ABOUT SCALING BEHAVIOR OF RANDOM BALLS MODELS

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Abstract. We study the limit of a generalized random field generated by uniformly scattered random balls as the mean radius of the balls tends to 0 or infinity. Assuming that the radius distribution has a power law behavior, we prove that the centered field, conveniently renormalized, admits a limit.

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1 Setting

We propose a first step to a unified frame including and extending both the situations of [4] and [2]. We start with a family of grains $X_j + B(0, R_j)$ in $\mathbb{R}^d$ built up from a Poisson point process $(X_j, R_j)_j$ in $\mathbb{R}^d \times \mathbb{R}^+$. Equivalently one can start with a Poisson random measure N on $\mathbb{R}^d \times \mathbb{R}^+$ and associate with each random point $(x, r) \in \mathbb{R}^d \times \mathbb{R}^+$ the random ball of center x and radius r. We assume that the intensity measure of N is given by $\mathrm{d}xF(\mathrm{d}r)$ where F is a σ-finite non-negative measure on $\mathbb{R}^+$ such that

$$\int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty. \tag{1}$$

1.1 Assumptions on F

We introduce the asymptotic power law behavior assumption on F, near 0 or at infinity, we will use in the following. For $\beta > 0$ with $\beta \neq d$,

$$\mathbf{H}(\beta) \ : F(\mathrm{d}r) = f(r)\mathrm{d}r \text{ with } f(r) \sim C_\beta r^{-\beta-1} \ , \text{ as } \ r \to 0^{d-\beta},$$

where by convention $0^\alpha = 0$ if $\alpha > 0$ and $0^\alpha = +\infty$ if $\alpha < 0$.

Let us remark that according to (1), it is natural to consider the asymptotic behavior of F around 0 for $\beta < d$ and at infinity for $\beta > d$.

1.2 Random field

We consider random fields defined on a space of measures, in the same spirit as the random functionals of [4] or the generalized random fields of [1] (see therein the links between "generalized random fields" and "ponctual random fields"). Let $\mathcal{M}^1$ denote the space of signed measures μ on $\mathbb{R}^d$ with finite total variation $||\mu||_1 = |\mu|(\mathbb{R}^d)$, with $|\mu|$ the total variation measure of μ. Since for all $\mu \in \mathcal{M}^1$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu(B(x,r))| \, \mathrm{d}x F(\mathrm{d}r) \leq |B(0,1)| \times ||\mu||_1 \times \int_{\mathbb{R}^+} r^d F(\mathrm{d}r) < +\infty \ ,$$

one can introduce the generalized random field X defined on $\mathcal{M}^1$ as

$$\mu \mapsto \langle X, \mu \rangle = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r)) \ N(\mathrm{d}x, \mathrm{d}r). \tag{2}$$

We introduce the following definition, which coincides with the usual definition of self-similar ponctual random fields.

Definition 1.1. *A random field X defined on $\mathcal{M}^1$ is said to be self-similar with index H if*

$$\forall s > 0 \ , \ \langle X, \mu_s \rangle \overset{fdd}{=} s^H \langle X, \mu \rangle \quad \textit{where} \quad \mu_s(A) = \mu(s^{-1}A).$$

2 Scaling limit

Let us introduce now the notion of "scaling", by which we indicate an action: a change of scale acts on the size of the grains. The following procedure is performed in [4] where grains of volume v are changed by shrinking into grains of volume $\rho\, v$ with a small parameter ρ ("small scaling" behavior). The same is performed in [2] in the homogenization section, but the scaling acts in the opposite way: the radii r of grains are changed into r/ε (which is a "large scaling" behavior). Note also that both scalings are performed in the case of α-stable measures in [3].

Let us multiply the intensity measure by $\lambda > 0$ and the radii by $\rho > 0$. We denote by $F_\rho(\mathrm{d}r)$ the image measure of $F(\mathrm{d}r)$ by the change of scale $r \mapsto \rho r$, and consider the associated random field on $\mathcal{M}^1$ given by

$$\int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r)) \ N_{\lambda,\rho}(\mathrm{d}x, \mathrm{d}r) \ ,$$

where $N_{\lambda,\rho}(\mathrm{d}x, \mathrm{d}r)$ is the Poisson random measure with intensity measure $\lambda \mathrm{d}x F_\rho(\mathrm{d}r)$ and $\mu \in \mathcal{M}^1$. Results are expected concerning the asymptotic behavior of this scaled random balls model when $\rho \to 0$ or $\rho \to +\infty$. We choose ρ as the basic model parameter, consider $\lambda = \lambda(\rho)$ as a function of ρ, and define on $\mathcal{M}^1$ the random field

$$X_\rho(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu(B(x,r)) \ N_{\lambda(\rho),\rho}(\mathrm{d}x, \mathrm{d}r) \ .$$

