A scaling limit process for the age-reproduction structure in a Markov population

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Abstract

The purpose of this paper is to introduce a model for a reproducing Markov population, set within terms of a fairly general Markovian framework and with arbitrary offspring mechanism, and study the time evolution of the joint empirical distribution of age and reproduction numbers.

The particular objective is to apply a diffusion approximation scaling as the population size grows and establish the existence of a non-degenerate measure-valued Markov process in the scaling limit. In general the limit process is not a superprocess. But it may be characterized by its log-Laplace function being the unique solution of a non-linear integral equation. Under additional assumptions the limit process is recognized as a superprocess, and its martingale characterization is obtained.

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1 Introduction

Our objective is to study the time evolution of the joint empirical distribution of age and fertility among individuals in a reproducing Markov population, and to obtain limit results under suitable scaling as the population size grows.

With each individual in the population we associate two random quantities, its lifelength and its reproduction process. The latter bears information on the succesive times during its life when this individual gives birth and the number of daughters born at each of these points in time. Lifelength and reproduction are not assumed to be independent.

In some detail we treat the measure-valued jump Markov process that describes the age-reproduction structure in the population arising from a single or a collection of individuals at time zero. This model is characterized by a non-linear integral equation which is satisfied by its family of Laplace transform functions and this property provides the basic tool for the further analyses of the model.

Our particular interest is the limiting behaviour of the process under a diffusion approximation type scaling. Out of several existing rescaling possibilities we wish to find one which carries enough intrinsic random fluctuations to bring about a nondegenerate measure-valued process in the limit, while at the same time it preserves or at least respects the age-reproduction space motion. Hence we consider a scheme of small mass and large density particle systems where time and lifelengths are scaled to balance each other and where the offspring mechanism is also modified in the sense that births are suppressed with dominating probability while only rarely is there an outbreak of a large number of daughter particles. For this specific time-mass-space-rate-offspring scaling we prove weak convergence to a non-degenerate measure-valued limit process. Admittedly the underlying reproduction process undergoes a certain degeneration under this procedure. However, we feel that this limit operation is worth investigating as we know of no other models closely related to this one and since to our knowledge a related limit process has not been obtained elsewhere.

In Section 2 we state the precise assumptions on the branching particle system and the main results. Section 3 provides preliminaries on the mean operator and in Section 4 we derive the basic non-linear integral equation for the branching model. Section 5 is devoted to the proof of convergence of the sequence of scaled systems via the corresponding integral equations obtained in their limits. In Section 6 we give some examples and discuss for comparison two other rescaling schemes which were not dealt with here. First a time-mass-space-rate scaling, which has the attractive feature that it preserves the age-reproduction motion. However, fluctuations on the measure-valued level are killed and consequently the process converges weakly to its expectation measure. Second the simple time-mass scaling, usually applied in the asymptotic theory of branching processes, which yields convergence in finite-dimensional distributions to the product of a real-valued continuous state branching process and a deterministic measure-valued process.

The paper may be viewed as an attempt to use technique and methodology from the theory of measure-branching processes, in particular from Dawson (1993) and from Kaj and Sagitov (1998), and apply them to a model which conceptually is more closely related to the models known as general branching processes as presented e.g. by Jagers (1989).

2 Model and statement of result

We begin by introducing a Markovian set-up which describes the lifespan of a single reproducing particle.

Suppose a probability space $(\Omega, F, (F_s), P, (\theta_s))$ is given equipped with a complete right-continuous filtration (F_s) and a family of shift operators given by $\theta_s\omega(r) = \omega(s+r)$, $\omega \in \Omega$. On this space consider an integer-valued, upward jump, in general time-inhomogeneous, adapted Markov process $N = \{N_s, s \geq 0\}$ with $N_0 = 0$. We suppose that a version of N has been chosen with càdlàg paths so that even the strong Markov property holds, and that the succesive jumps are totally inaccessible. On the same space let ℓ denote a strictly positive (F_s)-stopping time. Suppose furthermore that ℓ is a strong terminal time in the sense that, for each (F_s)-stopping time R

$$\ell = R + \ell \circ \theta_R \quad \text{almost surely on} \quad \{\ell > R\}. \tag{1}$$

We let the local characteristics of N_s be given by a pair (λ, π). Here $\lambda(s,z)$ is the intensity of a jump of N at time s if $N_{s-} = z$ and $\pi(s,z) = (\pi_k(s,z))_{k\geq 0}$ is the jump size distribution with $\pi_k(s,z)$ being the probability that $\Delta N_s := N_s - N_{s-} = k$ if a jump occurs at time s and $N_{s-} = z$.

The single particle offspring structure is captured by putting

$$\begin{aligned} h(s,z\,;q) &:= E(q^{\Delta N_s} | N_{s-} = z) = \sum_k q^k \pi_k(s,z), \quad 0 \leq q \leq 1 \\ m(s,z) &:= E(\Delta N_s | N_{s-} = z) = \sum_k k\pi_k(s,z). \end{aligned}$$

Initially we will make the following near minimal assumption. Denote by $C_{loc}(R_+ \times Z_+)$ the set of measurable functions $\phi(s,z)$ on $R_+ \times Z_+$ that are right-continuous in s and satisfies

$$\|\phi\|_b := \sup_{s\leq b,\, z\in Z_+} \phi(s,z) < \infty \;\text{ for all }\; b > 0.$$

Assume

$$\lambda(s,z),\;\; \pi_k(s,z),\; k\geq 1,\text{ and } m(s,z) \;\text{ belong to }\; C_{loc}(R_+\times Z_+). \tag{2}$$

It follows that

$$\widetilde{N}_s := N_s - \int_0^s \eta(r,N_r)\,dr\,, \qquad \eta(s,z):=m(s,z)\lambda(s,z)$$

is a $(P,(F_s))$-local martingale.

Now consider the corresponding time-homogeneous strong Markov process (s,N_s). The time variable s here and in the quantities introduced above is a label for age. This is in contrast to the absolute time variable $t\geq 0$ for which we define the age-function a_t by $a_0=a$ and $a_t=a_0+t$, $t>0$, where $a\geq 0$ is a given constant. Introduce the shifted filtration

$$\widetilde{F}_t = F_{a_t}, \quad t\geq 0$$

and define the $\widetilde{F}_t$-adapted, time-homogeneous Markov process $\{x_t, t\geq 0\}$, where

$$x_t=(a_t, N_{a_t}), \quad t\geq 0.$$

Define also a rightcontinuous $\widetilde{F}_t$-adapted process M by

$$M_t = 1_{\{\ell>a_t\}}, \quad t\geq 0,$$

and let

$$S=\inf\{t\geq 0: M_t=0\} = (\ell-a)^+.$$

Clearly $\ell-a$ and therefore S are $(\widetilde{F}_t)$-stopping times.

It is convenient to add a separate notation for the stopped reproduction process

$$\xi_t = N_{a_t\wedge\ell}$$

so

$$x_t=(a_t,\xi_t) \quad \text{on} \quad \{\ell>a_t\}=\{S>t\}.$$

Introduce $x=(a,z)$ as a generic notation for the state of a particle and let $\dagger$ denote a point adjoined to the state space $R_+\times Z_+$ of $t\mapsto x_t$. As our basic process describing the joint evolution of age and reproduction of a single particle we take the canonical subprocess $\{\widehat{x}_t, t\geq 0\}$ of $\{x_t, t\geq 0\}$ corresponding to M, which is defined by

$$\widehat{x}_t := \begin{cases} x_t & t<S \\ \dagger & t\geq S\,, \end{cases} \tag{3}$$

on a probability space $(\widehat{\Omega}, \widehat{F}, \widehat{P})$ such that

$$\widehat{E}_x[f(\widehat{x}_t)] = E_x[f(x_t)\, M_t]$$

for measurable functions f on $R_+ \times Z_+ \cup \{\dagger\}$. Here E_x and $\widehat{E}_x$ are the conditional expectations given $x_0 = x$. See Blumenthal and Getoor (1968), Ch III 3. In the sequel we will write for example $m(\widehat{x}_s)$, $h(\widehat{x}_s)$, and assume without further mention that such functions are defined on the extended state space by letting $m(\dagger) = h(\dagger) = 0$.

We now turn from the single particle setting into building a branching model for a population whose dynamics is described heuristically as follows:

- $\star$ each particle is alive for a random time which equals ℓ in distribution;
- $\star$ each particle reproduces according to the jump Markov process N_t, such that the total offspring in the period of time from s to t is given by the increase $\xi_t - \xi_s$;
- $\star$ the only interaction between particles is that the birthtime of a daughter occurs at a jump time of the ξ process for its mother;
- $\star$ at time $t = 0$ only one particle of age a which previously gave birth to z daughters is alive;

By alluding to the Ionescu-Tulcea Theorem and using the single particle characteristics (ℓ, N) as building blocks we may construct a product probability space $(\mathbf{\Omega}, \mathbf{F}, \mathbf{P})$, which provides a formal setting for the particle system described above. On this probability space we represent the state of the particle system at time t by the point measure

$$X_t = \sum_i \delta_{\hat{x}_t^i}\,,$$

where $\widehat{x}_t^i$ is the subprocess associated with the evolution of age and reproduction of particle i, and where we identify $\delta_\dagger$ with the zero measure. Since x_t is rightcontinuous we can always pick a rightcontinuous version of the function $t \mapsto X_t$. Write $\mathbf{E}_{\delta_x}$ for the expectation operator conditioned on $X_0 = \delta_x$ and denote by

$$\langle X_t, \phi\rangle = \int \phi(x)\, X_t(dx)$$

the integral of a measurable function ϕ on $R_+ \times Z_+$.

Let $\|\cdot\| = \|\cdot\|_\infty$ denote the supremum norm on $R_+ \times Z_+$ and let

$$\mathcal{C} = \{\phi \in C_{loc} : \|\phi\| < \infty\}.$$

The corresponding space of Radon measures on $R_+ \times Z_+$ is denoted by $\mathcal{M}$. It is well-known that $\mathcal{M}$ is Polish but not locally compact. Let $\mathcal{D}(R_+, \mathcal{M})$ denote the space of functions from R_+ into $\mathcal{M}$ which are right-continuous and have left limits. As usual $\mathcal{D}(R_+, \mathcal{M})$ is furnished with its Skorokhod topology.

