A SAMPLE PATH LARGE DEVIATION PRINCIPLE FOR L^2-MARTINGALE MEASURE PROCESSES

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Abstract. Gaussian White Noise, super-Brownian motion and the diffusion-limit Fleming-Viot process are examples of such infinite-dimensional Markov processes with continuous paths and L^2-martingale measures we study in this work as regards to their sample path large deviation probabilitites and their associated large deviation rate functions in the limit of small perturbations. We present a unified approach based on Girsanov transform techniques. We derive the rate function as a Lagrangian functional and, as an alternative representation, via some generalized derivatives in a 'Cameron-Martin space'.

1. Introduction

In Fleischmann, Gärtner and Kaj (1996) (referred to as [FGK] in the sequel), a sample path large deviation principle is derived for super-Brownian motion. In particular, for the case of a compact state space, a representation formula is shown for the rate function as an L^2-functional on a generalized 'Cameron-Martin space' of measure-valued paths. For the case of non-compact state space, the proof of the representation formula (regarding the lower bound) could be brought back to a 'Hypothesis of local blow-up', which was left unverified.

In Schied (1996), similar results to those in [FGK] are obtained involving other representations for the rate function.

In this paper we take a Hamiltonian approach to sample path large deviations for a general class of measure-valued processes as introduced in Dawson (1993), Ch 7, including super-Brownian motion. We attempt to prove directly upper and lower bounds of the large deviation probabilities using the Girsanov transform technique and other tools from stochastic analysis, which are available not only for the super-Brownian motion but also for general measure-valued processes. However, we restrict to continuous processes with L^2-martingale measures. Our main examples besides super-Brownian motion are the probability measure-valued Fleming-Viot process and the $\mathcal{S}'(R^d)$-valued cylindrical Brownian motion. We also mention a version of super-Brownian motion in a random environment studied by Mytnik (1995).

We establish large deviation principles for these models including a representation for the rate function in the style of [FGK]. The result is complete for the example of cylindrical Brownian motion. For the other examples and within our

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general set-up there is in the lower bound a restriction to sets of paths which are sufficiently regular solutions of the McKean-Vlasov equation corresponding to the model. This is a version of the large deviation lower bound which is parallell to recent results on the behaviour of sample path large deviation probabilities under hydrodynamical scaling in various particle models. In fact, Jona-Lasinio, Landim and Vares (1993) consider a nonlinear reaction-diffusion equation as the macroscopic description of a particle system with symmetric simple exclusion and spin flip dynamics. For the lower large deviation estimate only regular solutions of the reaction-diffusion equation are taken into account. Similarly, Benois, Kipnis and Landim (1995) and Landim and Yau (1995) obtain lower large deviation bounds from the hydrodynamical limit for the empirical measure of Ginzburg-Landau and zero-range processes in infinite volume, which also requires a restriction to regular solutions (smooth profiles) of a nonlinear equation.

In the next Section 2 we describe the class of L^2-martingale measure processes that we consider in this paper. In Section 3 we state the large deviation result including the relevant representations of the rate function, and list the four examples. The equivalence of several representations of the rate function are proved in Section 4. The topic of Section 5 is the Girsanov transform on which our large deviation estimates rely, and Section 6 is devoted to the estimates related to exponential tightness. With these preliminaries it is then relatively simple to complete the proof of the upper and lower large deviation bounds in Section 7. Remark 7.6 comments on the problem of avoiding in general the restriction to regular paths in the lower bound.

2. Measure-valued processes

The space (E,τ) with topology τ will denote a Polish space of measures or distributions on R^d, and Φ a related Banach space of continuous functions. Also, $\widetilde{\varphi}$ will denote a fixed strictly positive function in Φ which will have the role of a bounded Lyapunov function; we assume $\widetilde{\varphi} \le 1$. The Banach space norm on Φ is taken to be

$$\|f\| = \sup_{x \in R^d} \big| f(x)/\widetilde{\varphi}(x) \big|.$$

We will work with spaces from the following list:

a) $E = \mathcal{M}_1(R^d)$, the set of probability measures on R^d with the topology of weak convergence, and Φ the bounded continuous functions on R^d with the supremum norm. Hence μ_n converges to μ if and only if $\langle \mu_n, f\rangle$ converges to $\langle \mu, f\rangle$ for all $f \in \mathcal{C}_0(R^d)$, the continuous functions with compact support, and for $f = \widetilde{\varphi} \equiv 1$.

b) $E = \mathcal{M}^a(R^d)$, the set of a-tempered measures on R^d equipped with the a-vague topology with respect to the reference function

$$\widetilde{\varphi}(y) = \varphi_a(y) := (1 + |y|^2)^{-a/2}\,, \quad a > 0\,,$$

and $\{\Phi, \|\cdot\|\}$ the associated normed space of continuous functions, see e.g. Dawson (1993), Sect. 3.1.5 or [FGK] for the details on these spaces.

c) $E = \mathcal{M}^0(R^d)$, the set of finite measures on R^d with the weak topology, corresponding formally to case b) with $a = 0$ and $\widetilde{\varphi} \equiv 1$.

d) $E = \mathcal{S}'(R^d)$, the Schwartz distributions on R^d dual to the Schwartz space $\Phi = \mathcal{S}(R^d)$ of C^∞-functions on R^d which are rapidly decreasing at infinity. As Lyapunov function $\widetilde{\varphi}$ we can choose a function of the form $(1+|x|^n)^{-1}$ on R^d, for an integer n.

Let $\{S_t, t \geq 0\}$ be a one-parameter Feller semigroup on Φ and $\mathcal{A}$ the generator of $\{S_t\}$ with domain $D(\mathcal{A})$, which is then dense in Φ. Assume that

$$Q : E \to Q(R^d, R^d)\,, \quad \mu \mapsto Q(\mu\,; dx, dy)\,,$$

maps E into the collection $Q(R^d, R^d)$ of symmetric signed σ-finite measures on $R^d \times R^d$. Let

$$D(\mathcal{G}^\varepsilon) = \big\{F : F(\mu) = F(\langle \mu, f\rangle),\ F \in \mathcal{C}^2(R),\ \mu \in E,\ f \in D(\mathcal{A})\big\}\,,$$

be the domain of the infinitesimal measure generator $\mathcal{G}^\varepsilon$, which acts on elements in E by

$$\mathcal{G}^\varepsilon F(\mu) = \big\langle \mu, \mathcal{A}f\big\rangle\, F'(\langle \mu, f\rangle) + \frac{\varepsilon}{2} \iint f(x) f(y)\, Q(\mu\,; dx, dy)\, F''(\langle \mu, f\rangle)\,.$$