Then, we are looking for a normalization term $n(\rho)$ such that the centered field converges in distribution,

$$\frac{X_\rho(.) - \mathbf{E}(X_\rho(.))}{n(\rho)} \overset{fdd}{\rightarrow} W(.) \tag{3}$$

and we are interested in the nature of the limit field W. Let us remark that the random field X_ρ is linear on each vectorial subspace of $\mathcal{M}^1$ in the sense that for all $\mu_1, \ldots, \mu_n \in \mathcal{M}^1$ and $a_1, \ldots, a_n \in \mathbb{R}$, almost surely,

$$X_\rho \left(a_1\mu_1 + \ldots + a_n\mu_n\right) = a_1 X_\rho(\mu_1) + \ldots + a_n X_\rho(\mu_n).$$

Hence, the finite-dimensional distributions of the normalized field $\left(X_\rho(.) - \mathbf{E}(X_\rho(.))\right)/n(\rho)$ converge toward W whenever

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu) - \mathbf{E}(X_\rho(\mu))}{n(\rho)}\right)\right) \to \mathbb{E}\left(\exp\left(iW(\mu)\right)\right),$$

for all μ in a convenient subspace of $\mathcal{M}^1$.

To study the limiting behavior of the normalized field $\left(X_\rho(.) - \mathbf{E}(X_\rho(.))\right)/n(\rho)$ we need to impose some assumptions on the measure $\mu \in \mathcal{M}^1$. For $\alpha > 0$ with $\alpha \neq d$ let us define the space of measures

$$\mathcal{M}_\alpha = \begin{cases} \left\{\mu \in \mathcal{M}^1; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |z-z'|^{-(\alpha-d)}|\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') < +\infty\right\} & \text{if } \alpha > d \\ \left\{\mu \in \mathcal{M}^1; \int_{\mathbb{R}^d} |z|^{-(\alpha-d)}|\mu|(\mathrm{d}z) < +\infty \text{ and } \int_{\mathbb{R}^d}\mu(\mathrm{d}z) = 0\right\} & \text{if } \alpha < d \end{cases},$$

where $|\mu|$ is the total variation measure of μ. Let us consider the kernel on $\mathbb{R}^d \times \mathbb{R}^d$,

$$K_\alpha(z,z') = \begin{cases} |z-z'|^{-(\alpha-d)}, \text{ for } z \neq z' & \text{if } \alpha > d \\ \frac{1}{2}\left(|z|^{-(\alpha-d)} + |z'|^{-(\alpha-d)} - |z-z'|^{-(\alpha-d)}\right) & \text{if } \alpha < d \end{cases}.$$

Then, for $\alpha \in (d-1, 2d)$ with $\alpha \neq d$, for any $\mu \in \mathcal{M}_\alpha$, the kernel K_α is defined and non negative $|\mu| \times |\mu|$ everywhere, with

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_\alpha(z,z')|\mu|(\mathrm{d}z)|\mu|(\mathrm{d}z') < +\infty.$$

For $\beta \in (d-1, 2d)$ with $\beta \neq d$, we finally define the enlarged spaces

$$\overline{\mathcal{M}_\beta} = \begin{cases} \bigcup_{\alpha \in (\beta, 2d)} \mathcal{M}_\alpha & \text{if } \beta > d \\ \bigcup_{\alpha \in (d-1, \beta)} \mathcal{M}_\alpha & \text{if } \beta < d \end{cases}$$

Theorem 2.1. *Let F be a non-negative measure on $\mathbb{R}^+$ satisfying $\mathbf{H}(\beta)$ for $\beta \in (d-1, 2d)$ with $\beta \neq d$. For all positive functions λ such that $n(\rho) := \sqrt{\lambda(\rho)\rho^\beta} \underset{\rho \to 0^{\beta-d}}{\longrightarrow} +\infty$, the limit*

$$\frac{X_\rho(\mu) - \mathbf{E}(X_\rho(\mu))}{n(\rho)} \underset{\rho \to 0^{\beta-d}}{\overset{fdd}{\longrightarrow}} c_\beta W_\beta(\mu)$$

holds for all $\mu \in \overline{\mathcal{M}_\beta}$, in the sense of finite dimensional distributions of the random functionals. Here W_β is the centered Gaussian random linear functional on $\overline{\mathcal{M}_\beta}$ with

$$\mathbb{E}\left(W_\beta(\mu)W_\beta(\nu)\right) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_\beta(z,z')\mu(\mathrm{d}z)\nu(\mathrm{d}z'), \tag{4}$$

and c_β is a positive constant depending on β.

