Holomorphic Curve Theories in Symplectic Geometry

Lecture X

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Goal of lecture

Today:

- Weinstein manifolds and cocores
- Generalities about A_∞-categories:
    - Twisted complexes.
    - Generation.
- Generation by cocores for Weinstein manifolds.

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Section 2

Weinstein domains

Weinstein domains

Weinstein domains are classes of Liouville domains with a particularly well-behaved skeleton.

- The class of Weinstein domains is very rich, and their wrapped Fukaya categories realise many interesting algebraic structures;
- We know almost nothing about Liouville domains that do not admit Weinstein structures, but we have no reason to believe that they are rare.

Weinstein domains

Definition

A Weinstein domain is a triple (W, η, f) where

- $(W, d\eta)$ is a Liouville domain (in particular the Liouville vector field ζ is outwards transverse to ∂W);
- $f\colon W \to \mathbb{R}$ is a Morse function which is a pseudo-gradient for the Liouville vector-field ζ of η.

Since the Liouville flow expands the symplectic form while it contracts the stable manifolds of the critical points of ζ we get:

Lemma

The smooth part of $\operatorname{Skel}(W, \eta)$ *is isotropic, i.e.* $d\eta|_{T\operatorname{Skel}} \equiv 0$. *In particular, the maximum dimension of the cells in the skeleton is equal to* $n = \dim W/2$ *(these cells are Lagrangian).*

Weinstein domains

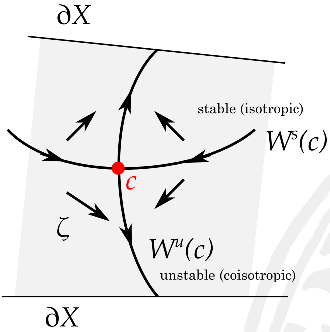

Figure: The stable manifold is isotropic (in particular it is at most half-dimensional) while the unstable manifold is coisotropic (in particular at least half-dimensional)

Weinstein domains

Example

A compact Stein-domain (X, J, ρ) with smooth boundary, i.e.

- J integrable;
- $i\partial\overline{\partial}\rho$ is symplectic;
- ∂X is a regular level-set of ρ;

gives rise to the Weinstein structure

$$(X, -d^c\rho/2, \rho)$$

in the case when ρ is a Morse function. (The Morse property can be assumed after a generic C^∞-small perturbation of ρ; the symplectic condition is stable under such perturbations.)

Weinstein domains

Example

- $(D^{2n}, -d^c\rho_0/2, \rho_0)$ where $\rho_0 = \|\mathbf{z}\|^2/2$ and
$$\eta_0 = -d^c\rho_0/2 = \frac{1}{2}\sum_i (x_i dy_i - y_i dx_i)$$
is Weinstein (as we saw: the skeleton is the origin).
- For any two Liouville (resp. Weinstein) domains X_1 and X_2, the product $(X_1 \times X_2, d\eta_1 \oplus d\eta_2)$, which has boundary with corners, can be smoothed to form a Liouville (resp. Weinstein) domain.

Subcritical Weinstein domains

Example

- In particular, the skeleton of $W = D^{2n_1} \times V^{2n_2}$ is of dim. at most $0 + n_2 < \dim W/2$ when V is Weinstein.
- More generally, a Weinstein domain W for which all critical points are of index $< \dim W/2$ is called *subcritical*.
- The wrapped Fukaya category of a subcritical Weinstein domain is quasi-equivalent to zero; All wrapped Floer complexes vanish by the vanishing criterion given at the end of [Lecture 9] (the skeleton is less than half-dimensional).

Weinstein structure on T^*M

Example

The standard Liouville form on the cotangent bundle θ_M has a Liouville vector field which is critical along the entire zero-section. A perturbation can be seen to be Weinstein: Take a Morse function $g \colon M \to \mathbb{R}$ which is a pseudo-gradient for a vector field $V \in \Gamma(TM)$ that generates the positive "gradient flow" $\psi^t \colon M \to M$. The domain

$$(DT^*M, \theta_M + d(p_i dq^i(V)), p^2/2 + f)$$

is Weinstein. The new Liouville vector-field is the Morsification $p\partial_p + V$ of the degenerate Liouville vector field $p\partial_p$ of θ_M, where the latter has a critical manifold equal to the zero section (the original Liouville vector field is non-degenerate in the Bott sense).

When is a Liouville domain Weinstein?

A generic Liouville structure is not Weinstein. *But*, there is a natural notion of equivalence of Liouville domains:

Definition

Two compact Liouville domains $(X, d\eta_0)$ and $(X, d\eta_1)$ are equivalent if there is a path of exact symplectic forms $d\eta_t$ that connect $d\eta_0$ and $d\eta_1$, such that $(X, d\eta_t)$, $t \in [0, 1]$, all are compact Liouville domains.

Except in dimension two, we know almost nothing about the question regarding which Liouville domains are equivalent to a Weinstein domain.