For notational simplicity we introduce the bracket symbol

$$\big\langle Q(\mu)\,, f_1 \otimes f_2 \big\rangle := \iint f_1(x) f_2(y)\, Q(\mu\,; dx, dy)\,.$$

We will consider continuous E-valued processes $t \mapsto X_t$ with distribution P^ε, which are defined as solutions of the following well-posed martingale problems (c.f. Dawson (1993), §7.2). Given (i), properties (ii) and (iii) are equivalent:

(i) At initial time $t = 0$, $\qquad P^\varepsilon\big[X_0 = \mu\big] = 1\,.$

(ii) For all $F \in D(\mathcal{G}^\varepsilon)$

$$M_t^F := F(X_t) - F(\mu) - \int_0^t \mathcal{G}^\varepsilon F(X_s)\, ds\,, \tag{2.1}$$

is a P^ε-local martingale.

(iii) For all $f \in D(\mathcal{A})$

$$M_t(f) := \langle X_t\,, f\rangle - \langle \mu, f\rangle - \int_0^t \langle X_s\,, \mathcal{A}f\rangle\, ds\,, \tag{2.2}$$

is a P^ε-local martingale with quadratic variation given by

$$\langle\!\langle M(f)\rangle\!\rangle_t := \varepsilon \int_0^t \big\langle Q(X_s)\,, f \otimes f\big\rangle\, ds\,. \tag{2.3}$$

As basic state space for the paths $t \mapsto X_t$ we take the set $\mathcal{C}(I, E)$ of uniformly continuous maps $t \mapsto \nu_t$ from an interval I to E. Normally $I = [0, T]$ with $T < \infty$ but occasionally we consider $I = [0, \infty)$.

Rather than using the set $D(\mathcal{A})$ we will work with other sets of continuous functions. Let $\Phi_I := \mathcal{C}(I, \Phi)$ denote the set of continuous maps $t \mapsto f_t$ of the interval $I = [0, T]$, $T > 0$, into the set Φ with the supremum norm $\|f\|_I := \sup_{t \in I} \|f_t\|$. Denote by $\mathcal{C}^\infty = \mathcal{C}^\infty(R^d)$ and $\mathcal{C}_I^\infty = \mathcal{C}_I^\infty(I \times R^d)$ the sets of real-valued functions on R^d or $I \times R^d$, respectively, possessing continuous derivatives of all orders. Moreover, introduce the following spaces of smooth functions,

$$\Phi^\infty = \Phi^\infty(R^d) := \{ f \in \mathcal{C}^\infty \cap D(\mathcal{A}) : f, \mathcal{A}f \in \Phi \},$$

$$\Phi_I^\infty = \Phi_I^\infty(I \times R^d) := \{ f \in \mathcal{C}_I^\infty \cap D(\mathcal{A}) : f, \dot{f}, \mathcal{A}f \in \Phi_I \}.$$

Note that for $f \in \Phi_I^\infty$ the martingale problem (2.3-4) takes the form

$$M_t(f) = \langle X_t\,, f_t \rangle - \langle \mu, f_0 \rangle - \int_0^t \langle X_s\,, \dot{f}_s + \mathcal{A} f_s \rangle \, ds$$

with

$$\langle\!\langle M(f) \rangle\!\rangle_t := \varepsilon \int_0^t \big\langle Q(X_s)\,, f_s \otimes f \big\rangle \, ds\,.$$

We define the Hamiltonian operator which corresponds to the generator $\mathcal{G}^\varepsilon$, by

$$\mathcal{H}^\varepsilon(\mu, f) := e^{-\langle \mu, f \rangle} \, \mathcal{G}^\varepsilon \, e^{\langle \mu, f \rangle}\,.$$

Hence

$$\mathcal{H}^\varepsilon(\mu, f) = \big\langle \mu, \mathcal{A} f \big\rangle + \frac{\varepsilon}{2} \big\langle Q(\mu)\,, f \otimes f \big\rangle\,.$$

Moreover, we introduce the scaled Hamiltonian

$$\begin{aligned} \mathcal{H}(\mu, f) &:= \lim_{\varepsilon \to 0} \varepsilon \, \mathcal{H}^\varepsilon(\mu, f/\varepsilon) \\ &= \big\langle \mu, \mathcal{A} f \big\rangle + \frac{1}{2} \big\langle Q(\mu)\,, f \otimes f \big\rangle\,. \end{aligned} \tag{2.4}$$

Furthermore, letting $\mathcal{D}$ denote the Schwartz space of $\mathcal{C}^\infty$-functions with compact support in R^d and $\mathcal{D}^*$ the dual space of real distributions on R^d, we introduce the associated Lagrangian functional $\mathcal{L}$, which acts on $\mathcal{D}^* \times E$, by

$$\mathcal{L}(\vartheta, \mu) = \sup_{\varphi \in \mathcal{D}} \big\{ \langle \vartheta, \varphi \rangle - \mathcal{H}(\mu, \varphi) \big\}\,. \tag{2.5}$$

We refer to Dawson and Gärtner (1987) or [FGK], §1.3, for an account on the notion of an absolutely continuous map $t \mapsto \vartheta_t \in \mathcal{D}^*$ with derivative $\dot{\vartheta}_t \in \mathcal{D}^*$. In addition, we need the Radon-Nikodym derivative $g = d\vartheta/d\mu$ of $\vartheta \in \mathcal{D}^*$ with respect to $\mu \in E$, which by definition is a function (equivalence class) g in $L^2(\mu)$, which satisfies

$$\langle \vartheta, \varphi \rangle = \langle \mu\,, g\varphi \rangle\,, \qquad \varphi \in \mathcal{D}\,,$$

see [FGK], Definition 1.3.2. For $\mu \in E$, let $\mathcal{A}^*\mu$ denote the element in $\mathcal{D}^*$ defined by

$$\langle \mathcal{A}^*\mu, \varphi \rangle = \langle \mu, \mathcal{A}\varphi \rangle\,, \qquad \varphi \in \mathcal{D}\,.$$