It will be shown in Lemma 6 that for any $\phi \in \mathcal{C}$ the expected value $\mathbf{E}_{\delta_x}\langle X_t, \phi\rangle <$ is finite, hence that X_t exists in the sense that it does not explode. Moreover, Lemma 10 pertains that this expected value as a function of x is itself an element of $\mathcal{C}$. Therefore we may follow the construction introduced by Iscoe (1986), namely extending to a locally compact space $\dot{\mathcal{M}}$ of measures defined on the one-point compactification of $R_+ \times Z_+$ and then showing that X_t only charges $\mathcal{M}$. The argument continues with an application to the $\dot{\mathcal{M}}$-valued process of the basic existence theorem for Markov processes, see e.g. Theorem 9.4 in Chapter 1 of Blumenthal and Getoor (1968), having as endresult the existence of a strong Markov process $t \mapsto X_t$ with paths in $\mathcal{D}(R_+, \mathcal{M})$.

We are now in the position to formulate the scaling limit result. To account for the scaling $t \mapsto nt$ of time, and therefore age, we denote by

$$\begin{aligned} x^{(n)} &= (an, z) \\ f^{(n)}(x) &= f(a/n, z) \qquad x = (a, z), \end{aligned}$$

the appropriate scaling of points in $R_+ \times Z_+$ and functions on $R_+ \times Z_+$, respectively.

Consider the sequence of offspring generating functions

$$h_n(x; u) = 1 - \frac{1}{n}(1 - \pi_0(x)) + \frac{1}{n}\sum_{k=1}^{\infty} u^{nk}\pi_k(x), \quad n \geq 1,$$

and let $X^{(n)}$ denote the process obtained after rescaling according to the scheme:

$$\begin{aligned} \ell &\mapsto n\ell \\ \lambda(\cdot) &\mapsto \frac{1}{n}\lambda^{(n)}(\cdot) \\ h(\cdot\,; u) &\mapsto h_n^{(n)}(\cdot\,; u). \end{aligned}$$

The above scalings essentially preserve the limiting mean mass while creating a non-degenerate variance in the limit.

Theorem 1 *In addition to the assumption (2), assume*

$$C_0 := \inf_x \int_0^\infty \eta(a_s, z) P_x(\ell > a_s)\, ds > 0, \tag{4}$$

$$C_1 := \sup_x \int_0^\infty \|\eta\|_{a_s} P_x(\ell > a_s)\, ds < \infty \tag{5}$$

and

$$C_2 := \sup_x \int_0^\infty \|\eta\|_{a_s}^2 P_x(\ell > a_s)\, ds < \infty. \tag{6}$$

Suppose that the scaled process $X^{(n)}$ satisfies

$$\frac{1}{n} X_0^{(n)}(dx^{(n)}) \to \mu.$$

Then the weak convergence of processes

$$\Big\{ \frac{1}{n}\, X_{nt}^{(n)}(dx^{(n)}),\, t \geq 0 \Big\} \longrightarrow \Big\{ Y_t(dx),\, t \geq 0 \Big\},$$

where $Y_0 = \mu$, holds in $\mathcal{D}(R_+, \mathcal{M})$. The mean

$$S_t\phi(x) = \mathbf{E}_{\delta_x}\langle Y_t, \phi\rangle \tag{7}$$

of the the limiting branching measure-valued Markov process Y_t given $Y_0 = \delta_x$ is the unique solution of the integral equation

$$S_t\phi(x) = \phi(a_t, z) P_x(\ell > a_t) + \int_0^t S_{t-s}\phi(0)\eta(a_s, z) P_x(\ell > a_s)\, ds \tag{8}$$

and the process Y_t is characterized by its log-Laplace function

$$V_t\phi(x) = -\log \mathbf{E}_{\delta_x} \exp -\langle Y_t, \phi\rangle,$$

being the unique solution of the nonlinear integral equation

$$V_t\phi(x) = S_t\phi(x) - \int_0^t S_s\left(\lambda\, \Psi(V_{t-s}\phi(0))\right)(x)\, ds, \tag{9}$$

where

$$\Psi(u) = \Psi(x\,; u) := h(x\,; e^{-u}) - 1 + m(x)u.$$

Remark 2 The assumption (4) is used only to prove the tightness part of the weak convergence statement. Assumption (5) provides the basic bound required to carry out the proof of convergence in distribution, except at one instance (proof of Lemma 14) where assumption (6) had to be added.

We will now investigate conditions under which the limit process Y_t is a superprocess, that is, there exists an underlying Markov process with state space $R_+ \times Z_+$ such that the operator S_t in (7) is the corresponding Markov semigroup.

Let L_t denote the renewal counting process obtained from a sequence of independent random variables $\ell_1, \ell_2, \ldots$ each with the same distribution as that of the lifelengths ℓ in the original population model. Let α_t, $t \geq 0$, denote the corresponding current age process defined by

$$\alpha_t = t - \sum_{k=1}^{L_t} \ell_k, \qquad \alpha_0 = a.$$

Corollary 3 *Suppose the lifelength ℓ is linked in exponential fashion to the offspring mean m and the reproduction intensity λ, via $\eta = m\lambda$, such that*

$$P_x(\ell > a_t) = e^{-\int_0^t \eta(a_s, z)\, ds} \quad x = (a, z). \tag{10}$$

In this case the limit process Y_t is the superprocess associated with the triple $((\alpha_t, z), \lambda, \psi)$, where (α_t, z), $t \geq 0$ is the underlying Markov process (with trivial second component), λ is the spatially non-homogeneous branching rate, and ψ denotes the non-local branching function given by

$$\psi(u\,; x) = \sum_{k=0}^{\infty} \pi_k(x) \left(e^{-\langle k\delta_0, u\rangle} - 1 + \langle k\delta_0, u\rangle \right).$$

Thus

$$S_t\phi(x) = E_x(\phi(\alpha_t, z))$$

and the log-Laplace function $V_t\phi$ is the unique solution to

$$V_t\phi(x) = E_x \left[\phi(\alpha_t, z) - \int_0^t \psi(V_{t-s}\phi\,; (\alpha_s, z))\lambda(\alpha_s, z)\, ds \right]. \tag{11}$$

In the case where

$$\lambda(x) \equiv 1$$

the limit process is characterized as the unique solution of the following martingale problem. For each real-valued, differentiable function f, and each locally bounded function $\phi(x) = \phi(a,z)$ on $R_+ \times Z_+$ which has a bounded derivative $\phi_a'(x)$ in the first coordinate, the process

$$f(\langle Y_t,\phi\rangle) - \int_0^t f'(\langle Y_s,\phi\rangle)\langle Y_s,\mathcal{L}\phi\rangle\,ds$$
$$-\int_0^t \sum_{k=0}^{\infty}\langle Y_s,\pi_k\rangle\left(f(\langle Y_s,\phi\rangle + k\phi(0)) - f(\langle Y_s,\phi\rangle) - f'(\langle Y_s,\phi\rangle)k\phi(0)\right)ds,$$

with

$$\mathcal{L}\phi(x) = \phi_a'(a,z) + m(x)\phi(0) - m(x)\phi(x),$$

is a local martingale. In particular,

$$H_t(\phi) = \langle Y_t,\phi\rangle - \int_0^t \langle Y_s,\mathcal{L}\phi\rangle\,ds$$

is a local martingale with quadratic variation process

$$\langle H(\phi)\rangle_t = \phi(0)^2\int_0^t \langle Y_s,m_2\rangle\,ds \qquad m_2(x) = \sum_{k=1}^{\infty}k^2\pi_k(x).$$

3 Properties of the expectation measure

In this section we study the expectation measure of X_t which we denote by

$$T_t\phi(x) = \mathbf{E}_{\delta_x}\langle X_t,\phi\rangle.$$

We begin by introducing the functional $C = \{C_t, t\geq 0\}$ defined by

$$C_t := \int_0^t \eta(x_r)\,M_r\,dr,$$

and summarize the properties of M, $\{x_t, t\geq 0\}$ and C as follows.

Lemma 4 *The family $M = \{M_t, t\geq 0\}$ defines a multiplicative functional of the strong Markov process $\{x_t, t\geq 0\}$, i.e.*

$$M_{t+s} = M_t(M_s\circ\theta_t) \quad \text{almost surely for each } t,s\geq 0.$$

The subprocess $\{\widehat{x}_t, t\geq 0\}$ has the strong Markov property. The family C defines a continuous additive functional of the pair process (x,M), i.e.

$$C_{t+s} = C_t + M_t(C_s\circ\theta_t) \quad \text{almost surely.}$$

Proof For the proof of the lemma we simply refer to Blumenthal and Getoor (1968), from which we quote the relevant results as follows.

First of all, the strong Markov property of $\{x_t, t \geq 0\}$ and the multiplicative property of M are immediate from their definitions. In view of the assumption (1) of ℓ being a strong (F_t)-terminal time, the random time S is a $(\widetilde{F}_t)$-strong terminal time. However, if and only if this holds, then M is even a strong multiplicative funtional, i.e

$$E_x[f(x_{t+T})M_{t+T}] = E_x E_{x_T}[f(x_t)M_t]M_T,$$

for all x, t, $(\widetilde{F}_t)$-stopping times T and bounded functions f; see III (4.26). But then the the strong Markov property of $\{x_t, t \geq 0\}$ carries over to its subprocess $\{\widehat{x}_t, t \geq 0\}$; Proposition III (3.12). The final part of the lemma is now an application of Proposition IV (1.6) with $R = \infty$ and $B_t = t$. We repeat that the quoted results in the proof refer to Blumenthal and Getoor (1968).

Recall that we have put $\xi_t = N_{a_t \wedge \ell}$ and $x_t = (a_t, \xi_t)$ on $\{\ell > a_t\}$.

Lemma 5 *There is a martingale $\widetilde{\xi}_t$ such that the continuous additive functional C_t is the $(\widetilde{F}_t)$-compensator of the process ξ_t,*

$$\xi_t - \xi_0 = C_t + \widetilde{\xi}_t\,. \tag{12}$$

The counting process

$$\Lambda_t = \sum_{s \leq t} 1_{\{\xi_s - \xi_{s-} > 0\}}$$

has the representation

$$\Lambda_t - \Lambda_0 = \int_0^t \lambda(x_r) M_r \, dr + \widetilde{\Lambda}_t, \tag{13}$$

where $\widetilde{\Lambda}_t$ is a $((\widetilde{F}_t), P)$-local martingale.