Weinstein structure on surfaces

Proposition

Any two-dimensional Liouville-domain $(X, d\eta)$ admits a Weinstein structure $(X, \eta + dh, \rho)$ for a suitable exact deformation $\eta + dh$ of the Liouville form.

In particular: $(X, d\eta)$ is equivalent to a Weinstein domain.

Proof (1/5).

Take a compatible integrable complex structure on X. (This we can do because of the assumption that $\dim X = 2$. In general we do not know if this can be done.) We may assume that it is cylindrical in the collar $(-\epsilon, 0] \times Y$ of ∂X, and hence we can write $\eta = -d^c e^t$ there (t is the coordinate on the collar).

Weinstein structure on surfaces

Proposition

Any two-dimensional Liouville-domain $(X, d\eta)$ admits a Weinstein structure $(X, \eta + dh, \rho)$ for a suitable exact deformation $\eta + dh$ of the Liouville form.

Proof (2/5).

We can inflate the Liouville domain in the collar by replacing η with $\eta_C = -d^c e^{\sigma_C(t)}$ where $\sigma_C''(t) \geq 0$, $\sigma_C(t) = t$ near $t = -\epsilon$, and $\sigma_C(0) = C \geq 0$.
By using the Liouville flow, can readily construct a diffeomorphism of X that pulls back η_C to $e^C\eta$. In other words, it suffices to construct a Weinstein structure $(X, \eta_C + dh, \rho_C)$.

Weinstein structure on surfaces

Proposition

Any two-dimensional Liouville-domain $(X, d\eta)$ admits a Weinstein structure $(X, \eta + dh, \rho)$ for a suitable exact deformation $\eta + dh$ of the Liouville form.

Proof (3/5).

The symplectic form $d\eta$ can be written as $d\eta = i\partial\overline{\partial}\rho$ by a standard argument:
We can write $\eta = \alpha^{0,1} + \overline{\alpha^{0,1}}$ since this is a real one-form. By Cartan's Theorem B (X is an open Riemann surface) we have $\alpha^{0,1} = \overline{\partial} f$ for some $f \colon X \to \mathbb{C}$. From this we compute

$$i\partial\overline{\partial}(-i)(f - \overline{f}) = \partial\overline{\partial} f - \partial\overline{\partial}\overline{f} = \partial\overline{\partial} f + \overline{\partial}\partial\overline{f} = d(\overline{\partial} f + \partial\overline{f}) = d\eta.$$

Weinstein structure on surfaces

Proposition

Any two-dimensional Liouville-domain $(X, d\eta)$ admits a Weinstein structure $(X, \eta + dh, \rho)$ for a suitable exact deformation $\eta + dh$ of the Liouville form.

Proof (4/5).

It follows that $\eta = -d^c(-i)(f - \overline{f})/2 + \gamma$ for some closed real one-form γ. Pick a holomorphic one-form $\beta^{1,0}$ such that $\gamma = \beta^{1,0} - dh_1$ (embed X in a closed Riemann surface). Cartan's Theorem B implies $\beta^{1,0} = \partial 2g$. Since ∂g and $-\overline{\partial} g$ are d-cohomologous, we get

$$\beta^{1,0} = (\partial - \overline{\partial})g = -id^c g.$$

I.e.: $\rho = (-i)(f - \overline{f} + g - \overline{g})$ satisfies $-d^c\rho/2 = \eta + dh$.

Weinstein structure on surfaces

Proposition

Any two-dimensional Liouville-domain $(X, d\eta)$ admits a Weinstein structure $(X, \eta + dh, f)$ for a suitable exact deformation $\eta + dh$ of the Liouville form.

Proof (5/5).

The Liouville vector field of the symplectic form $d(\eta + dh) = -d(d^c\rho/2)$ is not necessarily outwards pointing along ∂X. We amend this by deforming ρ near the by the formula

$$\rho_C := \rho + 2(e^{\sigma_C(t)} - e^t), \text{ for } C \gg 0.$$

Note that $-d^c\rho_C = \eta_C + dh$ is an exact deformation of the inflated Liouville form, from which the claim finally follows. □

Equivalence of Liouville structures

Theorem

If $(X, d\eta_0)$ and $(X, d\eta_1)$ are equivalent Liouville domains, then there is a quasi-equivalence $\mathcal{W}(X, \eta_0) \simeq \mathcal{W}(X, \eta_0)$ of their wrapped Fukaya categories.

Proof (1/3).

We only show how the objects (exact Lagrangians) are related in the two categories $\mathcal{W}(X, d\eta_0)$ and $\mathcal{W}(X, d\eta_1)$.
Complete $(X, d\eta_0)$ by gluing half a symplectisation

$$\overline{X} = X \cup ((0, +\infty) \times Y, d(e^t\alpha_0)), \;\; Y = \partial X, \;\; \alpha_0 = \eta_0|_{TY}.$$

along ∂X. Note that the coordinate t on the collar $(-\epsilon, 0] \times Y$ of ∂X defined by η_0 extends to the entire infinite cylinder $(-\epsilon, +\infty) \times Y$.