Given an absolutely continuous path $t \mapsto \nu_t$ in $\mathcal{C}(I,E)$ we introduce a Radon-Nikodym derivative g, $g_t := d(\dot{\nu}_t - \mathcal{A}^*\nu_t)/d\nu_t$, of $\dot{\nu}_t - \mathcal{A}^*\nu_t$ with respect to ν_t, requiring that for almost every $t \in I$

$$\left\langle \dot{\nu}_t - \mathcal{A}^*\nu_t\,,\varphi \right\rangle = \langle \nu_t\,, g_t\varphi\rangle\,, \qquad \varphi \in \mathcal{D}. \tag{2.6}$$

Note that the case $g \equiv 0$, almost everywhere, corresponds to $\nu = \eta$, where the "$\mathcal{A}$-flow" $t \mapsto \eta_t$ is a solution of the McKean-Vlasov equation

$$\langle \eta_t\,, f_t\rangle = \langle \mu, f_0\rangle + \int_0^t \left\langle \eta_s\,, \dot{f}_s + \mathcal{A}f_s \right\rangle ds\,, \quad t \in I \quad \dot{f} = \tfrac{\partial}{\partial s} f\,. \tag{2.7}$$

In greater generality than (2.6), we consider below functions $t \mapsto h_t$ which for almost every t satisfy the relation

$$\left\langle \dot{\nu}_t - \mathcal{A}^*\nu_t\,,\varphi \right\rangle = \left\langle Q(\nu_t)\,, h_t \otimes \varphi \right\rangle, \qquad \varphi \in \mathcal{D}. \tag{2.8}$$

3. Results

We repeat the assumption that we start out with a well-posed solution $P^\varepsilon = P^\varepsilon_\mu$, $\mu \in E$ of the martingale problems (2.1) and (2.2–3). In the boundary case $\varepsilon = 0$ the solution of the martingale problem is the degenerate distribution P^0 concentrated on the path η which solves the McKean-Vlasov equation (2.7). This suggests that as $\varepsilon \to 0$ the corresponding process $t \mapsto X_t$ on a fixed compact interval $I = [0,T]$ converges weakly in $\mathcal{C}(I,E)$ to η. The main result in this paper, Theorem 3.1, describes the behaviour of the large deviation probabilities of X from η in the limit $\varepsilon \to 0$.

There are various methods to obtain for a specific example the existence of such a well-posed solution, a well-known case is the technique to derive the super-Brownian motion as a limit of approximating particle systems. In each such example it is necessary to impose some restrictions on the remaining parameters of the model.

The following assumptions, which are natural also from the point of view of existence, are explicitly used in the derivation of our estimates of the large deviation probabilities, and are imposed from now on. We assume that:

(A1) For all $f \in \Phi$ the real-valued map

$$\nu \mapsto \int_0^T \left\langle Q(\nu_t)\,, f \otimes f \right\rangle dt$$

of $\nu \in \mathcal{C}(I,E)$ is continuous.

(A2) There are constants k_1, k_2 such that

$$|\mathcal{A}\widetilde{\varphi}| \le k_1 \widetilde{\varphi}$$
$$\left\langle Q(\mu)\,, \widetilde{\varphi} \otimes \widetilde{\varphi} \right\rangle \le k_2 \langle \mu\,, \widetilde{\varphi}\rangle\,, \quad \text{for any} \quad \mu \in E\,.$$

(A3) The locally uniform Lipschitz condition

$$\sup_{f \in \Phi\, \|f\| \le 1} \left| \left\langle Q(\nu_t) - Q(\nu'_t)\,, f \otimes f \right\rangle \right| \le C_T \sup_{\|f\| \le 1} \left| \langle \nu_t - \nu'_t\,, f\rangle \right|$$

holds for each pair ν and ν' of elements in $\mathcal{C}(I,E)$.

Following [FGK], Definition 1.4.2, we introduce for each fixed $\mu \in E$ a subset H_μ of $\mathcal{C}(I,E)$. Let H_μ denote the set of all paths $\nu \in \mathcal{C}(I,E)$ with

(i) $\nu_0 = \mu$;

(ii) the $\mathcal{D}^*$-valued map $t \mapsto \nu_t$ defined on I is absolutely continuous;

(iii) for almost all $t \in I$ there is a function h_t, which satisfies (2.8);

(iv) the map $t \mapsto h_t$ defined by (2.8) satisfies

$$\int_0^T \big\langle Q(\nu_t)\,, h_t \otimes h_t \big\rangle \, dt < \infty\,.$$

We also introduce the subspace

$$H_\mu^{\mathrm{reg}} = \big\{ \nu \in H_\mu : \{t \mapsto h_t\} \in \Phi_I^\infty \big\},$$

of H_μ, which consists of paths with the additional regularity that the function h, which is defined only for almost all $t \in I$, actually is a continuous function in Φ_I^∞.

Theorem 3.1. *Assume A1, A2 and A3. Fix $\mu \in E$. For $\nu \in \mathcal{C}(I,E)$ with $\nu_0 = \mu$, put*

$$S(\nu) = \begin{cases} \displaystyle\int_0^T \mathcal{L}(\dot\nu_t, \nu_t)\, dt\,, & \textit{if} \quad t \mapsto \nu_t \textit{ is absolutely continuous}\,, \\ \infty\,, & \textit{otherwise}\,. \end{cases}$$

The probabilities $\mathcal{P}^\varepsilon(A) = P^\varepsilon(X \in A)$, for measurable sets A in $\mathcal{C}(I,E)$, satisfy

(i) for each open subset G of $\mathcal{C}(I,E)$ $\quad \liminf\limits_{\varepsilon \to 0} \varepsilon \log \mathcal{P}^\varepsilon(G) \geq -\inf\limits_{\nu \in G \cap H_\mu^{\mathrm{reg}}} S(\nu)$,

(ii) for each closed subset F of $\mathcal{C}(I,E)$, $\quad \limsup\limits_{\varepsilon \to 0} \varepsilon \log \mathcal{P}^\varepsilon(F) \leq -\inf\limits_{\nu \in F} S(\nu)$.

(iii) The level sets $\{\nu \in \mathcal{C}(I,E) : S(\nu) \leq N\}$, $N > 0$, are compact.