Remark 2.2. *For $\beta \in (d-1, 2d)$ with $\beta \neq d$, the field W_β, defined on $\overline{\mathcal{M}_\beta}$, is $\frac{d-\beta}{2}$ self-similar with $\frac{d-\beta}{2} < 0$ for $\beta \in (d, 2d)$ and $\frac{d-\beta}{2} \in (0, 1/2)$ for $\beta \in (d-1, d)$.*

Proof. For $\mu \in \mathcal{M}^1$ let us define the functions φ_ρ and φ on $\mathbb{R}^+$ by

$$\varphi_\rho(r) = \int_{\mathbb{R}^d} \Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right) \mathrm{d}x, \quad \text{and } \varphi(r) = -\frac{1}{2}\int_{\mathbb{R}^d} \mu(B(x,r))^2 \mathrm{d}x,$$

where

$$\Psi(v) = e^{iv} - 1 - iv. \tag{5}$$

The characteristic function of the normalized field $(X_\rho(.) - \mathbf{E}(X_\rho(.)))/n(\rho)$ is then given by

$$\mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu) - \mathbf{E}(X_\rho(\mu))}{n(\rho)}\right)\right) = \exp\left(\int_{\mathbb{R}^+} \lambda(\rho)\varphi_\rho(r)F_\rho(\mathrm{d}r)\right).$$

According to the power law behavior of the density F we have the following result, which is inspired by Lemma 1 of [4].

Lemma 2.3. *Let F be a non-negative measure on $\mathbb{R}^+$ satisfying $\mathbf{H}(\beta)$ for $\beta > 0$ with $\beta \neq d$. Assume that g is a continuous function on $\mathbb{R}^+$ such that for some $0 < p < \beta < q$, there exists $C > 0$ such that*

$$|g(r)| \leq C\min(r^q, r^p).$$

Then

$$\int_{\mathbb{R}^+} g(r)F_\rho(\mathrm{d}r) \sim C_\beta \rho^\beta \int_0^\infty g(r) r^{-\beta-1}\mathrm{d}r \text{ as } \rho \to 0^{\beta-d}.$$

We apply Lemma 2.3 with the function φ. Since $\mu \in \mathcal{M}^1$, the function φ is continuous on $\mathbb{R}^+$. Moreover, the next lemma shows that φ satisfies the required upper bound.

Lemma 2.4. *Let $\alpha \in (d-1, 2d)$ with $\alpha \neq d$. If $\mu \in \mathcal{M}_\alpha$, then there exists $c > 0$ such that*

$$\int_{\mathbb{R}^d} \mu(B(x,r))^2 \mathrm{d}x \leq c\min(r^d, r^\alpha).$$

Therefore, for $\beta \in (d-1, 2d)$ with $\beta \neq d$, when $\mu \in \overline{\mathcal{M}_\beta}$

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \mu(B(x,r))^2 r^{-\beta-1}\mathrm{d}x\mathrm{d}r < +\infty.$$

Proof. Since $\mu \in \mathcal{M}^1$, $\int_{\mathbb{R}^d} \mu(B(x,r))^2 \mathrm{d}x \leq |B(0,1)| \, \|\mu\|_1^2 r^d$. We use Lemma 6 of [4] to conclude for $\mu \in \mathcal{M}_\alpha$, with $\alpha \in (d, 2d)$. When $\mu \in \mathcal{M}_\alpha$, with $\alpha \in (d-1, d)$, we conclude using the fact that, since $\int_{\mathbb{R}^d} \mu(\mathrm{d}z) = 0$, we can write

$$\int_{\mathbb{R}^d} \mu(B(x,r))^2 \mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{2}\left(\phi(z,r) + \phi(z',r) - \phi(z-z',r)\right)\mu(\mathrm{d}z)\mu(\mathrm{d}z'),$$

where

$$\phi(z,r) = \int_{\mathbb{R}^d} \left(\mathbf{1}_{B(x,r)}(z) - \mathbf{1}_{B(x,r)}(0)\right)^2 \mathrm{d}x, \tag{6}$$

is such that $\phi(z,r) \leq 2^d |B(0,1)| \, |z|^{d-\alpha} r^\alpha$, for all $(z,r) \in \mathbb{R}^d \times \mathbb{R}^+$. □