Proof Clearly

$$\widetilde{\xi}_t := \widetilde{N}_{a_t \wedge \ell} - \widetilde{N}_{a_0 \wedge \ell}$$

is a $((\widetilde{F}_t), P)$-local martingale. Hence

$$\begin{aligned} \xi_t - \xi_0 &= N_{a_t \wedge \ell} - N_{a_0 \wedge \ell} \\ &= 1_{\{\ell > a_0\}} \int_{a_0}^{a_t} \eta(s, N_s) 1_{\{\ell > s\}} \, ds + \widetilde{\xi}_t \end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \eta(a_r, N_{a_r}) 1_{\{\ell > a_r\}}\, dr + \widetilde{\xi}_t \\
&= \int_0^t \eta(x_r)\, M_r\, dr + \widetilde{\xi}_t = C_t + \widetilde{\xi}_t\,. \qquad (14)
\end{aligned}
$$

A similar argument shows the second statement.

We now introduce the characteristic of the additive functional C, defined as

$$
A_x(t) := E_x(\xi_t - \xi_0) = E_0(N_{(a+t)\wedge\ell} - N_a \mid \ell > a, N_a = z) = E_x C_t, \qquad (15)
$$

the expected number of offspring up to time t for a particle with initial state $x = (a, z)$, and define

$$
U(t) = \sum_{k=0}^{\infty} A_0^{*k}(t) \quad \text{and} \quad V_x(t) = A_x * U(t), \qquad (16)
$$

where $A^{*0} = 1$ and A^{*k}, $k \geq 1$, is the k-fold convolution of A with itself. We remark that since C_t is absolutely continuous, all functions $A_x(t)$, $U(t)$ and $V_x(t)$ are absolutely continuous functions of t, even if this is not made explicit below where we write $A_x(dt)$, etc. In particular

$$
A_x(t) = E_x \int_0^t \eta(x_s) M_s\, ds.
$$

Lemma 6 *For any function $\phi \in C_{loc}(R_+ \times Z_+)$ we have $T_t\phi \in C_{loc}(R_+ \times Z_+)$ for each t and the mean value operator is given by*

$$
T_t\phi(x) = \widehat{E}_x(\phi(\widehat{x}_t)) + \int_0^t \widehat{E}_0(\phi(\widehat{x}_{t-s}))\, V_x(ds).
$$

Proof Fix an initial state x. If the initial particle is alive at time t, there is an atom δ_{x_t} which is part of X_t. In other words, under $\mathbf{P}_{\delta_x}$ the initial particle adds $\delta_{(a+t,\xi_{a+t})}$ to the measure-valued state X_t. Any other contribution to X_t originates from jumps of ξ_s in the interval $0 \leq s \leq t$, that is, jumps of N_s for $a \leq s \leq a_t$. Each such jump, if any, produces $\Delta\xi_s := \xi_s - \xi_{s-}$ copies of the process starting with a newborn particle at time s (of age 0 and with no previous offspring) and evolving over the time span $[s, t]$. Hence

$$
T_t\phi(x) = E_x(\phi(x_t)\,;\, \ell > a_t) + E_x \int_0^t T_{t-s}\phi(0)\, d\xi_s, \qquad (17)
$$

which can be rewritten as

$$T_t\phi(x) = E_x(\phi(x_t)\, M_t) + \int_0^t T_{t-s}\phi(0)\, A_x(ds). \tag{18}$$

For $x = 0 = (0,0)$ this is a renewal type equation associated with the possibly improper function $A_0(t)$. Since $\eta \in C_{loc}(R_+ \times Z_+)$ and therefore $A_0(t) < \infty$ for each t there exists a bounded solution. Upon solving at $x = 0$,

$$T_t\phi(0) = \int_0^t E_0(\phi(x_{t-s})\, M_{t-s})\, U(ds) = \int_0^t \widehat{E}_0(\phi(\widehat{x}_{t-s}))\, U(ds),$$

the desired representation follows immediately for any $x \geq 0$. Moreover, it is simple to check that both $E_{(a,z)}(\phi(x_t)\,;\, \ell > a_t)$ and, using (15), $A_{(a,z)}(t)$ are rightcontinuous in a. The rightcontinuity of $T_t\phi(a,z)$ in a then follows from (17).

We wish to verify now the continuity in t and the semigroup property of T_t.

Lemma 7 *For each ϕ in $C_{loc}(R_+ \times Z_+)$ the map $t \mapsto T_t\phi$ is right continuous on any compact interval.*

Proof Fix a time interval $[0, \tau]$. For any x putting

$$\Delta_h(t,x) = |T_{t+h}\phi(x) - T_t\phi(x)|, \quad t + h \leq \tau,$$

we obtain from (18)

$$\begin{aligned} \Delta_h(t,x) \leq\ & |E_x[\phi(x_{t+h})\, M_{t+h} - \phi(x_t)\, M_t]| \\ & + \int_t^{t+h} T_{t+h-s}\phi(0)\, A_x(ds) + \int_0^t \Delta_h(t-s,0)\, A_x(ds). \end{aligned}$$

By Lemma 6,

$$T_t\phi(x) \leq \|\phi\|_{a_\tau}(1 + V_x(\tau)) \quad t \in [0,\tau].$$

Then the imposed regularity imply that the three upper bounds

$$|E_x[(\phi(x_{t+h}) - \phi(x_t))\, M_{t+h}]| \leq E_x|\phi(x_{t+h}) - \phi(x_t)|,$$

$$E_x[\phi(x_t)\,(M_t - M_{t+h})] \leq \|\phi\|_{a_\tau}\, P_x(a_t < \ell \leq a_{t+h}),$$

and

$$\int_t^{t+h} T_{t+h-s}\phi(0)\, A_x(ds) \le \|\phi\|_{a_\tau} (1+V_x(\tau))\, (A_x(t+h)-A_x(t))$$

can all be made arbitrarily small for small h. Thus for $\varepsilon > 0$ and sufficiently small h,

$$\Delta_h(t,x) \le \varepsilon + \int_0^t \Delta_h(t-s,0)\, A_x(ds).$$

We have $A_x(\tau) < \infty$. Hence if we can show that $\sup_{t\le\tau} \Delta_h(t,0) \to 0$ as $h \to 0$ then $\Delta_h(t,x) \to 0$ for any $t \le \tau$, which is the claim of the lemma. However, for $x = 0$

$$\Delta_h(t,0) \le \varepsilon + \int_0^t \Delta_h(t-s,0)\, A_0(ds),$$

hence

$$\Delta_h(t,0) \le \varepsilon \exp A_0(\tau), \quad 0 \le t \le \tau,$$

by Gronwall's lemma, finishing the proof.

Lemma 8 *For each function ϕ in $C_{loc}(R_+ \times Z_+)$ and $s,t \ge 0$,*

$$T_t(T_s\phi)(x) = T_{t+s}\phi(x). \tag{19}$$

Proof Since $s \mapsto T_{t-s}\phi$ is left continuous according to Lemma 7, in equation (17) we can replace the integral with respect to ξ_t with an integration over its compensator C_t, c.f. (14),

$$\begin{aligned} T_t\phi(x) &= E_x(\phi(x_t)\,;\, \ell > a_t) + E_x \int_0^t T_{t-s}\phi(0)\, d\xi_s \\ &= E_x\Big[\phi(x_t)\, M_t + \int_0^t T_{t-s}\phi(0)\, \eta(x_s) M_s\, ds\Big]. \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned} T_t(T_s\phi)(x) &= E_x\Big[T_s\phi(x_t)\, M_t + \int_0^t T_{t-r}(T_s\phi)(0)\, dC_r\Big] \\ &= E_x\Big[E_{x_t}[\phi(x_{t+s})\, M_{t+s}]\, M_t + \int_0^t T_{t-r}(T_s\phi)(0)\, dC_r \\ &\qquad + E_{x_t} \int_0^s T_{s-r}\phi(0)\, dC_r(\theta_t)\, M_t\Big]. \end{aligned}$$

The first term on the right side simplifies by the strong Markov property. Moreover,

$$E_x E_{x_t} \int_0^s T_{s-r}\phi(0)\, dC_r(\theta_t)\, M_t = E_x \int_t^{t+s} T_{t+s-r}\phi(0)\, dC_r$$

by the last relation in Lemma 4. Thus

$$\begin{aligned} T_t(T_s\phi)(x) \; &= E_x\Big[\phi(x_{t+s})\, M_{t+s} + \int_0^t T_{t-r}(T_s\phi)(0)\, dC_r \\ &\quad + \int_t^{t+s} T_{t+s-r}\phi(0)\, dC_r\Big]. \end{aligned}$$

On the other hand

$$T_{t+s}\phi(x) = E_x\Big[\phi(x_{t+s})\, M_{t+s} + \int_0^{t+s} T_{t+s-r}\phi(0)\, dC_r\Big].$$

Hence the function defined by

$$T_u^s\phi(x) = T_{t-u}T_s\phi(x) \quad u \le t \qquad \text{and} \qquad T_{t+s-u}\phi(x) \quad u > t,$$

solves the same equation as does T_u, $u \ge 0$. Clearly the solution must be unique, therefore $T_t(T_s) = T_{t+s}$.

We turn to properties of T_t which are based not only on (2) but also on the assumption (5), which was not in use up to this point. We introduce

$$J_v(x) := \int_v^\infty E_x[\eta(x_s)M_s]\, ds \qquad J(x) := J_0(x),$$

in particular

$$J(0) := \int_0^\infty E_0[\eta(x_s)M_s]\, ds,$$

and call the process X subcritical if $J(0) < 1$, critical if $J(0) = 1$ and supercritical if $J(0) > 1$ and finite.

It turns out that $\langle X_t, J\rangle$ behaves nicely. A useful fact is e.g. that in the critical case this process is a martingale. We remark here that similar martingales have been used successfully in the theory of general branching processes, c.f. Nerman (1984) and Jagers (1989).

By (5) the uniform bounds

$$A_x(t) \le J(x) \le \|J\| < \infty \tag{21}$$

and

$$V_x(t) \le U(t)A_x(t) \le U(t)J(x) \le U(t)\|J\| < \infty \tag{22}$$

hold.