Moreover, $S(\nu) < \infty$ if and only if $\nu \in H_\mu$, and if $\nu \in H_\mu$ then

$$S(\nu) = \frac{1}{2} \int_0^T \big\langle Q(\nu_t)\,, h_t \otimes h_t \big\rangle \, dt\,. \tag{3.2}$$

Examples 3.3 The four examples a)-d) below refer respectively to the list a)-d) of measure state spaces in Section 2.

a) For the Fleming-Viot process $\mathcal{A}$ is called the mutation operator and the covariance is given by

$$Q(\mu\,; dx, dy) = \gamma\big(\delta_x(dy)\mu(dx) - \mu(dx)\mu(dy)\big)\,,$$

with $\gamma > 0$ a sampling-resampling coefficient. By (2.8),

$$\big\langle \dot\nu_t - \mathcal{A}^*\nu_t\,, \varphi \big\rangle = \gamma \int \big(\varphi - \langle \nu_t, \varphi \rangle\big)\big(h_t - \langle \nu_t, h_t \rangle\big)\, \nu_t\,(dx)\,.$$

With $\psi = \varphi - \langle \nu_t, \varphi \rangle \in \mathcal{D}$, this can be written

$$\left\langle \dot{\nu}_t - \mathcal{A}^* \nu_t \,, \psi \right\rangle = \gamma \left\langle \nu_t \,, \psi \left(h_t - \langle \nu_t, h_t \rangle \right) \right\rangle ,$$

which shows that

$$S(\nu) = \frac{1}{2\gamma} \int_0^T \left\langle \nu_t \,, g_t^2 \right\rangle dt \,,$$

where $g_t = h_t - \langle \nu_t, h_t \rangle$ equals $d(\dot{\nu}_t - \mathcal{A}^* \nu_t)/d\nu_t$ as in (2.6) and, moreover, $\langle \nu_t, g_t \rangle = 0$ almost surely.

b) Super-Brownian motion, which can be viewed as a measure-valued process in the state space $\mathcal{M}^a(R^d)$, $a > d$, is obtained by choosing the generator of the spatial motion as $\mathcal{A} = \kappa \Delta$ and the covariance operator

$$Q(\mu\,; dx, dy) = 2\varrho\, \delta_x(dy) \mu(dx) \,,$$

where ϱ is the branching intensity. In fact, the limit $\varepsilon \to 0$ corresponds to a vanishing branching effect in the model whereas the spatial motion of mass is preserved. In this case relation (2.8) simplifies to (2.6) so that

$$S(\nu) = \frac{1}{2} \int_0^T \left\langle \nu_t \,, g_t^2 \right\rangle dt = \frac{1}{4\varrho} \int_0^T \left\langle \nu_t \,, \left(d(\dot{\nu}_t - \mathcal{A}^* \nu_t)/d\nu_t \right)^2 \right\rangle dt \,,$$

which yields the representation (1.4.4) in [FGK].

More general $\mathcal{A}$-superprocesses can be treated in the same manner. For example, the $(\alpha, d, 1)$-superprocess has $\mathcal{A} = \mathcal{A}_\alpha$, the generator of a symmetric α-stable motion, and Q as above. It is well-known that this process can be considered in $\mathcal{M}^a(R^d)$, $d < a < d + \alpha$, and that assumption $A2$ is fulfilled for this range of parameters. Hence the $(\alpha, d, 1)$-superprocess also satisfies the large deviation principle. This case was not covered in [FGK].

c) In Mytnik (1995), a limit process is introduced in a way which mirrors the construction of super-Brownian motion from a branching particle system, with the novelty that the offspring distribution in this case is affected by a random environment. The offspring medium is a time and space stationary random field with given covariance function $G(x,y)$, $x, y \in R^d$ (notation $g(x,y)$ in Mytnik (1995)). The limit process is constructed in $\mathcal{M}^0(R^d)$ and is given by $\mathcal{A} = \Delta$ and

$$Q(\mu\,; dx, dy) = \delta_x(dy)\mu(dx) + G(x,y)\, \mu(dx)\mu(dy) \,.$$

Our assumptions A1–A3 are fulfilled if we suppose that G is bounded on $R^d \times R^d$. In this example the solution h_t of the variational problem (2.5) satisfies

$$h_t(x) + \left\langle \nu_t, G(x, \cdot) h_t \right\rangle = \frac{d(\dot{\nu}_t - \Delta^* \nu_t)}{d\nu_t}(x) \,, \quad a.a.\ x \,.$$

d) For the Cylindrical Brownian motion we have $\mathcal{A} = 0$ and

$$Q(\mu\,; dx, dy) = \sigma^2 \delta_x(dy)\, dx \,,$$

therefore

$$S(\nu) = \frac{1}{2\sigma^2} \int_0^T \int \left(d\dot{\nu}_t/dx\right)^2 dx\, dt\,.$$

As we show in Proposition 7.4, in this example H_μ^{reg} is dense in H_μ and therefore the lower bound has the optimal form

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathcal{P}^\varepsilon(G) \geq -\inf_{\nu\in G} S(\nu).$$

4. Representations of the rate function

Before the first lemma we introduce some additional notation which is used throughout the paper. Recall that in each of the cases $a)$ to $d)$ listed in Section 2, E denotes a Polish space and Φ a Banach space of continuous functions. Note that Φ is convergence-determining for E and that $E \subset \Phi^*$, the dual Banach space of Φ. In analogy with the Banach space norm on elements in Φ^* we measure distance between two elements μ, μ' in E using

$$\|\mu-\mu'\| = \sup_{f\in\Phi,\, \|f\|\leq 1} \left|\langle \mu-\mu', f\rangle\right|.$$

For given $\nu \in \mathcal{C}(I,E)$ we introduce the Hilbert space

$$L^2_I(\nu,Q) := L^2\big(I\times R^d \times R^d\,;\, Q(\nu_t\,;\, dx,dy)\, dt\big)\,,$$

and denote its norm

$$\|f\|_{Q(\nu)} = \Big(\int_0^T \big\langle Q(\nu_s), f_s\otimes f_s\big\rangle\, ds\Big)^{1/2}.$$

For $\nu\in\mathcal{C}(I,E)$, $f\in\Phi_I^\infty$ and $0\leq s\leq t\leq T$, define

$$\ell_{s,t}(\nu,f) = \langle \nu_t, f_t\rangle - \langle \nu_s, f_s\rangle - \int_s^t \big\langle \nu_r, \dot{f}_r + \mathcal{A} f_r\big\rangle\, dr\,, \tag{4.1}$$

and

$$\begin{aligned} J_{s,t}(\nu,f) &= \ell_{s,t}(\nu,f) - \frac{1}{2}\int_s^t \big\langle Q(\nu_r)\,, f\otimes f\big\rangle\, dr \\ &= \langle \nu_t, f\rangle - \langle \nu_s, f\rangle - \int_s^t \langle \nu_r, \dot{f}_r\rangle\, dr - \int_s^t \mathcal{H}(\nu_r, f)\, dr\,. \end{aligned} \tag{4.2}$$

We now prove the nonvariational form (3.2) of $S(\nu)$ stated in Theorem 3.1. The proof exhibits an alternative representation for $S(\nu)$ displayed as (4.4) below.