When F satisfies $\mathbf{H}(\beta)$ for $\beta \in (d-1, 2d)$ with $\beta \neq d$, let us choose $\mu \in \overline{\mathcal{M}_\beta}$. According to Lemma 2.4, there exists $\alpha > 0$ such that $|\varphi(r)| \leq c\min(r^d, r^\alpha)$, with $\alpha > \beta$ if $\beta > d$ and $\alpha < \beta$ if $\beta < d$. Therefore φ satisfies the assumptions of Lemma 2.3 and

$$\lim_{\rho \to 0^{\beta-d}} \int_{\mathbb{R}^+} \varphi(r)\lambda(\rho)n(\rho)^{-2}F_\rho(\mathrm{d}r) = C_\beta \int_{\mathbb{R}^+} \varphi(r) r^{-\beta-1}\mathrm{d}r,$$

since $n(\rho) = \sqrt{\lambda(\rho)\rho^\beta}$.

We define the function $\Delta_\rho(r) = \lambda(\rho)\varphi_\rho(r) - \lambda(\rho)n(\rho)^{-2}\varphi(r)$ and observe that

$$\Delta_\rho(r) = \lambda(\rho)\int_{\mathbb{R}^d}\left(\Psi\left(\frac{\mu(B(x,r))}{n(\rho)}\right) + \frac{1}{2}\left(\frac{\mu(B(x,r)}{n(\rho)}\right)^2\right)\mathrm{d}x. \tag{7}$$

The following result, inspired by Lemma 2 of [4], will play the role of the Lebesgue's Theorem to ensure convergence of the integrals.

Lemma 2.5. *Let F be a non-negative measure on $\mathbb{R}^+$ satisfying $\mathbf{H}(\beta)$ for $\beta > 0$ with $\beta \neq d$. Let g_ρ be a family of continuous functions on $\mathbb{R}^+$. Assume that*

$$\lim_{\rho\to 0^{\beta-d}} \rho^\beta g_\rho(r) = 0, \quad \textit{and} \quad \rho^\beta|g_\rho(r)| \leq C\min(r^p, r^q),$$

for some $0 < p < \beta < q$ and $C > 0$. Then

$$\lim_{\rho\to 0^{\beta-d}} \int_{\mathbb{R}^+} g_\rho(r)F_\rho(\mathrm{d}r) = 0.$$

Let us verify that Δ_ρ given by (7) satisfies the assumptions of Lemma 2.5. For $\mu \in \mathcal{M}^1$, the function Δ_ρ is continuous on $\mathbb{R}^+$. Because $\left|\Psi(v) - \left(-\frac{v^2}{2}\right)\right| \leq \frac{|v|^3}{6}$ and

$$\int_{\mathbb{R}^d} \mu\left(B(x,r)\right)^3 \mathrm{d}x \leq \|\mu\|_1^2 \int_{\mathbb{R}^d} |\mu\left(B(x,r)\right)| \mathrm{d}x \leq C_d\|\mu\|_1^3 r^d,$$

we get

$$\left|\lambda(\rho)^{-1}n(\rho)^2\Delta_\rho(r)\right| \leq \frac{C_d\|\mu\|_1^3}{6}n(\rho)^{-1}r^d.$$

Moreover, since $|\Psi(v)| \leq \frac{|v|^2}{2}$, by Lemma 2.4 when $\mu \in \overline{\mathcal{M}_\beta}$, there exists $\alpha \in (\beta, 2d)$ if $\beta > d$, $\alpha \in (d-1, \beta)$ if $\beta < d$ such that

$$\left|\lambda(\rho)^{-1}n(\rho)^2\Delta_\rho(r)\right| \leq cr^\alpha.$$

Therefore, when $\mu \in \overline{\mathcal{M}_\beta}$ for $\beta \in (d-1, 2d)$, with $\beta \neq d$,

$$\lim_{\rho\to 0^{\beta-d}} \int_{\mathbb{R}^+} \Delta_\rho(r)F_\rho(\mathrm{d}r) = 0.$$