Equivalence of Liouville structures

Theorem

If $(X, d\eta_0)$ and $(X, d\eta_1)$ are equivalent Liouville domains, then there is a quasi-equivalence $\mathcal{W}(X, \eta_0) \simeq \mathcal{W}(X, \eta_0)$ of their wrapped Fukaya categories.

Proof (2/4).

One can readily construct a smooth isotopy of X supported in $(-\epsilon, 0] \times Y$ which "straightens" the Liouville vector fields ζ_s defined by $\iota_{\zeta_s} d\eta_s = \eta_s$ in the collar, so that $\zeta_s \equiv \zeta_0 = \partial_t$ holds there for all $s \in [0, 1]$. Consequently, the forms η_s are all of the form $e^t\alpha_s$ in the same neighbourhood. (Hence $\alpha_s \in \Omega^1(Y)$ is a smooth family of contact one-forms, and η_s extend smoothly to the completion $\overline{X}$ by the formula $e^t\alpha_s$.) From now on we assume without loss of generality that all η_s are of this form.

Equivalence of Liouville structures

Theorem

If $(X, d\eta_0)$ and $(X, d\eta_1)$ are equivalent Liouville domains, then there is a quasi-equivalence $\mathcal{W}(X, \eta_0) \simeq \mathcal{W}(X, \eta_0)$ of their wrapped Fukaya categories.

Proof (3/4).

By deforming the family $e^t\alpha_s$ on the collar by a smooth bump function $e^t\alpha_{s\rho(t)}$ where $\rho(t) \in [0, 1]$, $\rho(t) = 1$ for $t \leq 0$, $\rho(0) = 1$ for all $t \gg 0$, and $|\rho'(t)|$ sufficiently small, we get a new family of *Liouville forms* whose Liouville vector fields all have a positive component of ∂_t in the infinite collar $(-\epsilon, +\infty) \times Y$, and which all coincide with η_0 outside of a compact subset.

Equivalence of Liouville structurs

Theorem

If $(X, d\eta_0)$ and $(X, d\eta_1)$ are equivalent Liouville domains, then there is a quasi-equivalence $\mathcal{W}(X,\eta_0) \simeq \mathcal{W}(X,\eta_0)$ of their wrapped Fukaya categories.

Proof (4/4).

Call the resulting compactly supported path of Liouville forms $\tilde{\eta}_s$, where $\tilde{\eta}_0 = \eta_0$. An application of Moser's trick produces a compactly supported smooth isotopy $\psi_s \colon \overline{X} \to \overline{X}$ for which $\psi_s^* d\tilde{\eta}_s = d\eta_0$. Now, any exact $L \subset (\overline{X}, d\eta_0)$ which is cylindrical inside $(-\epsilon, +\infty) \times Y$, produces

$$\phi_{\tilde{\zeta}_1}^{-N}(\psi^1(L)) \cap X, \text{ for } N \gg 0$$

which is an exact Lagrangian of the same type in $(X, d\eta_1)$. □

Lagrangian cocores

- Except in the case of cotangent bundles, the skeleton of a Weinstein manifold is singular. This makes Floer homology difficult to define.
- While closed Lagrangians seemingly are very rare, there exist plenty of exact Lagrangians with Legendrian boundary in every Liouville domain; However, as we saw, we have no guarantees that they give rise to interesting objects in the wrapped Fukaya category.
- It turns out that the embedded exact *Lagrangian cocore discs* will play a crucial role in the wrapped Fukaya category.

Lagrangian cocores

Let $\dim W = 2n$. We again point out the fact that, since $(\phi_\zeta^t)^* d\eta = e^t d\eta$ gives a positive rescaling of the symplectic form, it follows that:

- The stable manifolds W^s of the critical points of ζ are *isotropic*, i.e. $\omega|_{TW^s} \equiv 0$ or equivalently $TW^s \subset (TW^s)^\omega$. Consequently, the critical points c of ζ have index that satisfies $\operatorname{index} c = \dim W^s(c) \leq n$.
- The unstable manifolds W^u of the critical points of ζ are *coisotropic*, i.e. the ω-orthogonal complement satisfies the inclusion $(TW^u)^\omega \subset (TW^u)$; Observe that $\dim W^u = 2n - \dim W^s$ in this case.

Lagrangian cocores

The Lagrangian cocore discs

They are the unstable manifolds of the critical points of f of Morse index $n = \dim W/2$, i.e. the top index critical points.

- Coisotropic and half-dimensional implies Lagrangian.
- The cocores thus consistute a finite number $D_1, \ldots, D_k$ of disjoint *exact Lagrangian* discs inside W which are cylindrical near ∂W.
- For a subcritical Weinstein manifold, there are no Lagrangian cocore discs.
- However, one can always introduce cancelling handles to introduce more cocores, while keeping the equivalence class of the Liouville structure.