Lemma 4.3. *The identity*

$$S(\nu) = \sup_{f\in\Phi_I^\infty} J_{0,T}(\nu,f) \tag{4.4}$$

holds. Moreover, $S(\nu)<\infty$ *if and only if* $\nu\in H_\mu$ *, and if* $\nu\in H_\mu$ *then*

$$S(\nu) = \frac{1}{2}\int_0^T \big\langle Q(\nu_t)\,, h_t\otimes h_t\big\rangle\, dt\,. \tag{4.5}$$

Proof. Since

$$S(\nu)=\int_0^T \sup_{\varphi\in\mathcal{D}}\big\{\langle\dot\nu_t,\varphi\rangle-\mathcal{H}(\nu_t,\varphi)\big\}\,dt\,,$$

we have for each f in Φ_I^∞ which in addition is compactly supported on E

$$S(\nu)\geq\int_0^T\big\{\langle\dot\nu_t,f_t\rangle-\mathcal{H}(\nu_t,f_t)\big\}\,dt=J_{0,T}(\nu,f)\,.$$

Therefore

$$S(\nu)\geq\sup_{f\in\Phi_I^\infty}J_{s,t}(\nu,f)\,, \tag{4.6}$$

hence if the expression on the right side is infinite so is $S(\nu)$.

Now suppose that $\sup_{f\in\Phi_I^\infty}J_{0,T}(\nu,f)=:K_T<\infty$. We follow step by step the proof of [FGK], Lemma 4.2.3, observing that all arguments hold with $\langle\nu_t,f_t^2\rangle$ replaced by $\big\langle Q(\nu_t),f_t\otimes f_t\big\rangle$. Indeed, for all real c

$$c\,\ell_{s,t}(\nu,f)-c^2\int_s^t\big\langle Q(\nu_r)\,,f_r\otimes f_r\big\rangle\,dr=J_{s,t}(\nu,cf)\leq K_T\,,$$

thus

$$\ell_{s,t}(\nu,f)^2\leq 4K_T\int_s^t\big\langle Q(\nu_r)\,,f_r\otimes f_r\big\rangle\,dr,$$

and therefore $\ell_{s,t}(\nu,\cdot)$ defines a linear bounded functional on the closure of Φ_I^∞ in $L_I^2(\nu,Q)$, restricted to $[s,t]$. It then follows from the Riesz Representation Theorem that there exists a function h in the closure of Φ_I^∞ in $L_I^2(\nu,Q)$ such that

$$\ell_{0,t}(\nu,f)=\int_0^t\big\langle Q(\nu_r)\,,f_r\otimes h_r\big\rangle\,dr\,. \tag{4.7}$$

Some details in the argument leading to (4.7) are purposedly not repeated here.

As in the quoted proof it follows directly from (4.7) that $\nu:I\to\mathcal{D}^*$ is absolutely continuous and that for $\varphi\in\mathcal{D}$ and Lebesgue almost all $t\in I$

$$\langle\dot\nu_t-\mathcal{A}^*\nu_t,\varphi\rangle=\big\langle Q(\nu_t),h_t\otimes\varphi\big\rangle\,. \tag{4.8}$$

Therefore

$$\begin{aligned}S(\nu)&=\int_0^T\sup_{\varphi\in\mathcal{D}}\Big\langle Q(\nu_t)\,,h_t\otimes\varphi-\frac12\varphi\otimes\varphi\Big\rangle\,dt\\&=\frac12\int_0^T\big\langle Q(\nu_t)\,,h_t\otimes h_t\big\rangle\,dt\\&\qquad-\frac12\int_0^T\inf_{\varphi\in\mathcal{D}}\Big\langle Q(\nu_t)\,,(h_t-\varphi)\otimes(h_t-\varphi)\Big\rangle\,dt\\&\leq\frac12\int_0^T\big\langle Q(\nu_t)\,,h_t\otimes h_t\big\rangle\,dt\,.\end{aligned} \tag{4.9}$$

Now use (4.7) to obtain for every $f \in \Phi_I^\infty$

$$J_{0,T}(\nu, f) = \int_0^T \big\langle Q(\nu_r)\,, f_r \otimes h_r \big\rangle \, dr - \frac{1}{2} \int_0^T \big\langle Q(\nu_r)\,, f_r \otimes f_r \big\rangle \, dr \,,$$

that is,

$$J_{0,T}(\nu, f) = \frac{1}{2} \int_0^T \big\langle Q(\nu_r)\,, h_r \otimes h_r \big\rangle \, dr - \frac{1}{2} \|h - f\|^2_{Q(\nu)} \tag{4.10}$$

However, since $h \in L_I^2(\nu, Q)$ can be approximated in $L_I^2(\nu, Q)$ by functions in Φ_I^∞,

$$\sup_{f \in \Phi_I^\infty} J_{0,T}(\nu, f) = \frac{1}{2} \int_0^T \big\langle Q(\nu_r)\,, h_r \otimes h_r \big\rangle \, dr \geq S(\nu)\,, \tag{4.11}$$

by (4.9). This together with (4.6) finishes the proof of (4.4).

Fix $\nu \in \mathcal{C}(I, E)$ such that $S(\nu) < \infty$. As we have just seen, under this assumption the above proof entails not only relation (4.4) but also the conclusion that ν is absolutely continuous and, according to (4.6) and (4.11), the identity (4.5). Here, by (4.8), the function $h \in L_I^2(\nu, Q)$ satisfies (2.8), hence $\nu \in H_\mu$.

The converse, that if $\nu \in H_\mu$ then $S(\nu) < \infty$, is obvious in view of (4.9). $\square$

We continue with a demonstration that the Lipschitz condition A3 suffices for uniqueness in equation (4.7).

Lemma 4.12. *For each fixed $\phi \in \Phi_I^\infty$ and $\mu \in E$ there exists a unique weak solution to the McKean-Vlasov equation associated with Q, that is, a unique path $\xi \in \mathcal{C}(I, E)$ which satisfies $\xi_0 = \mu$ and for $t \in I$*

$$\langle \xi_t, \varphi \rangle = \langle \mu, \varphi \rangle + \int_0^t \langle \xi_s, \mathcal{A}\varphi \rangle + \int_0^t \big\langle Q(\xi_s)\,, \phi_s \otimes \varphi \big\rangle \, ds, \quad \varphi \in \mathcal{D}.$$

Proof. Existence of a solution to this equation is a consequence of Lemma 7.2 below. It is obtained as the $\varepsilon \to 0$ limit of X under the measure $\widetilde{P}_\phi^\varepsilon$ which is constructed in Section 5.

