Phi\colon CF(\phi^1_{H^{wr}_\lambda}(L), L') \to CF(\phi^1_{H^{wr}_\lambda}(L), \phi^N_\zeta(L'))$$

where ϕ^t_ζ is the Liouville flow. (This is not a Hamiltonian flow, but it preserves exact Lagrangians!)
Since L is disjoint from the skeleton, its perturbation L' can be pushed out of X by the Liouville flow ϕ_ζ. An action argument shows that, if $\phi^N_\zeta(L') \subset \overline{X} \setminus X$, then e_L is in the kernel of Φ for $\lambda \gg 0$. $\square$

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