Lemma 9 *The process $\{\langle X_t, J\rangle, t \geq 0\}$ defines a* $\mathbf{P}$*-supermartingale, a* $\mathbf{P}$*-martingale or a* $\mathbf{P}$*-submartingale in the subcritical, the critical and the supercritical case, respectively.*

Proof We compute

$$
\begin{aligned}
T_t(J_v)(x) &= E_x[J_v(x_t)M_t] + \int_0^t E_0[J_v(x_{t-s})M_{t-s}]\, V_x(ds) \\
&= \int_v^\infty E_x[M_t E_{x_t}[\eta(x_s)M_s]]\, ds \\
&\quad + \int_0^t \int_v^\infty E_0[M_{t-s}E_{x_{t-s}}[\eta(x_r)M_r]]\, dr\, V_x(ds) \\
&= \int_v^\infty E_x[\eta(x_{s+t})M_{s+t}]\, ds + \int_0^t \int_v^\infty E_0[\eta(x_{r+t-s})M_{r+t-s}]\, dr\, V_x(ds) \\
&= J_{v+t}(x) + \int_0^t J_{v+t-s}(0)\, V_x(ds),
\end{aligned}
$$

using the strong Markov property of $\widehat{x}_t$ from Lemma 4. Observe that $J_v(x) = A_x(\infty) - A_x(v)$. Thus

$$
\begin{aligned}
T_t(J_v)(x) &= J_{v+t}(x) + \int_0^t (A_0(\infty) - A_0(v+t-s))\, V_x(ds) \\
&= J_{v+t}(x) + A_0(\infty)V_x(t) - \int_0^{v+t} A_0(v+t-s)\, V_x(ds) \\
&\quad + \int_t^{v+t} A_0(v+t-s)\, V_x(ds).
\end{aligned}
$$

Now $A_0 * V_x = A_0 * A_x * U_0 = A_x * (U_0 - 1) = V_x - A_x$, hence

$$
T_t(J_v)(x) = J(x) + J(0)V_x(t) - V_x(t+v) + \int_t^{v+t} A_0(v+t-s)\, V_x(ds).
$$

Taking $v = 0$ we obtain

$$
T_t(J)(x) = J(x) + (J(0) - 1)\, V_x(t).
$$

Hence

$$
\begin{aligned}
\mathbf{E}[\langle X_{t+s}\,, J\rangle |\, \mathbf{F}_s] &= \langle X_s\,, T_t(J)\rangle \\
&= \langle X_s\,, J\rangle + (J(0) - 1)\, \langle X_s\,, V_\cdot(t)\rangle, \qquad (23)
\end{aligned}
$$

which immediately shows the martingale properties of $\langle X_t\,, J\rangle$.

Lemma 10 *We have*

$$\|T_t\phi\| \leq \|\phi\|\Big(1 + U(t)\|J\|\Big), \quad \phi \in \mathcal{C},$$

and the map $t \mapsto T_t$ *defines a bounded operator on* $\mathcal{C}$ *into itself. Furthermore*

$$\int_0^t T_s(\eta)(x)\,ds \leq A_x(t)\Big(1 + \|\eta\|_t \int_0^t U(s)\,ds\Big).$$

Proof By Lemma 6,

$$|T_t\phi(x)| \leq |E_x(\phi(x_t)\,M_t)| + \|\phi\|_t\,V_x(t), \quad \phi \in C_{loc}(R_+ \times Z_+), \tag{24}$$

Together with (22) this yields the bound for $\|T_t\phi\|$. In combination with Lemma 6 applied to functions $\phi \in \mathcal{C}$ it shows the boundedness of T_t.

Next integrate (24) with ϕ taken to be the nonnegative function η. Then

$$\int_0^t T_s(\eta)(x)\,ds \leq A_x(t) + \|\eta\|_t \int_0^t V_x(s)\,ds.$$

Use again (22) to get the final estimate stated in the lemma.

4 Nonlinear integral equation

Our first aim in this section is to derive a preliminary form of an integral equation satisfied by the function

$$Q_t\phi(x) = 1 - \mathbf{E}_{\delta_x} \exp -\langle X_t, \phi\rangle,$$

where we restrict to nonnegative functions ϕ. We point out that throughout this section only the assumption (2) is used and not (4)-(6).

Lemma 11 *For each nonnegative function* $\phi \in C_{loc}(R_+, Z_+)$ *the function* $Q_t\phi$ *satisfies the equation*

$$\begin{aligned} Q_t\phi(x) &= \widehat{E}_x(1 - e^{-\phi(\hat{x}_t)}) + \int_0^t Q_{t-s}\phi(0)\,A_x(ds) \\ &\quad -\widehat{E}_x \int_0^t Q_{t-s}\phi(\widehat{x}_s)\Big\{1 - h(\widehat{x}_s\,; 1 - Q_{t-s}\phi(0))\Big\}\lambda(\widehat{x}_s)\,ds \\ &\quad -\widehat{E}_x \int_0^t \Big\{h(\widehat{x}_s\,; 1 - Q_{t-s}\phi(0)) - 1 + Q_{t-s}\phi(0)m(\widehat{x}_s)\Big\}\lambda(\widehat{x}_s)\,ds. \end{aligned}$$

Proof Fix x and let $\tau_1, \tau_2, \ldots$ denote the succesive times between jumps of $\xi_t - \xi_0$, $t \geq 0$. In the case $\xi_\ell = 0$, we let $\tau_i = \infty$, $i \geq 1$. Clearly,

$$\begin{aligned} 1 - Q_t\phi(x) &= E_x(e^{-\phi(x_t)}\,;\tau_1 > t, \ell > a_t) \\ &\qquad + P_x(\tau_1 > t, \ell \leq a_t) + \mathbf{E}_{\delta_x}\, e^{-\langle X_t, \phi\rangle}\, 1_{\{\tau_1 \leq t\}}\,. \end{aligned}$$

Let $X_t(x)$ denote the measure X_t given $X_0 = \delta_x$. On the set $\{\tau_1 \leq t\}$ we have the basic decomposition

$$X_t(x) = X_{t-\tau_1}(x_{\tau_1}) + \sum_{j=1}^{\Delta\xi_{\tau_1}} X^j_{t-\tau_1}(0)\,,$$

where the additional upper index $j \geq 1$ signifies independent copies. Hence

$$\mathbf{E}_{\delta_x}\, e^{-\langle X_t, \phi\rangle} 1_{\{\tau_1 \leq t\}} = E_x(1 - Q_{t-\tau_1}\phi(x_{\tau_1}))(1 - Q_{t-\tau_1}\phi(0))^{\Delta\xi_{\tau_1}}\, 1_{\{\tau_1 \leq t\}},$$

so

$$\begin{aligned} Q_t\phi(x) &= E_x(1 - e^{-\phi(x_t)}\,;\ell > a_t, \tau_1 > t) + P_x(\tau_1 \leq t) \\ &\qquad - E_x(1 - Q_{t-\tau_1}\phi(x_{\tau_1}))(1 - Q_{t-\tau_1}\phi(0))^{\Delta\xi_{\tau_1}}\, 1_{\{\tau_1 \leq t\}}. \end{aligned}$$

Invoking the generating function $h(x\,;q)$, the third term on the right side in the above equation may be written

$$E_x(1 - Q_{t-\tau_1}\phi(x_{\tau_1}))\, h(x_{\tau_1}\,; 1 - Q_{t-\tau_1}\phi(0))\, 1_{\{\tau_1 \leq t\}}.$$

Thus, rewriting,

$$\begin{aligned} Q_t\phi(x) &= E_x Q_{t-\tau_1}\phi(x_{\tau_1})\, 1_{\{\tau_1 \leq t\}} + E_x(1 - e^{-\phi(x_t)}\,;\ell > a_t, \tau_1 > t) \qquad (25) \\ &\qquad + E_x(1 - Q_{t-\tau_1}\phi(x_{\tau_1}))\{1 - h(x_{\tau_1}\,; 1 - Q_{t-\tau_1}\phi(0))\}\, 1_{\{\tau_1 \leq t\}}. \end{aligned}$$

We shall iterate the first term on the right side in (25). First we have

$$\begin{aligned} Q_{t-\tau_1}\phi(x_{\tau_1}) &= E_{x_{\tau_1}} Q_{t-\tau_1-\tau_2}\phi(x_{\tau_1+\tau_2})\, 1_{\{\tau_2 \leq t-\tau_1\}} \\ &\qquad + E_{x_{\tau_1}}(1 - e^{-\phi(x_{t-\tau_1})}\,;\ell - a_{\tau_1} > t - \tau_1, \tau_2 > t - \tau_1) \\ &\qquad + E_{x_{\tau_1}}(1 - Q_{t-\tau_1-\tau_2}\phi(x_{\tau_1+\tau_2})) \\ &\qquad\qquad \times\{1 - h(x_{\tau_1+\tau_2}; 1 - Q_{t-\tau_1-\tau_2}\phi(0))\}\, 1_{\{\tau_2 \leq t-\tau_1\}}. \end{aligned}$$

Hence, writing $S_k = \sum_{i=1}^k \tau_i$ (the succesive jump times of the process $\xi_t - \xi_0$, $t \geq 0$),

$$\begin{aligned} E_x Q_{t-\tau_1}\phi(x_{\tau_1})\, 1_{\{\tau_1 \leq t\}} &= E_x E_{x_{S_1}} Q_{t-S_2}\phi(x_{S_2})\, 1_{\{S_2 \leq t\}} \\ &\quad + E_x E_{x_{S_1}}(1 - e^{-\phi(x_{t-S_1})}\,;\ \ell > a_t, S_1 \leq t < S_2) \\ &\quad + E_x E_{x_{S_1}}(1 - Q_{t-S_2}\phi(x_{S_2}))\{1 - h(x_{S_2}; 1 - Q_{t-S_2}\phi(0))\}\, 1_{\{S_2 \leq t\}}. \end{aligned}$$