Lagrangian cocore in T^*S^1

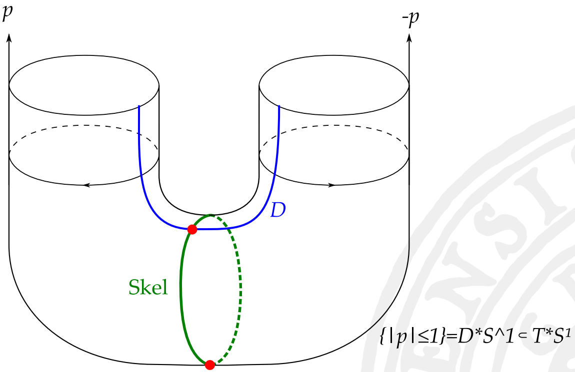

Figure: In general, the cocore(s) in any D^*M with the above Weinstein can be identified with the cotangent fibre. The depicted case is the cocore in D^*S^1.

Cocores in the punctured torus

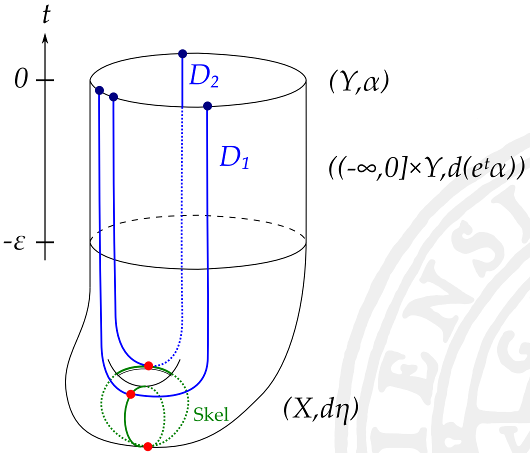

Figure: The two Lagrangian cocores for the standard handle decomposition on the punctures torus.

Section 3

A generation result for the wrapped Fukaya category of a Weinstein manifold

Generation by cocores

We have already seen that: If there are no Lagrangian cocores, then the critical points of ζ are all of index at most $n-1 < n = \dim W/2$. Hence W is subcritical, and the wrapped Fukaya category $\mathcal{W}(W,\eta)$ is quasi-equivalent to the trivial category.

Definition

A *quasi-equivalence* between two A_∞-categories $\{f_d\}\colon \mathcal{A} \to \mathcal{B}$ is an A_∞-functor (generalisation of morphism of A_∞-algebra, f_0 map of objects) for which f_1 induces an isomorphism

$$[f_1]\colon H(Hom_{\mathcal{A}}(L_0,L_1)) \xrightarrow{\cong} H(Hom_{\mathcal{B}}(f_0(L_0),f_0(L_1)))$$

on the level of homology.

Generation by cocores

The remaining part of this lecture will be devoted to making the following statement meaningful:

Theorem ([CRGG19], [GPS19])

For a Liouville domain $(X, d\eta)$, and set of Lagrangian cocores for an equivalent Weinstein structure generate the wrapped Fuakaya category $\mathcal{W}(X, \eta)$.

Enlarging the wrapped Fukaya category

In order to formulate the generation we need to consider the following enlargement of A_∞-categories.

$$\mathcal{W}(X,\eta) \subset Tw\mathcal{W}(X,\eta) \subset \Pi(Tw\mathcal{W}(X,\eta)).$$

These notions all appear in the work [Sei08] by Seidel (which concerns the Fukaya category for closed manifolds).

Remark

In fact, the generation result presented here only needs the first enlargement.

Categories and Algebras

A category (resp. A_∞-category) is like an algebra (resp. A_∞-algebra), except that:

- One is usually not allowed to multiply elements; i.e. compose morphisms) unless the composition makes sense; In an additive category, this can be amended by passing to sums of objects.
- When there is infinitely many objects, then this trick does still not produce a unital algebra: an infinite direct sum of unital algebras is not unital.
- Nevertheless, the category still behaves as an algebra in many respects, and sometimes it is even equivalent in a certain technical sense to an algebra.

Categories and Algebras

To pass from an A_∞-subcategory $\mathcal{B} \subset \mathcal{A}$ to an A_∞-subcategory $Tw\mathcal{B} \subset Tw\mathcal{A}$ is analogous to

- Passing from a subcategory $\mathcal{B} \subset ModA$ of A to the additive closure

$$add(\mathcal{B}) \subset ModA$$

  in its module category (i.e. $\mathcal{B} = \{A\}$ produces the subcategory of finitely generated free modules);
- Even better: Passing from a subcategory $\mathcal{B} \subset \mathcal{C}^b_{dg}(A)$ of bounded DG-modules over a DG-algebra A to its *triangulated envelope* inside the triangulated category $\mathcal{C}^b_{dg}(A)$.