Since

$$\lim_{\rho\to 0^{\beta-d}} \rho^{-\beta}\int_{\mathbb{R}^+} \varphi(r)F_\rho(\mathrm{d}r) = C_\beta\int_{\mathbb{R}^+} \varphi(r)r^{-\beta-1}\mathrm{d}r,$$

we get

$$\lim_{\rho\to 0^{\beta-d}} \mathbb{E}\left(\exp\left(i\frac{X_\rho(\mu) - \mathbf{E}(X_\rho(\mu))}{n(\rho)}\right)\right) = \mathbb{E}\left(\exp\left(iW(\mu)\right)\right),$$

where $W(\mu)$ is the centered Gaussian random variable with

$$\mathbb{E}\left(W(\mu)^2\right) = C_\beta\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \mu\left(B(x,r)\right)^2 r^{-\beta-1}\mathrm{d}x\mathrm{d}r, \tag{8}$$

which is finite using Lemma 2.4. Let us prove that the covariance of W satisfies (4). By linearity, it is enough to compute the variance of W. When $\beta \in (d, 2d)$, for $\mu \times \mu$ almost all $(z, z') \in \mathbb{R}^d \times \mathbb{R}^d$, the function $r \mapsto |B(z,r) \cap B(z',r)|$ is in $L^1\left(\mathbb{R}^+, r^{-\beta-1}\mathrm{d}r\right)$. Therefore we can define the kernel,

$$K(z,z') = \int_{\mathbb{R}^+} |B(z,r) \cap B(z',r)|\, r^{-\beta-1}\mathrm{d}r = C(\beta)K_\beta(z,z').$$

By changing the order of integration in (8), we conclude $W \overset{fdd}{=} cW_\beta$ with $c = (C_\beta C(\beta))^{1/2}$. When $\beta \in (d-1, d)$, the function $r \mapsto |B(z,r) \cap B(z',r)|$ is not in $L^1\left(\mathbb{R}^+, r^{-\beta-1}\mathrm{d}r\right)$ anymore. However, since $\mu \in \overline{\mathcal{M}_\beta}$, we have $\int_{\mathbb{R}^d} \mu(\mathrm{d}z) = 0$, and we can write

$$\int_{\mathbb{R}^d} \mu\left(B(x,r)\right)^2 \mathrm{d}x = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{1}{2}\left(\phi(z,r) + \phi(z',r) - \phi(z-z',r)\right)\mu(\mathrm{d}z)\mu(\mathrm{d}z'),$$

where $\phi(z,r)$, given by (6) is in $L^1\left(\mathbb{R}^+, r^{-\beta-1}\mathrm{d}r\right)$, for all $z \in \mathbb{R}^d$. Therefore we can define the kernel $K(z,z') = \frac{1}{2}\left(\kappa(z) + \kappa(z') - \kappa(z-z')\right)$, with

$$\kappa(z) = \int_{\mathbb{R}^+} \phi(z,r) r^{-\beta-1}\mathrm{d}r = C(\beta)K_\beta(z,z).$$

Then, also in this case $W \overset{fdd}{=} cW_\beta$ with $c = (C_\beta C(\beta))^{1/2}$, which concludes the proof.

□

Let us mention that similar arguments allow us to state an intermediate scaling result.

Theorem 2.6. *Under the assumptions of Theorem 2.1, when* $\lambda(\rho)\rho^\beta \underset{\rho\to 0^{\beta-d}}{\longrightarrow} \sigma_0^{d-\beta}$, *for some* $\sigma_0 > 0$, *the following limit holds in the sense of finite dimensional distributions of the random functionals*

$$X_\rho(\mu) - \mathbf{E}(X_\rho(\mu)) \overset{fdd}{\to} J_\beta(\mu_{\sigma_0}),$$

for all $\mu \in \overline{\mathcal{M}_\beta}$. *Here* J_β *is the centered random linear functional on* $\overline{\mathcal{M}_\beta}$ *defined as*

$$J_\beta(\mu) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^+} \mu\left(B(x,r)\right)\widetilde{N_\beta}(\mathrm{d}x, \mathrm{d}r),$$

where $\widetilde{N_\beta}$ *is a compensated Poisson random measure with intensity* $C_\beta \mathrm{d}x r^{-\beta-1}\mathrm{d}r$, *and* μ_{s_0} *is defined by* $\mu_{\sigma_0}(A) = \mu\left(\sigma_0^{-1}A\right)$.

We should also obtain what is called the small-grain scaling in [4]. In this particular case, the limit is an independently scattered β-stable random measure on $\mathbb{R}^d$ with Lebesgue measure and unit skewness.

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