To prove uniqueness recall that S_t denotes the one-parameter Feller semigroup associated with the generator $\mathcal{A}$. In terms of the semigroup the McKean-Vlasov equation under consideration may be written

$$\langle \xi_t, \varphi \rangle = \langle \mu, S_t \varphi \rangle + \int_0^t \big\langle Q(\xi_s)\,, \phi_s \otimes S_{t-s}\varphi \big\rangle \, ds, \quad \varphi \in \mathcal{D}.$$

Let ξ and $\widehat{\xi}$ denote two solutions of the above integral equation. Then

$$\big|\langle \xi_t - \widehat{\xi}_t, \varphi \rangle\big| \leq \int_0^t \big|\big\langle Q(\xi_s) - Q(\widehat{\xi}_s)\,, \phi_s \otimes S_{t-s}\varphi \big\rangle\big| \, ds.$$

Recall the polarization identity

$$\big\langle Q(\xi_s)\,, \phi_s \otimes \varphi \big\rangle = \frac{1}{4} \big\langle Q(\xi_s)\,, (\phi_s + \varphi) \otimes (\phi_s + \varphi) \big\rangle - \frac{1}{4} \big\langle Q(\xi_s)\,, (\phi_s - \varphi) \otimes (\phi_s - \varphi) \big\rangle,$$

concluding from A3

$$\begin{aligned}\left|\left\langle Q(\xi_s) - Q(\widehat{\xi}_s)\,,\phi_s \otimes S_{t-s}\varphi\right\rangle\right| &\le \frac{C_T}{4}\,\|\xi_s - \widehat{\xi}_s\|\big(\|\phi_s + S_{t-s}\varphi\| + \|\phi_s - S_{t-s}\varphi\|\big)\\ &\le \frac{C_T}{2}\,\|\xi_s - \widehat{\xi}_s\|\big(\|\phi_s\| + \|\varphi\|\big).\end{aligned}$$

Thus

$$\|\xi_t - \widehat{\xi}_t\| \le \frac{C_T}{2}(1 + \|\phi\|_I)\int_0^t \|\xi_s - \widehat{\xi}_s\|\,ds$$

so that $\sup_{t\in I}\|\xi_t - \widehat{\xi}_t\| = 0$, by Gronwall's Lemma. □

5. Change-of-measure

For fixed $\phi \in \Phi_I^\infty$ we are going to consider the perturbed time-inhomogeneous infinitesimal generator $\widetilde{\mathcal{G}}_\phi^\varepsilon$ acting on $F(\mu) = F(\langle\mu, f\rangle)$, defined by

$$\widetilde{\mathcal{G}}_\phi^\varepsilon F(t,\mu) := \mathcal{G}^\varepsilon F(\mu) + \left\langle Q(\mu)\,,\phi_t \otimes f\right\rangle F'(\langle\mu, f\rangle)\,,$$

and show that the martingale problem $(D(\widetilde{\mathcal{G}}_\phi^\varepsilon), \widetilde{\mathcal{G}}_\phi^\varepsilon)$ is well-posed. This will be done by showing that the associated probability measure $\widetilde{P}_\phi^\varepsilon$ on path space is absolutely continuous with respect to the original measure P^ε. The relevant change-of-measure is given in the following lemma due to Dawson, see Lemma 10.1.2.1 in Dawson (1993). We include the proof for the sake of completeness of this paper and since there are some minor variations.

Lemma 5.2. *(Dawson). For each $\phi \in \Phi_I^\infty$ the process*

$$Z_t^\varepsilon(\phi) := \exp\left\{\frac{1}{\varepsilon}M_t(\phi) - \frac{1}{2\varepsilon^2}\langle\!\langle M(\phi)\rangle\!\rangle_t\right\} = \exp\,\varepsilon^{-1}J_{0,t}(X,\phi)\,, \qquad t \ge 0\,,$$

is a true $(P^\varepsilon, \mathcal{F}_t)$-martingale. For each bounded and predictable function a_t, $t \ge 0$,

$$Z_t^{a,\varepsilon}(\phi) := \exp\left\{\frac{1}{\varepsilon}\int_0^t a_s\,dM_s(\phi) - \frac{1}{2\varepsilon^2}\langle\!\langle M(a\phi)\rangle\!\rangle_t\right\}$$

is a true $(P^\varepsilon, \mathcal{F}_t)$-martingale.

Proof. Recall that $\widetilde{\varphi}$ denotes a fixed Lyapunov function and let

$$\tau_n := \inf\left\{t > 0 : \langle X_t, \widetilde{\varphi}\rangle + \int_0^t \langle X_s, \widetilde{\varphi}\rangle\,ds > n\right\}.$$

We claim that

$$\tau_n \nearrow \infty \quad P^\varepsilon - a.s. \tag{5.3}$$

In fact,

$$t \mapsto \langle X_t, \widetilde{\varphi}\rangle + \int_0^t \langle X_s, \widetilde{\varphi}\rangle\,ds \quad \text{is continuous.}$$

Hence for all $t > 0$,

$$\langle X_t, \widetilde{\varphi}\rangle + \int_0^t \langle X_s, \widetilde{\varphi}\rangle\,ds < \infty\,. \tag{5.4}$$

Thus if $\tau_n \to \tau < \infty$ then, $P^\varepsilon - a.s.$,

$$\langle X_{\tau_n}, \widetilde{\varphi}\rangle + \int_0^{\tau_n} \langle X_s, \widetilde{\varphi}\rangle\, ds \longrightarrow \langle X_\tau, \widetilde{\varphi}\rangle + \int_0^{\tau} \langle X_s, \widetilde{\varphi}\rangle\, ds = \infty$$

which contradict (5.4), hence proving (5.3).