Triangulated categories

A *triangulated* category satisfies a number of axioms that we do not have time to describe. Roughly, it prescribes:

- An endofunctor Σ called "suspension"; In our situation, this functor simply shifts grading of modules, i.e.

$$(\Sigma M)_* = M_{*+1} = M[1].$$

- A set of *exact triangles* such that each morphism $x \in Hom(L_0, L_1)$ can be completed to an exact triangle

$$L_0 \xrightarrow{x} L_1 \to Cone(x) \to L_0[1].$$

(A typical example is the mapping cone construction in homological algebra.)

Categories and Algebras: Twisted complexes

The constructions of

$$Tw\mathcal{B} \subset Tw\mathcal{A} \subset Mod\,\mathcal{A}$$

can be performed via a closure inside a module category:

- Take the triangulated envelope of the images

  $$\mathcal{Y}_r(\mathcal{B}) \subset \mathcal{Y}_r(\mathcal{A}) \subset Mod\,\mathcal{A}$$

  of the categories $\mathcal{B} \subset \mathcal{A}$ under the fully faithful Yoneda embedding

  $$\mathcal{Y}_r \colon \mathcal{A} \to Mod\,\mathcal{A}$$

  into the category of A_∞-category modules over $\mathcal{A}$.
- We will instead give an explicit construction of this enlargement below, which bypasses the Yoneda embedding.

Categories and Algebras: Twisted complexes

The construction of the further enlargements

$$\Pi(Tw\mathcal{B}) \subset \Pi(Tw\mathcal{A}) \subset Mod\,\mathcal{A}$$

needs an additional step

- Add all summands that correspond to idempotents. (I.e. take the split-closure.)

Example

Analogy with modules over an algebra A: The triangulated envelope of A yields bounded complexes of *free* modules. Adding all summands that correspond to idempotents yields the bounded complexes of *projective* modules, i.e. $Perf(A)$.

Precise generation result

We are now ready to reformulate the generation result in the following manner:

Theorem ([CRGG19], [GPS19])

For a Liouville domain $(X, d\eta)$*, and the full subcategory* $\mathcal{D} \subset \mathcal{W}(X, \eta)$ *whose objects consist of the Lagrangian cocores for an equivalent Weinstein structure, we have a natural quasi-equivalence*

$$Tw\mathcal{D} \overset{\simeq}{\subset} Tw\mathcal{W}(X, \eta)$$

of A_∞*-categories.*

An equivalent formulation: every object $L \in \mathcal{W}(X, \eta)$ is isomorphic inside $Tw\mathcal{W}(X, \eta)$ to an iterated cone built from the cocores $\{D_1, \ldots, D_k\}$.

Consequences of generation

The generation result makes $\mathcal{W}(X,\eta)$ of a Weinstein manifold possible to compute by understanding the full A_∞-subcategory $\mathcal{B} = \{D_1,\ldots,D_k\} \subset \mathcal{W}(X,\eta)$ consisting of the Lagrangian cocores:

- There is a quasi-equivalence between $Tw\mathcal{B}$ and the triangulated envelope of the $\mathcal{B}$-modules $\mathcal{Y}_r(D_1),\ldots,\mathcal{Y}_r(D_k) \subset Mod\mathcal{B}$ induced by the Yoneda embedding; see [Sei08][Lemmas 3.34,3.36].
- Since $\mathcal{B}$ can be seen as an A_∞-algebra, this has an even more concrete formulation: $Tw\mathcal{B}$ is quasi-equivalent to the *triangulated envelope* of the A_∞-modules $End(D_i) = Hom(D_i, D_i) \in ModB$ over the A_∞-algebra $B = End(D_1 \oplus \ldots \oplus D_k)$.

Consequences of generation

It is sometimes useful to replace the subcategory $\mathcal{B}$ consisting of the cocores by something which is quasi-equivalent:

- A quasi-equivalence $\mathcal{B}_1 \simeq \mathcal{B}_2$ of A_∞-categories extends to a quasi-equivalence $Tw\mathcal{B}_1 \simeq Tw\mathcal{B}_2$ of the corresponding twisted complexes [Sei08][Lemma 3.25].
- **In particular:** A quasi-isomorphism $B_2 \simeq B_2$ of A_∞-algebras induces a quasi-isomorphism of the triangulated envelopes of $B_1 \in ModB_1$ and $B_2 \in ModB_2$. (Recall that A_∞-algebras are A_∞-categories with a unique object.)

The closed exact case

Of course, for all we may know, $\mathcal{W}(X, \eta)$ may be quasi-equivalent to the zero category. This is not always the case; indeed, there are plenty of examples of interesting wrapped Fukaya categories. We present one here:

Theorem (Abouzaid [Abo12])

For the standard Weinstein structure on a connected cotangent bundle D^*M, the unique cocore D satisfies

$$(Hom(D, D), \{\mu_d\}) \cong C_*\Omega M$$

where the right-hand side is the DG-algebra of singular chains in the based loop-space of M equipped with the Pontryagin product.