The intermediate expectations $E_{x_{S_1}}$ drop out by the strong Markov property of x_t. Hence

$$\begin{aligned} E_x Q_{t-S_1}\phi(x_{S_1})\, 1_{\{S_1 \le t\}} &= E_x Q_{t-S_2}\phi(x_{S_2})\, 1_{\{S_2 \le t\}} \\ &\quad + E_x(1 - e^{-\phi(x_t)}\,;\, \ell > a_t, S_1 \le t < S_2) \\ &\quad + E_x(1 - Q_{t-S_2}\phi(x_{S_2}))\{1 - h(x_{S_2}; 1 - Q_{t-S_2}\phi(0))\}\, 1_{\{S_2 \le t\}}. \end{aligned}$$

Reinsert this into expression (25) to obtain

$$\begin{aligned} Q_t\phi(x) &= E_x Q_{t-S_2}\phi(x_{S_2})\, 1_{\{S_2 \le t\}} + E_x(1 - e^{-\phi(x_t)}\,;\, \ell > a_t, S_2 > t) \\ &\quad + E_x \sum_{j=1}^{2} (1 - Q_{t-S_j}\phi(x_{S_j}))\{1 - h(x_{S_j}; 1 - Q_{t-S_j}\phi(0))\}. \end{aligned}$$

Since $\xi_t < \infty$ a.s. for each t, if for some j we have $t < S_j < \infty$ then also $a_t < \ell$. Therefore, by repeating the iterative steps over all jumps of ξ, we get

$$\begin{aligned} Q_t\phi(x) &= E_x(1 - e^{-\phi(x_t)}\,; \ell > a_t) \\ &\quad + E_x \sum_{j=1}^{\infty} (1 - Q_{t-S_j}\phi(x_{S_j}))\{1 - h(x_{S_j}; 1 - Q_{t-S_j}\phi(0))\} \\ &= E_x\Big[(1 - e^{-\phi(x_t)})\, M_t \\ &\quad + \int_0^t (1 - Q_{t-s}\phi(x_{s-}))\{1 - h(x_{s-}; 1 - Q_{t-s}\phi(0))\} d\Lambda_s\Big]. \end{aligned}$$

Along the same line as in Lemma 7, it can be shown from the above relation that $t \mapsto Q_t\phi$ is right continuous on any compact interval. In the last term we then apply relation (13) in order to replace the counting process Λ with its compensator,

$$\begin{aligned} Q_t\phi(x) &= E_x\Big[(1 - e^{-\phi(x_t)})\, M_t \\ &\quad + \int_0^t (1 - Q_{t-s}\phi(x_s))\{1 - h(x_s\,; 1 - Q_{t-s}\phi(0))\}\lambda(x_s) M_s\, ds\Big]. \end{aligned}$$

By adding and subtracting $\int_0^t Q_{t-s}\phi(0)\, A_x(ds)$ and rewriting in terms of the subprocess measure this equation takes the form stated in the lemma.

We proceed by elaborating on the equation for $Q_t\phi$ just obtained by using a renewal equation type argument. The final form of the integral equation we are going to use exhibits more clearly the role in this context of the semigroup T_t. Introduce the notation

$$\Phi(u) = \Phi(x\,;u) := h(x\,;1-u) - 1 + m(x)u.$$

Proposition 12 *The function $Q_t\phi$ satisfies the equation*

$$\begin{aligned} Q_t\phi(x) = T_t(1-e^{-\phi})(x) - \int_0^t T_s\left(\lambda\,\Phi(Q_{t-s}\phi(0))\right)(x)\,ds \\ - \int_0^t T_s\left(\lambda\,Q_{t-s}\phi\,\{1-h(\cdot\,;1-Q_{t-s}\phi(0))\}\right)(x)\,ds. \end{aligned}$$

Proof Define

$$\begin{aligned} K(t,x) \;:= & \;\widehat{E}_x \int_0^t Q_{t-s}\phi(\widehat{x}_s)\Big\{1-h(\widehat{x}_s\,;1-Q_{t-s}\phi(0))\Big\}\lambda(\widehat{x}_s)\,ds \\ & +\widehat{E}_x \int_0^t \Phi(\widehat{x}_s\,;Q_{t-s}\phi(0))\lambda(\widehat{x}_s)\,ds. \end{aligned} \tag{26}$$

The equation in Lemma 11 is

$$Q_t\phi(x) = \widehat{E}_x\Big[1-e^{-\phi(\widehat{x}_t)}\Big] + \int_0^t Q_{t-s}\phi(0)\,A_x(ds) - K(t,x)\,.$$

Solving the renewal equation at $x=0$,

$$Q_t\phi(0) = \int_0^t \widehat{E}_0\Big[1-e^{-\phi(\widehat{x}_{t-s})}\Big]\,U(ds) - K(\cdot,0)*U(t),$$

whence

$$\begin{aligned} Q_t\phi(x) \;= & \;\widehat{E}_x\Big[1-e^{-\phi(\widehat{x}_t)}\Big] + \int_0^t \widehat{E}_0\Big[1-e^{-\phi(\widehat{x}_{t-s})}\Big]\,V_x(ds) \\ & -K(t,x) - K(\cdot,0)*V_x(t). \end{aligned}$$

In view of Lemma 6 we have shown

$$Q_t\phi(x) = T_t(1-e^{-\phi})(x) - K(t,x) - K(\cdot,0)*V_x(t). \tag{27}$$

In order to compute the convolution term $K(\cdot,0)*V_x(t)$, consider a locally bounded function $F(t,x)$, $t\geq 0$, $x\in R_+\times Z$, and put

$$\widetilde{K}(t,x) = \widehat{E}_x \int_0^t F(t-s,\widehat{x}_s)\,ds.$$

Then

$$\begin{aligned}\widetilde{K}(\cdot,0)*V_x(t) &= \widehat{E}_0\int_0^t\int_0^{t-r} F(t-r-s,x_s)\,ds\,V_x(dr)\\ &= \widehat{E}_0\int_0^t\int_r^t F(t-s,x_{s-r})\,ds\,V_x(dr)\\ &= \int_0^t\int_0^s \widehat{E}_0[F(t-s,x_{s-r})]\,V_x(dr)\,ds.\end{aligned}$$

Therefore, again using Lemma 6,

$$\begin{aligned}&\widetilde{K}(t,x)+\widetilde{K}(\cdot,0)*V_x(t)\\ &\quad=\int_0^t\Big\{\widehat{E}_x[F(t-s,\widehat{x}_s)]+\int_0^s\widehat{E}_0[F(t-s,\widehat{x}_{s-r})]\,V_x(dr)\Big\}ds\\ &\quad=\int_0^t T_s(F(t-s,\cdot))(x)\,ds.\end{aligned}$$

By an application of this relation to the function $K(t,x)$ in (26), we obtain from Equation (27) the final form of the integral equation which is stated in the proposition.

5 Proof of the main result

Recall that under the rescaling scheme given in Section 2, the evolution of the empirical measure process $X_{nt}^{(n)}$, $t\geq 0$, of the particle motions $\widehat{x}_{nt}^{(n)}$ is governed by the dynamics generated by the scaled parameters $n\ell$, $n^{-1}\lambda^{(n)}$ and $h_n^{(n)}$. We cover in this section proofs of the convergence of finite-dimensional distributions and the verification of tightness in $\mathcal{M}$. For similar proofs see e.g. Dawson (1993), Chapter 4, and Kaj and Sagitov (1998).

Uniform convergence of one-dimensional distributions.

The first lemma in this section shows that the law of $t\mapsto\widehat{x}_t$ is degenerate in the scaling limit. Indeed, given $x_0^{(n)}=(an,z)$ we have $x_{nt}^{(n)}=(n(a+t),\xi_{nt}^{(n)})$ on $\{n\ell>a_{nt}\}=\{\ell>a_t\}$ and we will show that the second component converges to its initial value z. In fact, the lemma shows that if $\xi_0^{(n)}=z$ then $\xi_{nt}^{(n)}$ converges to z distribution under both P and $\widehat{P}$ for each $t\geq 0$ as $n\to\infty$.

Lemma 13 *For each t and $n\geq 2C_1$*

$$P_{x^{(n)}}(\xi_{nt}^{(n)}>z)\leq 2C_1/n$$

and

$$P_{x^{(n)}}(\xi_{nt}^{(n)} > z\,;\,\ell > a_t) \le 2t\|\eta\|_{a_t} P_x(\ell > a_t)/n.$$

Proof Fix $x = (a, z)$. For each $k \ge 1$ let $\Lambda_t[k]$ denote the process which counts the number of jumps of $\xi_t - \xi_0$ of size k. Then

$$\Lambda_t = \sum_k \Lambda_t[k], \qquad \Lambda_t[k] := \sum_{s \le t} 1_{\{\Delta\xi_s = k\}}$$

and

$$\xi_t - \xi_0 = \sum_{k=1}^{\infty} k\Lambda_t[k],$$

where the $\Lambda[k]$ are independent by construction and where

$$\Lambda_t[k] - \int_0^t \pi_k(x_r)\lambda(x_r)M_r\,dr, \quad k \ge 1 \tag{28}$$

are $(P, (\widetilde{F}_t))$-local martingales.

Now

$$P_{x^{(n)}}(\xi_{nt}^{(n)} = z) = \prod_{k=1}^{\infty} P_{x^{(n)}}(\Lambda_{nt}^{(n)}[nk] = 0)$$

so

$$-n \log P_{x^{(n)}}(\xi_{nt}^{(n)} = z) = \sum_{k=1}^{\infty} -n \log(1 - P_{x^{(n)}}(\Lambda_{nt}^{(n)}[nk] \ge 1)).$$

With intensity $n^{-1}\lambda^{(n)}$, $\xi^{(n)}$ performs a jump which is of size nk with probability $n^{-1}\pi_k$ and therefore

$$\begin{aligned} E_{x^{(n)}}\Lambda_{nt}^{(n)}[nk] &= E_{x^{(n)}} \int_0^{nt} n^{-2}\pi_k^{(n)}(x_r^{(n)})\lambda^{(n)}(x_r^{(n)})M_r^{(n)}\,dr \\ &= \frac{1}{n}E_x \int_0^t \pi_k(a_r, \xi_{nr}^{(n)})\lambda(a_r, \xi_{nr}^{(n)})M_r\,dr, \quad k \ge 1. \end{aligned}$$

Hence

$$nP_{x^{(n)}}(\Lambda_{nt}^{(n)}[nk] \ge 1) \le c_k(n) := E_x \int_0^t k\pi_k(a_r, \xi_{nr}^{(n)})\lambda(a_r, \xi_{nr}^{(n)})M_r\,dr$$

(the factor k in the term on the right side is introduced only for convenience). Moreover, by (5)

$$c_k(n) \le \sum_{k=1}^{\infty} c_k(n) = E_x \int_0^t \eta(a_r, \xi_{nr}^{(n)})M_r\,dr \le J^{(n)}(x) \le C_1.$$

Now, for $n \geq 2C_1$, by Taylor expansion,

$$\begin{aligned} -n\log(1 - P_{x^{(n)}}(\Lambda_{nt}^{(n)}[nk] \geq 1)) &\leq -n\log(1 - n^{-1}c_k(n)) \\ &\leq c_k(n)\sum_{j=0}^{\infty}\frac{1}{j+1}(c_k(n)/n)^j \\ &\leq c_k(n)(1 - C_1/n)^{-1} \leq 2c_k(n). \end{aligned}$$

Thus

$$-n\log P_{x^{(n)}}(\xi_{nt}^{(n)} = z) \leq 2\sum_{k=1}^{\infty} c_k(n) = 2J^{(n)}(x) \leq 2C_1,$$

which proves the first inequality stated in the lemma.