By (A2),

$$\frac{1}{2\varepsilon^2}\langle\!\langle M(\phi)\rangle\!\rangle_{\tau_n} \le \|\phi\|_I^2\, \frac{k_2}{2\varepsilon} \int_0^{\tau_n} \langle X_s, \widetilde{\varphi}\rangle\, ds \le \|\phi\|_I^2\, \frac{k_2}{2\varepsilon}\, n\,.$$

Therefore $\{Z^\varepsilon_{t\wedge\tau_n}, t \ge 0\}$ is a true P^ε-martingale and hence, for fixed t and letting $B_n = 1_{\{\tau_n \le t\}}$,

$$1 = E^\varepsilon\big[Z^\varepsilon_{t\wedge\tau_n}\big] = E^\varepsilon\big[Z^\varepsilon_{\tau_n}\,; B_n\big] + E^\varepsilon\big[Z^\varepsilon_t\,; B_n^c\big]\,.$$

Here $E^\varepsilon\big[Z^\varepsilon_t\,; B^c_n\big]$ converges to $E^\varepsilon[Z^\varepsilon_t] = 1$ as $n \to \infty$ since $B^c_n \nearrow \Omega$ by (5.3). In order to show the martingale property of Z^ε_t it remains to show $E^\varepsilon\big[Z^\varepsilon_{\tau_n}\,; B_n\big] \to 0$ as $n \to \infty$.

To this end note first that $\widetilde{P}^\varepsilon_\phi = Z^\varepsilon_t(\phi)P^\varepsilon$, solves the local martingale problem $(\widetilde{\mathcal{G}}^\varepsilon_\phi, D(\widetilde{\mathcal{G}}^\varepsilon_\phi))$. Indeed, it follows from the properties of Radon-Nikodym derivatives that for any real number r and function $f \in \Phi_I^\infty$ the quotient $Z^\varepsilon_t(\phi + rf)/Z^\varepsilon_t(\phi)$ is a $(P^\varepsilon, \mathcal{F}_t)$-local martingale. Differentiate twice at $r = 0$ to get that for every $f \in \Phi_I^\infty$

$$N_t(f) := M_t(f) - \int_0^t \big\langle Q(X_s)\,, \phi_s \otimes f_s\big\rangle\, ds\,, \quad t \ge 0\,,$$

is a $(\widetilde{P}^\varepsilon_\phi, \mathcal{F}_t)$-local martingale with quadratic variation

$$\langle\!\langle N(f)\rangle\!\rangle_t = \varepsilon \int_0^t \big\langle Q(X_s)\,, f_s \otimes f_s\big\rangle\, ds\,.$$

Hence for each n

$$\begin{aligned}\widetilde{E}^\varepsilon_\phi\langle X_{t\wedge\tau_n}, \widetilde{\varphi}\rangle - \langle \mu, \widetilde{\varphi}\rangle &= \widetilde{E}^\varepsilon_\phi\Big[\int_0^{t\wedge\tau_n} \big\langle X_s, \mathcal{A}\widetilde{\varphi}\big\rangle\, ds + \varepsilon\int_0^{t\wedge\tau_n} \big\langle Q(X_s)\,, \phi_s \otimes \widetilde{\varphi}\big\rangle\, ds\Big] \\ &\le \big(k_1 + \varepsilon k_2\|\phi\|_I\big)\, \widetilde{E}^\varepsilon_\phi \int_0^{t\wedge\tau_n} \big\langle X_s, \widetilde{\varphi}\big\rangle\, ds\,,\end{aligned}$$

by A2. Set $K = k_1 + \varepsilon k_2\|\phi\|_I$. By Gronwall's inequality

$$\widetilde{E}^\varepsilon_\phi\big\langle X_{t\wedge\tau_n}, \widetilde{\varphi}\big\rangle\, ds \;\le \langle \mu, \widetilde{\varphi}\rangle\, e^{Kt}\,.$$

Also

$$\widetilde{E}^\varepsilon_\phi \int_0^{t\wedge\tau_n} \big\langle X_s, \widetilde{\varphi}\big\rangle\, ds \le \langle \mu, \widetilde{\varphi}\rangle\, K^{-1}(e^{Kt} - 1)\,. \tag{5.5}$$

Next, note that

$$\begin{aligned}E^\varepsilon\big[Z^\varepsilon_{\tau_n}\,; B_n\big] &= \widetilde{P}^\varepsilon_\phi(\tau_n \le t) \\ &\le \widetilde{P}^\varepsilon_\phi\Big(\sup_{s\le t\wedge\tau_n} \langle X_s, \widetilde{\varphi}\rangle \ge \frac{n}{2}\Big) + \widetilde{P}^\varepsilon_\phi\Big(\int_0^{t\wedge\tau_n} \langle X_s, \widetilde{\varphi}\rangle\, ds > \frac{n}{2}\Big).\end{aligned}$$

By Chebyshev's inequality and (5.5)

$$\widetilde{P}^{\varepsilon}_{\phi}\Big(\int_0^{t\wedge\tau_n}\langle X_s,\widetilde{\varphi}\rangle\,ds>\frac{n}{2}\Big)\le\frac{2}{n}\,\langle\mu,\widetilde{\varphi}\rangle\,K^{-1}(e^{Kt}-1)\,. \tag{5.6}$$

Moreover, the Burkholder-Davies-Gundy inequality implies, also using A2 and (5.5),

$$\begin{aligned}\widetilde{E}^{\varepsilon}_{\phi}\big[\sup_{s\le t\wedge\tau_n}\big|N_s(\widetilde{\varphi})\big|^2\big]&\le\mathrm{const}\,\widetilde{E}^{\varepsilon}_{\phi}\langle\!\langle N(\widetilde{\varphi})\rangle\!\rangle_{t\wedge\tau_n}\\&\le\mathrm{const}\,\varepsilon\,k_2\,\widetilde{E}^{\varepsilon}_{\phi}\int_0^{t\wedge\tau_n}\langle X_s,\widetilde{\varphi}\rangle\,ds\\&\le\mathrm{const}\,\varepsilon\,k_2\,\langle\mu,\widetilde{\varphi}\rangle\,K^{-1}(e^{Kt}-1)\,.\end{aligned} \tag{5.7}$$

Thus, since

$$|\langle X_{t\wedge\tau_n},\widetilde{\varphi}\rangle|\le\langle\mu,\widetilde{\varphi}\rangle+K\int_0^{t\wedge\tau_n}\big\langle X_s,\widetilde{\varphi}\big\rangle\,ds+\big|N_{t\wedge\tau_n}(\widetilde{\varphi})\big|,$$

by Chebyshev's inequality, using (5.6) and (5.7), we deduce that

$$\widetilde{P}^{\varepsilon}_{\phi}\Big(\sup_{s\le t\wedge\tau_n}\langle X_s,\widetilde{\varphi}\rangle\ge\frac{n}{2}\Big)\to 0\,,\quad n\to\infty\,.$$

Hence

$$E^{\varepsilon}\big[Z^{\varepsilon}_{\tau_n}\,;B_n\big]\to 0\,,\quad n\to\infty\,,$$

therefore $E^{\varepsilon}\big[Z^{\varepsilon}_{\tau_n}\big]=1$.