(Abouzaid also proved the generation result in the particular case of the cotangent bundle: [Abo11a])

The closed exact case

In particular, $\mathcal{W}(D^*M, \theta_M)$ is quasi-equivalent to full-subcategory of the *semifree* DG-modules, i.e. the triangulated envelope of $C_*\Omega M$ inside its category $\mathcal{C}h^b_{dg}(C_*\Omega M)$ of DG-modules.

Theorem (Abouzaid [Abo11b])

The A_∞-algebra $CF(L, L)$ for a closed exact Lagrangian L with $\mathbb{F}$-coefficients is quasi-isomorphic (as an A_∞-algebra) to the unital differential graded algebra $C^*(L, \mathbb{F})$ of singular chains (this is an A_∞-algebra with $\mu_d = 0$ for all $d \geq 3$).

The original proof goes via an A_∞-structure which is constructed on the Morse complex of the compact manifold L. Instead, we take a different path here which uses algebraic topology and homological algebra.

The closed exact case

Proof (1/2).

- Since the Lagrangian L is closed and exact, on can compute its A_∞-structure inside its Weinstein neighbourhood $(D^*L, d\theta_L)$. We consider L as an object inside the wrapped Fukaya category $\mathcal{W}(D^*L, \theta_L)$.
- Since $D \cap L$ intersects transversely in a single point $Hom(D, L) = \mathbb{F}$. The Yoneda embedding identifies the object $L \in \mathcal{W}(D^*L, \theta_L)$ with the one-dimensional $Hom(D, D)$-module $Hom(D, L)$.
- The Yoneda embedding is fully faithful, so there is a quasi-isomorphism of A_∞-algebras

$$Hom(L, L) \simeq Hom_{ModHom(D,D)}(Hom(D, L), Hom(D, L)).$$

The closed exact case

Proof (2/2).

- $Hom(D, D)$ is quasi-isomorphic to $C_*\Omega M$ by Abouzaid's result. This identifies the $Hom(D, D)$-module $Hom(D, L)$ with a semifree resolution of the $C_*\Omega M$-module $\mathbb{F}$ (with module multiplication on $\mathbb{F}$ defined by the DGA-morphism $C^*(M) \to C^*\{pt\} = \mathbb{F}$ induced by $\{\mathrm{pt}\} \subset M$).
- Hence

$$Hom(L, L) \simeq Rhom_{C_*\Omega M}(\mathbb{F}, \mathbb{F})$$

and hence the classical result

$$Rhom_{C_*\Omega M}(\mathbb{F}, \mathbb{F}) \simeq C^*(M)$$

from e.g. [FHT95][Theorem 7.2(ii)] then finishes the claim.

□

Section 4

Twisted complexes and A_∞-modules

Modules over algebras

It is useful to use the category $ModA$ of (right) A-modules to understand an algebra A, even if this category is a gadget that in some sense is much larger than the algebra itself; for instance

$A \in ModA$ is an object with $Hom_{ModA}(A, A) \cong A$.

The same is true for A_∞-modules (to be defined below)

Modules over categories

- A module over a category $\mathcal{A}$ is a *functor*

$$\mathcal{F}\colon \mathcal{A} \to \mathit{Vect}(\mathbb{F})$$

  to the category of vector spaces;
- What this means: $x\colon b \to c$ in the category is sent to an element

$$\mathcal{F}(x) \in \mathit{Hom}_{\mathbb{F}}(\mathcal{F}(b), \mathcal{F}(c)),$$

  i.e. we have a map

$$\mathcal{F}(b) \otimes \mathit{Hom}_{\mathcal{A}}(a, b) \to \mathcal{F}(c),$$
$$(m \otimes x) \mapsto \mathcal{F}(x)(m)$$

  i.e. the module multiplication.

Modules over categories

- This construction works for A_∞-categories as well, but one has to replace functor with A_∞-functor (generalisation of A_∞-morphisms from algebras to categories).
- Moreover, we want to consider DG-modules, so the correct definition is the following:

Definition

A module over an A_∞-category $\mathcal{A}$ is an A_∞-functor

$$\mathcal{F} \colon \mathcal{A} \to \mathcal{C}h(\mathbb{F})$$

to the DG-category of chain complexes

A_∞-modules

The concrete formulas in the case of an A_∞-category with one object, i.e. an A_∞-algebra A, is the following:
An A_∞ A-module is a vector space M together with operations

$$\nu_d \colon M \times A^d \to M, \;\; d = 1, 2, 3 \ldots$$

that satisfy

$$\sum_n (-1)^{\maltese} \nu_{n+1}(\nu_{d-n}(m, a_{d-1}, \ldots, a_{n+1}), \ldots, a_1) + \sum_{m,n} (-1)^{\maltese} \nu_{m-d+1}(m, a_d, \ldots, \nu_m(a_{n+m}, \ldots, a_{n+1}), \ldots, a_1) = 0$$

Twisted complexes

Since one can take cones of modules, and shift their grading, they form a triangulated category. Twisted complexes is an abstract way to enhance an A_∞-category by adding these cones.
We start with the **shift functor**:

- There is a shift of grading $L[i]$ of the objects, where

$$Hom^*(L_0[i], L_1[j]) = Hom^{*+i-j}(L_0, L_1)$$

- For chain complexes we have $C^*[i] = C^{*-i}$ (and graded *Hom*s get shifted as above).
- In the wrapped Fukaya category the shift is geometrically induced by a choice of Maslov potential. (We did not talk about this.)