The other inequality follows from a simple variation of the above argument. Using

$$c_k'(n) = E_x\Big[\int_0^t k\pi_k(a_r, \xi_{nr}^{(n)})\lambda(a_r, \xi_{nr}^{(n)})\, dr \Big| \, \ell > a_t\Big]$$

and observing that $\|\eta^{(n)}\|_b = \|\eta\|_b$, one obtains

$$-n\log P_{x^{(n)}}(\xi_{nt}^{(n)} = z \mid \ell > a_t) \leq 2\sum_{k=1}^{\infty} c_k'(n) \leq 2t\,\|\eta\|_{a_t}.$$

Hence

$$P_{x^{(n)}}(\xi_{nt}^{(n)} > z\,;\, \ell > a_t) \leq P_x(\ell > a_t)\Big(1 - e^{-2t\|\eta\|_{a_t}/n}\Big).$$

Define

$$T_t^{(n)}\phi(x) = \mathbf{E}_{\delta_x}\langle X_t^{(n)}, \phi\rangle.$$

Lemma 14 *For any $\phi \in C_{loc}(R_+ \times Z_+)$ the equation*

$$S_t\phi(x) = \phi(a_t, z)P_x(\ell > a_t) + \int_0^t S_{t-s}\phi(0)\eta(a_s, z)P_x(\ell > a_s)\, ds$$

has a unique semigroup solution $S_t\phi$. The difference

$$\Delta_t^{(n)}[\phi](x) := |T_{nt}^{(n)}\phi^{(n)}(x^{(n)}) - S_t\phi(x)|$$

satisfies, with $\phi \in \mathcal{C}$ and fixed t,

$$\|\Delta_t^{(n)}[\phi]\| \leq const\,\frac{1}{n},$$

and furthermore

$$\int_0^t \|\Delta_s^{(n)}[\eta]\, ds\| \leq const\,\frac{1}{n}.$$

Proof The equation for $S_t\phi$ has the same form as equation (18) for $T_t\phi$. The main difference is that $A_x(t)$ is here replaced by the simpler function $\int_0^t \eta(a_s,z)P_x(\ell > a_s)\,ds$. Hence its unique solution may even be obtained in the convolution form of Lemma 6. Clearly Lemma 7, Lemma 8 and Lemma 10 also hold for $S_t\phi$.

By equation (17)

$$\begin{aligned} T_{nt}^{(n)}\phi^{(n)}(x^{(n)}) &= E_{x^{(n)}}(\phi^{(n)}(x_{nt}^{(n)})\,;n\ell > a_{nt}) + E_{x^{(n)}}\int_0^{nt} T_{nt-s}^{(n)}\phi^{(n)}(0)d\xi_s^{(n)} \\ &= E_x\Big[\phi(a_t,\xi_{nt}^{(n)})M_t + \int_0^t T_{n(t-s)}^{(n)}\phi^{(n)}(0)\,d\xi_{ns}^{(n)}\Big]. \end{aligned}$$

The mean reproduction rate is preserved under scaling since $h_n'(\cdot,1) = m(\cdot)$, therefore

$$\begin{aligned} E_{x^{(n)}}\xi_{ns}^{(n)} - z &= E_{x^{(n)}}\int_0^{ns} n^{-1}\lambda^{(n)}(x_r^{(n)})m^{(n)}(x_r^{(n)})M_r^{(n)}\,dr \\ &= E_x\int_0^s \eta(a_r,\xi_{nr}^{(n)})M_r\,dr. \end{aligned}$$

Hence

$$T_{nt}^{(n)}\phi^{(n)}(x^{(n)}) = E_x\Big[\phi(a_t,\xi_{nt}^{(n)})M_t + \int_0^t T_{n(t-s)}^{(n)}\phi^{(n)}(0)\eta(a_s,\xi_{ns}^{(n)})M_s\,ds\Big]$$

and hence

$$\begin{aligned} \Delta_t^{(n)}[\phi](x) \;\le\; & \Big|E_x(\phi(a_t,\xi_{nt}^{(n)}) - \phi(a_t,z)\,;\,\ell > a_t)\Big| \\ & + E_x\int_0^t \Delta_{t-s}^{(n)}[\phi](0)\,\eta(a_s,\xi_{ns}^{(n)})M_s\,ds \\ & + \int_0^t S_{t-s}\phi(0)\Big|E_x(\eta(a_s,\xi_{ns}^{(n)}) - \eta(a_s,z)\,;\,\ell > a_s)\Big|\,ds. \end{aligned}$$

We have the uniform bounds

$$\begin{aligned} \Big|E_x(\phi(a_t,\xi_{nt}^{(n)}) - \phi(a_t,z)\,;\,\ell > a_t)\Big| &\le 2\|\phi\|\,P_x(\xi_{nt}^{(n)} > z\,,\,\ell > a_t) \\ &\le 2\|\phi\|\,P_x(\xi_{nt}^{(n)} > z) \le 4C_1\|\phi\|/n \end{aligned}$$

and

$$\begin{aligned} \Big|E_x(\eta(a_s,\xi_{ns}^{(n)}) - \eta(a_s,z)\,;\,\ell > a_s)\Big| &\le 2\|\eta\|_{a_s}\,P_x(\xi_{ns}^{(n)} > z\,,\,\ell > a_s) \\ &\le 4s\,\|\eta\|_{a_s}^2\,P_x(\ell > a_s)/n, \end{aligned} \tag{29}$$

by the previous Lemma 13. At this point we apply the assumption (6). It follows that for sufficiently large n, putting $C' = 4C_1\|\phi\| + 4t\sup_{r\le t} S_r\phi(0)C_2$,

$$\Delta_t^{(n)}[\phi](x) \le C'/n + \int_0^t \Delta_{t-s}^{(n)}[\phi](0)\, E_x[\eta(a_s, \xi_{ns}^{(n)})M_s]\, dr.$$

By Gronwall's inequality, as in the proof of Lemma 7,

$$\Delta_t^{(n)}[\phi](0) \le \frac{1}{n}C'e^{J^{(n)}(0)} \le \frac{1}{n}C'e^{C_1},$$

and thus, using again (5)

$$\|\Delta_t^{(n)}[\phi]\| \le C'\left(1 + C_1 e_1^C\right)/n.$$

To obtain the remaining estimate stated in the lemma it suffices to substitute the particular function η instead of ϕ in the above estimates of $\Delta_s^{(n)}[\phi]$, note that only the norm $\|\eta\|_{a_s}$ appears and observe that the integration over s yields terms that are bounded due to (6).

Define

$$Q_t^{(n)}\phi(x) = 1 - \mathbf{E}_{\delta_x}\exp -\langle X_t^{(n)}, \phi\rangle, \quad V_t^{(n)}\phi(x) = nQ_{nt}^{(n)}(\phi^{(n)}/n)(x^{(n)}).$$

Lemma 15 *The solution $V_t\phi$ of equation (9) is unique and forms a one-parameter semigroup.*

If the sequence ϕ_n converges uniformly to ϕ in $\mathcal{C}$, $\|\phi_n - \phi\| \to 0$, then for each t

$$\|V_t^{(n)}(\phi_n) - V_t\phi\| \to 0, \quad n \to \infty.$$

Proof Uniqueness follows from Gronwall's lemma. The semigroup property is immediate from the corresponding property of $S_t\phi$.

We prove

$$\Pi_t^{(n)} := \|V_t^{(n)}\phi - V_t\phi\| \to 0, \quad n \to \infty.$$

The more general statement in the lemma is then a straightforward variation. By Proposition 12

$$\begin{aligned} V_t^{(n)}\phi(x) &= nT_{nt}^{(n)}(1 - e^{-\phi^{(n)}/n})(x^{(n)}) \\ &\quad - n\int_0^t T_{ns}^{(n)}\left(\lambda^{(n)}\Phi_n^{(n)}(n^{-1}V_{t-s}^{(n)}\phi(0))\right)(x^{(n)})\, ds \\ &\quad - \int_0^t T_{ns}^{(n)}\left(\lambda^{(n)}\, V_{t-s}^{(n)}\phi\,\{1 - h_n^{(n)}(1 - n^{-1}V_{t-s}^{(n)}\phi(0))\}\right)(x^{(n)})\, ds. \end{aligned}$$

Observe that by the definition of h_n, for any $0 \le v \le n$,

$$1 - h_n(\cdot\,; 1 - n^{-1}v) = \frac{1}{n}\sum_{k=0}^{\infty} \pi_k(\cdot)\Big[1 - (1 - n^{-1}v)^{nk}\Big] \le \frac{1}{n}\, m(\cdot),$$

hence the last term in the equation for $V_t^{(n)}\phi$ is bounded above by

$$\frac{1}{n}\int_0^t T_{ns}^{(n)}\Big(\eta^{(n)}\, V_{t-s}^{(n)}\phi\Big)\,(x^{(n)})\,ds \le \frac{1}{n}\int_0^t \|V_{t-s}^{(n)}\phi\|\, T_{ns}^{(n)}(\eta^{(n)})(x^{(n)})\,ds. \quad (30)$$

Using Lemma 10 we continue estimating this term arriving at the upper bound

$$\frac{1}{n}\sup_{s\le t}\|V_s^{(n)}\phi\|\,\|J^{(n)}\|\Big(1 + \|\eta^{(n)}\|_t \int_0^t U^{(n)}(s)\,ds\Big).$$

It is clear, using the obvious inequality

$$V_t^{(n)}\phi(x) \le T_{nt}^{(n)}\phi^{(n)}(x^{(n)})$$

and in view of Lemma 10, that a few more steps will result in a uniform upper bound of the order $1/n$. We denote this remainder term $\mathcal{O}(1/n)$.

Considering now the equations for $V_t^{(n)}\phi$, obtained in this proof, and for $V_t\phi$ stated in Theorem 1 we find, exploiting the notation $\Delta^{(n)}[\phi]$ introduced in Lemma 14,

$$\begin{aligned}
\Pi_t^{(n)}(x) \le\ & \Big|T_{nt}^{(n)}\Big\{n(1 - e^{-\phi^{(n)}/n}) - \phi^{(n)}\Big\}(x^{(n)})\Big| + \Delta_t^{(n)}[\phi](x) \\
&+ \int_0^t T_{ns}^{(n)}\Big\{\lambda^{(n)}\Big|n\Phi_n^{(n)}(n^{-1}V_{t-s}^{(n)}\phi(0)) - \Psi^{(n)}(V_{t-s}^{(n)}\phi(0))\Big|\Big\}(x^{(n)})\,ds\Big| \\
&+ \Big|\int_0^t T_{ns}^{(n)}\Big\{\lambda^{(n)}\Big(\Psi^{(n)}(V_{t-s}^{(n)}\phi(0)) - \Psi^{(n)}(V_{t-s}\phi(0))\Big)\Big\}(x^{(n)})\,ds\Big| \\
&+ \int_0^t \Delta_s^{(n)}[\lambda\Psi(V_{t-s}\phi(0))](x)\,ds + \mathcal{O}(\frac{1}{n}).
\end{aligned}$$

By an elementary estimate and Lemma 14 the first two terms on the right side may be merged together with the term already denoted $\mathcal{O}(1/n)$. Observe furthermore that

$$\Phi_n(\cdot\,; v/n) = \frac{1}{n}\Big[\sum_{k=0}^{\infty}\pi_k(\cdot)(1 - n^{-1}v)^{nk} - 1 + m(\cdot)v\Big], \quad 0 \le v \le n,$$

hence

$$\begin{aligned}|n\Phi_n(\cdot\,;v/n)-\Psi(\cdot\,;v)| &= |h(\cdot\,;(1-v/n)^n)-h(\cdot\,;e^{-v})|\\ &\le m(\cdot)\Big|(1-v/n)^n-e^{-v}\Big| \le m(\cdot)\frac{v^2}{n}, \quad n\ge 2v.\end{aligned}$$

Using this bound, the Lipschitz property $|\Psi(u)-\Psi(v)|\le m|u-v|$ and the estimate $\Psi(u)\le mu$ we continue with

$$\begin{aligned}\Pi_t^{(n)}(x) &\le \mathcal{O}(\frac{1}{n})+\frac{1}{n}\int_0^t T_{ns}^{(n)}[\eta^{(n)}](x^{(n)})\,V_{t-s}^{(n)}\phi(0)^2\,ds\\ &+\int_0^t \Pi_{t-s}^{(n)}(0)\,T_{ns}^{(n)}(\eta^{(n)})(x^{(n)})\,ds\\ &+\int_0^t \Delta_s^{(n)}[\eta](x)\,V_{t-s}\phi(0)\,ds.\end{aligned}$$

Here the term with a factor $1/n$ in front is immediately disposed of into $\mathcal{O}(1/n)$ since the squared factor in the integrand does not affect the argument which set out from (30) above. Moreover, the final statement of Lemma 14 shows that the last term on the right side is of order $\mathcal{O}(1/n)$ as well. Hence

$$\Pi_t^{(n)}(x)\le \mathcal{O}(\frac{1}{n})+\int_0^t \Pi_{t-s}^{(n)}(0)\,T_{ns}^{(n)}(\eta^{(n)})(x^{(n)})\,ds.$$

which we recognize as a Gronwall inequality concluding that the proof be completed in the same way as in Lemma 14.

Convergence of finite-dimensional distributions.

For each sequence of time points $0<t_1\le\ldots\le t_k$ and functions $\phi_1,\ldots\phi_k$ in $\mathcal{C}$, introduce

$$\begin{aligned}Q_{t_1,\ldots,t_k}^{(n)}(\phi_1,\ldots,\phi_k)(x) &= 1-\mathbf{E}_{\delta_x}\exp-\sum_{i=1}^k\langle X_{t_i}^{(n)},\phi_i\rangle,\\ V_{t_1,\ldots,t_k}^{(n)}(\phi_1,\ldots,\phi_k)(x) &= nQ_{nt_1,\ldots,nt_k}^{(n)}(n^{-1}\phi_1^{(n)},\ldots,n^{-1}\phi_k^{(n)})(x^{(n)}).\end{aligned}$$

For $k\ge 2$ define inductively

$$V_{t_1,\ldots,t_k}(\phi_1,\ldots,\phi_k)(x)=V_{t_1}\Big(\phi_1+V_{t_2-t_1,\ldots,t_k-t_1}(\phi_2,\ldots,\phi_k)\Big)(x),$$

in terms of the solution $V_t\phi$ of equation (9). Put

$$\begin{aligned}\psi_n^{(k)}(x) &= -n\log\Big(1-n^{-1}V_{t_1,\ldots,t_k}^{(n)}(\phi_1,\ldots,\phi_k)(x)\Big)\\ \psi^{(k)}(x) &= V_{t_1,\ldots,t_k}(\phi_1,\ldots,\phi_k)(x)\end{aligned}$$

and

$$R_n = \mathbf{E}\Big[\exp -\frac{1}{n}\langle X_0^{(n)}, \psi_n^{(k)}\rangle - \exp -\frac{1}{n}\langle X_0^{(n)}, \psi^{(k)}\rangle\Big]. \tag{31}$$

It follows from Lemma 10 that $\psi^{(k)} \in \mathcal{C}$ and from the semigroup property of $V_t\phi$ that the finite-dimensional distributions of the Markov process Y_t are given by

$$-\log \mathbf{E}\exp -\sum_{i=0}^{k}\langle Y_{t_i}, \phi_i\rangle = \langle \mu\,, \psi^{(k)}\rangle.$$

We have

$$\mathbf{E}\exp -\frac{1}{n}\sum_{i=1}^{k}\langle X_{t_i}^{(n)}, \phi_i\rangle = R_n + \mathbf{E}\exp -\frac{1}{n}\langle X_0^{(n)}, \psi^{(k)}\rangle,$$

where

$$\mathbf{E}\exp -\frac{1}{n}\langle X_0^{(n)}, \psi^{(k)}\rangle \longrightarrow \langle \mu, \psi^{(k)}\rangle, \quad n\to\infty,$$

since it is assumed that $n^{-1}X_0^{(n)}$ converges weakly to μ. Hence to obtain the desired convergence in distribution it suffices to prove that the difference term R_n vanishes as $n\to\infty$.

For any $r > 0$ split the expectation in expression (31) according to $\langle X_0^{(n)}, 1\rangle > nr$, or the opposite. Then,

$$\begin{aligned}|R_n| &\le \mathbf{P}(\langle X_0^{(n)}, 1\rangle > nr) + \mathbf{E}\Big[n^{-1}\Big\langle X_0^{(n)}, |\psi_n^{(k)} - \psi^{(k)}|\Big\rangle;\ \langle X_0^{(n)}, 1\rangle \le nr\Big]\\ &\le \mathbf{P}(n^{-1}\langle X_0^{(n)}, 1\rangle > r) + r\,\|\psi_n^{(k)} - \psi^{(k)}\|.\end{aligned}$$

Hence, if $\psi_n^{(k)}$ converges uniformly to $\psi^{(k)}$, then, letting first n and then r tend to infinity, we obtain the convergence of R_n to 0. But

$$\|\psi_n^{(k)} - \psi^{(k)}\| = \| - n\log\Big(1 - n^{-1}V_{t_1}^{(n)}(\phi_1 + \psi_n^{(k-1)})\Big) - V_{t_1}(\phi_1 + \psi^{(k-1)})\|,$$

where the superindex $k-1$ refers to the time points $t_2, \dots, t_k$. Therefore it follows directly from Lemma 15 and an elementary inequality, assuming as induction hypothesis the convergence of $k-1$ dimensional distributions, that $\|\psi_n^{(k)} - \psi^{(k)}\| \to 0$ for each k as $n\to\infty$. The proof of finite-dimensional convergence is complete.

Tightness in Skorokhod topology.

To prove the theorem it remains to show that the sequence $n^{-1}X^{(n)}$ is tight in $\mathcal{M}$. We apply the method based on Aldous' criterion, Aldous (1978), originally suggested by Dawson and Fleischmann (1988) and elaborated on by e.g. Dawson (1993), Sagitov (1994) and Kaj and Sagitov (1998). Fix $\phi \in \mathcal{C}$ and denote

$$\eta_n(t) = n^{-1}\langle X_t^{(n)}, \phi\rangle.$$

Let ε_n denote a non-random sequence and τ_n a sequence of bounded stopping times with respect to the filtrations generated by the η_n's, and denote

$$F_n(\varepsilon_n) = \mathbf{E}\exp\Big\{-\lambda_1\eta_n(\tau_n) - \lambda_2\eta_n(\tau_n+\varepsilon_n)\Big\}, \quad \lambda_1,\ \lambda_2 \geq 0.$$

We refer to the references above for the detailed arguments which bring back the tightness property on the validity of the following final lemma. Indeed, it turns out that the lemma implies the convergence in $\mathbf{P}$-probability

$$\eta_n(\tau_n+\varepsilon_n) - \eta_n(\tau_n) \to 0, \quad n\to\infty, \quad \varepsilon_n \to 0,$$

which we recognize as Aldous tightness criterion.