Twisted complexes

We then proceed by **sums of objects**:

- Enlarge the A_∞-category by adding the finite sums of shifts of objects

$$\mathbf{L} = L_{i_1}[j_1] \oplus \ldots \oplus L_{i_k}[j_k]$$

with

$$Hom(\mathbf{L}, T) = Hom^{*+j_1}(L_{i_1}, T) \oplus \ldots \oplus Hom^{*+j_k}(L_{i_k}, T),$$
$$Hom(T, \mathbf{L}) = Hom^{*-j_1}(T, L_{i_1}) \oplus \ldots \oplus Hom^{*-j_k}(T, L_{i_k}).$$

The A_∞-operations are defined by additively extending.

Example

$$\begin{aligned} Hom^*(L_0[i] \oplus L_1[j], L_0[i] \oplus L_1[j]) = \\ = \quad End^*(L_0) \oplus Hom^{*+i-j}(L_0, L_1) \oplus Hom^{*+j-i}(L_1, L_0) \oplus End^*(L_1) \end{aligned}$$

Twisted complexes

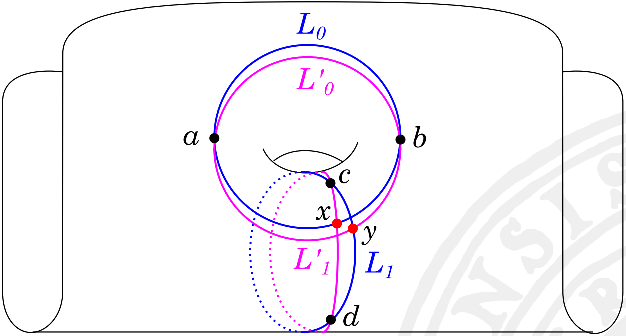

Figure: In the wrapped Fukaya category, sums of objects has a natural geometric explanation: immersions given by disjoint unions of Lagrangian embeddings, e.g. $L_0 \cup L_1$ and its perturbation $L'_0 \cup L'_1$ shown in the figure. Here $x \in Hom(L_0, L'_1) \subset Hom(L_0 \oplus L_1, L'_0 \oplus L'_1)$, while $y \in Hom(L_1, L'_0) \subset Hom(L_0 \oplus L_1, L'_0 \oplus L'_1)$.

Twisted complexes

What remains is to add **cones**.

- This is done by *twisting* the above direct sums by solutions to the Maurer–Cartan equation which satisfy a certain filtration property.
- One can do this iteratively by defining cones between sums of two objects:

Twisted complexes

Definition

The object $Cone(x)$ for a closed morphism $x \in Hom(L_0, L_1)$ is the object $L_0[1] \oplus L_1$ with A_∞-operations “twisted” by the element x via

$$\mu_d^x(a_d, \ldots, a_1) = \sum_{k \geq 0} mu_{d+k}(\ldots, a_d, \ldots, a_{d-1}, \ldots, x, \ldots, a_1, \ldots)$$

where the element x has been inserted in all possible ways.

(The above sum is finite since x is not an endomorphism.)

Example

$$\mu_1^x(a) = \mu_1(a) + \mu_2(a, x) + \mu_2(x, a)$$

Twisted complexes

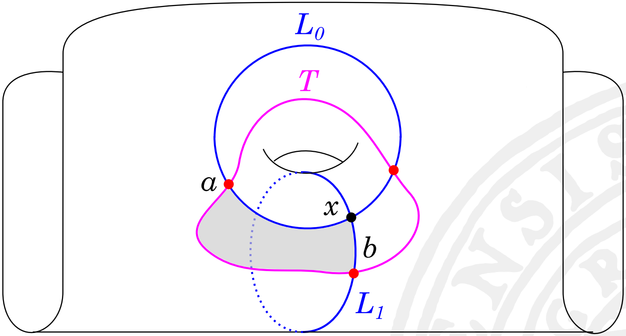

Figure: Twisting by a cycle $x \in Hom(L_1, L_0)$ as depicted in the figure yields $\langle \mu_1^x(a), b\rangle = \langle \mu_2(a, x), b\rangle = 1$, where $a \in Hom(L_0, T)$ and $b \in Hom(L_1, T)$.

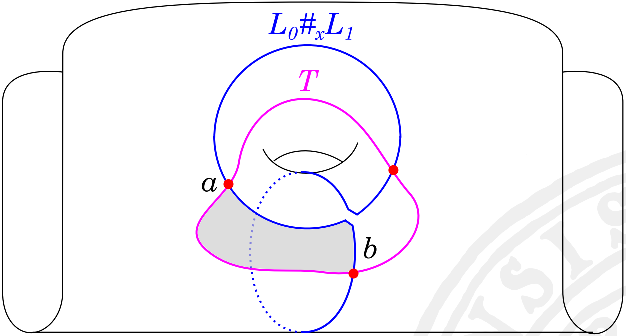

Figure: There is also a geometric explanation: performing surgery at the double point $x \in L_0 \cup L_1$ to produce $L_0\#_x L_1$; again $\langle \mu_1(a), b\rangle = 1$, where $a, b \in Hom(L_0\#_x L_1, T)$. Note that the "input corner" of $x \in Hom(L_1, L_0)$ has been rounded.

Twisted complexes

Remark

If there are more than one intersection point between L_0 and L_1, then the result $L_0 \#_x L_1$ is connected but typically not embedded. This makes Floer homology difficult to define.

Isomorphism and cones

Recall that:

- If L_0 and L_1 are Hamiltonian isotopic Lagrangian submanifolds, then they are isomorphic object in the Donaldson category, with an isomorphism given by a continuation element $[c_{L_0,H}] \in H(Hom(L_0, L_1))$.
- In general, two objects in a classical category are isomorphic in the category if and only if there exists a morphism $x \in Hom(L_0, L_1)$ for which left and right composition induces isomorphisms

$$l_x \colon Hom(L_0, L_0) \xrightarrow{\cong} Hom(L_0, L_1)$$
$$r_x \colon Hom(L_0, L_1) \xrightarrow{\cong} Hom(L_1, L_1).$$

of morphisms sets. (Check that the above two properties ensure left and right invertibility of the morphism x.)

Isomorphism and cones

- To every A_∞-category one can associate its homology category $H\mathcal{A}$ which consists of the same objects, but where $Hom_{H\mathcal{A}}(L_0, L_1) = H(Hom(L_0, L_1))$.
- pause $H\mathcal{A}$ a classical category which is equal to the Donaldson category in the case when $\mathcal{A}$ is the Fukaya category.

We have the following relation between isomorphism in $H\mathcal{A}$ and the acyclicity of cones in $Tw\mathcal{A}$:

Lemma

For a cycle $x \in Hom(L_0, L_1)$ in an A_∞-category $\mathcal{A}$, the object $Cone(x) \in Tw\mathcal{A}$ is acyclic, i.e. $H(End(Cone(x))) = 0$, if and only if x is an isomorphism in the homology category $H\mathcal{A}$. Moreover, in this case the two A_∞-algebras $Hom(L_0, L_0)$ and $Hom(L_1, L_1)$ are quasi-isomorphic.

Isomorphism and cones

Proof (1/3).

- Technical assumption which can be achieved after quasi-equivalence: all operations μ_d, $d \geq 3$, involving a unit e_L vanish. (So called *strict unitality*.)
- The property for x to be an isomorphism in $H\mathcal{A}$ and acyclicity $H(End(Cone(x)))$ are equivalent for the following reason: $H(End(Cone(x))) = 0$ is equivalent to the unit in $Cone(x)$, i.e. the cycle given by

$$e_{Cone(x)} = e_{L_0} \oplus e_{L_1} \in End(L_0) \oplus End(L_1) \subset End(Cone(x)),$$

being a *boundary*. (That the sum of units is the unit follows from strict unitality.)

Isomorphism and cones

Proof (2/3).

We now show that $End(L_0)$ and $End(L_1)$ are quasi-isomorphic when l_x and r_x induces an isomorphism between morphism spaces in $H\mathcal{A}$:

- Consider the A_∞-subalgebra

  $$C := End(L_0) \oplus Hom(L_0, L_1) \oplus End(L_1) \subset End(Cone(x)).$$

  There are obvious A_∞-morphism from C to both A_∞-algebras $End(L_i)$ given by the canonical projections

  $$\pi_0 \colon C \to End(L_0),$$
  $$\pi_1 \colon C \to End(L_1).$$

- In fact, all f_d, $d \geq 2$, vanish for these A_∞-morphisms.

Isomorphism and cones

Proof (3/3).

- These projections are quasi-isomorphism since their kernels

$$\ker \pi_0 = Hom(L_0, L_1) \oplus End(L_1) \subset C \subset End(Cone(x)),$$
$$\ker \pi_1 = End(L_0) \oplus Hom(L_0, L_1) \subset C \subset End(Cone(x)).$$

both are acyclic cones themselves.

- Namely, $[\pi_i]$ is an isomorphism by the long exact sequences in homology arising from the short exact sequences

$$0 \to \ker \pi_i \to C \xrightarrow{\pi_i} End(L_i) \to 0$$

of complexes.

□

Twisted complexes

Thank you!

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