Lemma 16 *Under the conditions of the theorem,*

$$F_n(\varepsilon_n) - F_n(0) \to 0, \quad n\to\infty, \quad \varepsilon_n\to 0.$$

Proof By the strong Markov property

$$F_n(\varepsilon_n) = \mathbf{E}\exp -\frac{1}{n}\Big\langle X_{\tau_n}^{(n)}, \lambda_1\phi + W_{\varepsilon_n}^{(n)}(\lambda_2\phi)\Big\rangle,$$

where

$$W_t^{(n)}\phi(x) = -n\log(1-n^{-1}V_t^{(n)}\phi(x)).$$

Consider the scaled version

$$J^{(n)}(x) = E_{x^{(n)}}\int_0^\infty \eta(a_s, \xi_s^{(n)})M_s\,ds$$

of the quantity J introduced in section 3 and put

$$\bar{J}(x) = \int_0^\infty \eta(a_s, z)P_x(\ell > a_s)\,ds.$$

Simple estimates, using Lemma 13, (6) and (29), show that for any t

$$\begin{aligned} |J^{(n)}(x) - \bar{J}(x)| &\le \frac{1}{n} 4t \int_0^t \|\eta\|_{a_s}^2 P(\ell > a_s)\, ds + 2 \int_t^\infty \|\eta\|_{a_s} P(\ell > a_s)\, ds \\ &\le \frac{1}{n} 4t C_2 + 2 \int_t^\infty \|\eta\|_{a_s} P(\ell > a_s)\, ds. \end{aligned}$$

By (5) we may choose first t and then n so large that the right hand side is less than $C_0/2$. Thus, in view of the assumption (4), $\inf_x J^{(n)}(x) \ge C_0/2$ and therefore

$$|F_n(\varepsilon_n) - F_n(0)| \le \sup_n P_x\Big(n^{-1}\langle X^{(n)}_{\tau_n}, J^{(n)}\rangle > r\Big) + 2C_0^{-1}\, r\, \|W^{(n)}_{\varepsilon_n}(\lambda_2\phi) - \lambda_2\phi\|$$

for any $r > 0$. By Lemma 15,

$$\|W^{(n)}_{\varepsilon_n}(\lambda_2\phi) - \lambda_2\phi\| \le \|V^{(n)}_{\varepsilon_n}(\lambda_2\phi) - \lambda_2\phi\| + \mathcal{O}(1/n)$$

vanishes as $n \to \infty$. Hence it remains to prove

$$\lim_{r\to\infty} \sup_n P_x\Big(n^{-1}\langle X^{(n)}_{\tau_n}, J^{(n)}\rangle > r\Big) = 0.$$

To this end we recall the martingale properties of $M_t^{(n)} := \langle n^{-1} X_t^{(n)}, J^{(n)}\rangle$ obtained in Lemma 9 and apply maximal inequalities. Let $\tau_n \le t_0$ be a uniform bound for the stopping times and take $s > 0$. For each n such that $M_t^{(n)}$ is a submartingale, i.e. $J^{(n)}(0) \ge 1$,

$$\mathbf{P}\Big(\sup_{0\le t\le t_0} M_t^{(n)} > r\,,\, M_0^{(n)} \le s\Big) \le r^{-1}\mathbf{E}[M_{t_0}^{(n)}\,;\, M_0^{(n)} \le s] \le \mathit{const}\, s/r,$$

where the inequality $\mathbf{E}[M_{t_0}^{(n)} \,|\, X_0] \le \mathit{const}\,\mathbf{E}M_0^{(n)}$, for some constant independent of n, is obtained from (5) and (23). For each n such that $M_t^{(n)}$ is a supermartingale, i.e. $J^{(n)}(0) \le 1$,

$$\mathbf{P}_x\Big(\sup_{0\le t\le t_0} M_t^{(n)} > r\,;\, M_0^{(n)} \le s\Big) \le r^{-1}\mathbf{E}[M_0^{(n)}\,;\, M_0^{(n)} \le s] \le s/r.$$

Thus

$$\sup_n \mathbf{P}\Big(\sup_{0\le t\le t_0} M_t^{(n)} > r\Big) \le (1 + \mathit{const})\frac{s}{r} + \sup_n \mathbf{P}(M_0^{(n)} > s).$$

Let r and then s tend to infinity to complete the proof of Theorem 1.

Proof of Corollary 3.

Recall the renewal process L_t and its current age process α_t introduced before Corollary 3. Fix t and condition on whether the first renewal event occurs after or before time t. Using assumption (10),

$$E_x(\phi(\alpha_t, z)) = \phi(a_t, z)P_x(\ell > a_t) + \int_0^t E_0(\phi(\alpha_{t-s}, z))\eta(a_s, z)P_x(\ell > a_s)\, ds,$$

where $x = (\alpha_0, z) = (a, z)$. Hence the function $E_x(\phi(\alpha_t, z))$ solves Equation (7), and is therefore uniquely identified as the mean function $S_t\phi(x)$ of the measure-valued process Y_t. The identification of Y_t as a superprocess now proceeds in the usual manner, c.f. Dynkin (1994), Chapter 1.2.

For the statement of the martingale problem in the case $\lambda \equiv 1$ we refer to e.g. Dawson (1993), Chapter 6.

6 Further remarks

The main efforts of this paper have been directed towards obtaining a genuine measure process in the limit by considering a sequence of small-mass-large-density particle systems scaled in time and space and with appropriately modified offspring mechanism. To further motivate and illuminate this choice we proceed to discuss, on an informal level, two other possible scalings.

First, consider the same situation as in Theorem 1 with the only exception that the offspring generating function h stays independent of n. The scaled counting processes $\Lambda_{nt}^{(n)}[k]$ are now such that, cf (28),

$$\Lambda_{nt}^{(n)}[k] - \int_0^t \pi_k(a_r, \xi_{nr}^{(n)})\lambda(a_r, \xi_{nr}^{(n)})M_r\, dr, \quad k \geq 1$$

are $(P, (\widetilde{F}_t))$-local martingales. By uniqueness of solutions to the martingale problems, the law of the processes $\Lambda_t[k]$, hence of ξ_t, is preserved under the rescaling. This furthermore implies invariance of the expectation measure in the sense that now

$$T_{nt}^{(n)}\phi^{(n)}(x^{(n)}) = T_t\phi(x).$$

If we keep the notation $V_t^{(n)}\phi(x)$ for the Laplace function corresponding to this alternative scaling, the integral equation that derives from Proposition

12 takes the form

$$V_t^{(n)}\phi(x) = nT_t(1-e^{-\phi/n})(x) - n\int_0^t T_s\left(\lambda\Phi(n^{-1}V_{t-s}^{(n)}\phi(0))\right)(x)\,ds$$
$$-\int_0^t T_s\left(\lambda\, V_{t-s}^{(n)}\phi\,\{1-h(1-n^{-1}V_{t-s}^{(n)}\phi(0))\}\right)(x)\,ds.$$

However,

$$1-h(\cdot\,;1-n^{-1}v) = 1-\sum_{k=0}^{\infty}\pi_k(\cdot)(1-n^{-1}v)^k \to 0, \quad n\to\infty$$

and

$$n\Phi(\cdot\,;v/n) = \sum_{k=0}^{\infty}\pi_k(\cdot)\Big\{(1-n^{-1}v)^k - 1 + kn^{-1}v\Big\} \to 0, \quad n\to\infty.$$

Therefore the nonlinear terms are bound to disappear and we obtain the limiting equation $V_t\phi = T_t\phi$, consequently $\langle Y_t,\phi\rangle \equiv T_t\phi$.

As the second alternative scaling we discuss the process obtained simply by assigning mass $1/n$ to each particle, start the evolution by assuming the weak convergence $n^{-1}X_0^{(n)} \to \mu$ and run the system until time nt. We assume that the process is critical, $J(0)=1$, and that the average child-bearing age is finite, i.e.

$$\int_0^\infty sA_x(ds) < \infty.$$

Under such assumptions, by renewal theory, the limit $T_\infty\phi(x)$ exists. The limiting process under this scaling is degenerate in the sense that the fluctuations in space are killed yielding a product form: the time evolution is given by a continuous state branching process while a deterministic measure governs the distribution in space. In fact, the convergence of finite-dimensional distributions holds,

$$\Big\{Z_0^{(n)}, Z_{t_1}^{(n)},\ldots,Z_{t_k}^{(n)}\Big\} \longrightarrow \Big\{\mu, Z_{t_1},\ldots,Z_{t_k}\Big\},$$

where

$$\langle Z_t,\phi\rangle = \chi_t\,T_\infty\phi(0)$$

and $t\mapsto\chi_t$ is a continuous state branching process with $\chi_0 = \langle\mu,J\rangle$.

There is a nontrivial problem in this situation to extend to weak convergence. A simple exponential model is studied in Bose and Kaj (1995) with emphasis on establishing the tightness property required for weak convergence. This is achieved under additional assumptions on the initial sequence $X_0^{(n)}$.

References

[1] Bose, A. and Kaj, I. (1995) Diffusion approximation for an age-structured population. *Ann. Appl. Probab.* **5** 140-157.

[2] Blumenthal, R.M. and Getoor, R.K. (1968) *Markov processes and potential theory.* Academic Press, New York.

[3] Dawson, D.A. (1993) Measure-valued Markov Processes. École d'été de probabilités de Saint Flour 1991. *Lecture Notes in Math.* **1541**, pp. 1-260. Springer, Berlin-Heidelberg-New York.

[4] Dawson, D.A. and Fleischmann, K. (1988) Strong clumping of critical space-time branching models in subcritical dimensions. *Stoch. Proc. Appl.* **30** 193-208.

[5] Dynkin, E.B. (1994) An introduction to branching measure-valued processes. CRM Monograph Series, Vol 6, American Mathematical Society, Providence, RI.

[6] Iscoe, I. (1986) A weighted occupation time for a class of measure-valued critical branching Brownian motion. *Probab. Th. Rel Fields* **71** 85-116.

[7] Jagers, P. (1989) General branching processes as Markov fields. *Stoch. Proc. Appl.* **32** 183-212.

[8] Kaj, I. and Sagitov, S. (1998) Limit processes for age-dependent branching particle systems. *J. Th. Prob.* **11** 225-257.

[9] Nerman, O. (1984) The stable pedigrees of critical branching populations *J. Appl. Probab.* **21** 447-463.

[10] Sagitov, S. (1994) Measure-branching renewal processes. *Stoch. Proc. Appl.* **52** 293-307.

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