The second statement is a standard variation of the first. □

At our disposal we have now for every $\mu\in E$ and every $\phi\in\Phi_I^{\infty}$ a probability measure $\widetilde{P}^{\varepsilon}_{\phi}$, which is absolutely continuous with respect to P^{ε} with Radon-Nikodym derivative

$$Z^{\varepsilon}_t(\phi)=\frac{\widetilde{P}^{\varepsilon}_{\phi}}{P^{\varepsilon}}\Big|_{\mathcal{F}_t},$$

and which is the unique solution to the martingale problem $(D(\widetilde{\mathcal{G}}^{\varepsilon}_{\phi}),\widetilde{\mathcal{G}}^{\varepsilon}_{\phi})$. Hence, given

(i) at initial time $t=0$, $\quad\widetilde{P}^{\varepsilon}_{\phi}\big[X_0=\mu\big]=1$,

the next relations (ii) and (iii) are equivalent:

(ii) For all $F\in D(\mathcal{G}^{\varepsilon})$,

$$M^F_t:=F(X_t)-F(\mu)-\int_0^t\widetilde{\mathcal{G}}^{\varepsilon}_{\phi}F(X_s)\,ds$$

is a $\widetilde{P}^{\varepsilon}_{\phi}$-local martingale.

(iii) For all $f\in D(\mathcal{A})$,

$$N_t(f):=\langle X_t\,,f\rangle-\langle\mu,f\rangle-\int_0^t\langle X_s\,,\mathcal{A}f\rangle\,ds-\int_0^t\big\langle Q(X_s)\,,\phi_s\otimes f\big\rangle\,ds$$

is a $\widetilde{P}^{\varepsilon}_{\phi}$-local martingale with quadratic variation

$$\langle\!\langle N(f)\rangle\!\rangle_t := \varepsilon \int_0^t \big\langle Q(X_s)\,, f\otimes f\big\rangle\, ds\,.$$

The Hamiltonian operator associated with the generator $\widetilde{\mathcal{G}}^{\varepsilon}_{\phi}$ is

$$\widetilde{\mathcal{H}}^{\varepsilon}_{\phi}(\mu,f) := e^{-\langle\mu,f\rangle}\,\widetilde{\mathcal{G}}^{\varepsilon}_{\phi}\,e^{\langle\mu,f\rangle},$$

so that

$$\widetilde{\mathcal{H}}^{\varepsilon}_{\phi}(\mu,f) = \mathcal{H}^{\varepsilon}(\mu,f) + \big\langle Q(\mu)\,, f\otimes\phi\big\rangle,$$

the scaled Hamiltonian therefore

$$\begin{aligned}\widetilde{\mathcal{H}}_{\phi}(\mu,f) &:= \lim_{\varepsilon\to 0}\,\varepsilon\,\widetilde{\mathcal{H}}^{\varepsilon}_{\phi}(\mu,f/\varepsilon)\\ &= \mathcal{H}(\mu,f) + \big\langle Q(\mu)\,, f\otimes\phi\big\rangle.\end{aligned}$$

Let

$$\widetilde{\mathcal{L}}_{\phi}(\vartheta,\mu) = \sup_{\varphi\in\mathcal{D}}\big\{\langle\vartheta,\varphi\rangle - \widetilde{\mathcal{H}}_{\phi}(\mu,\varphi)\big\}.$$

Related to the shifted measure $\widetilde{P}^{\varepsilon}_{\phi}$ introduce for $\xi\in\mathcal{C}(I,E)$

$$\widetilde{S}_{\phi}(\xi) = \begin{cases}\displaystyle\int_0^T \widetilde{\mathcal{L}}_{\phi}(\dot{\xi}_t,\xi_t)\,dt\,, & \text{if}\quad t\mapsto\xi_t \text{ is absolutely continuous}\,,\\ \infty\,, & \text{otherwise}\,.\end{cases}$$

Furthermore, in analogy to (4.1) and (4.2)

$$\begin{aligned}\widetilde{\ell}^{\phi}_{s,t}(\xi,f) &= \ell_{s,t}(\xi,f) - \int_s^t \langle Q(\xi_r),\phi_r\otimes f_r\rangle\,dr,\\ \widetilde{J}^{\phi}_{s,t}(\xi,f) &= \langle\xi_t,f_t\rangle - \langle\xi_s,f_s\rangle - \int_s^t \widetilde{\mathcal{H}}_{\phi}(\xi_r,f_r)\,dr\\ &= J_{s,t}(\xi,\phi+f) - J_{s,t}(\xi,\phi)\\ &= \widetilde{\ell}^{\phi}_{s,t}(\xi,f) - \frac{1}{2}\int_s^t \langle Q(\xi_r), f_r\otimes f_r\rangle\,dr\,, \qquad f\in\Phi^{\infty}_I.\end{aligned}$$

Recall that the function h in (3.2) depends on the particular choice of $\nu\in H_\mu$ under consideration. Occasionally we emphasize this by writing h^{ν}, h^{ξ}, etc. For each $\xi\in H_\mu$, by Lemma 4.3

$$\begin{aligned}\widetilde{S}_{\phi}(\xi) &= \sup_{f\in\Phi^{\infty}_I}\widetilde{J}^{\phi}_{0,T}(\xi,f) = \sup_{f\in\Phi^{\infty}_I} J_{0,T}(\xi,\phi+f) - J_{0,T}(\xi,\phi)\\ &= S(\xi) - J_{0,T}(\xi,\phi)\,,\end{aligned}$$

where

$$S(\xi) = \frac{1}{2}\int_0^T \big\langle Q(\xi_t)\,, h^{\xi}_t\otimes h^{\xi}_t\big\rangle\,dt\,.$$

In addition, by (4.7)

$$J_{0,T}(\xi,\phi)=\int_0^T \left\langle Q(\xi_r)\,,\phi_r\otimes h_r^\xi\right\rangle dr-\frac{1}{2}\int_0^T \left\langle Q(\xi_r)\,,\phi_r\otimes\phi_r\right\rangle dr\,.$$

Hence

$$\widetilde{S}_\phi(\xi)=\frac{1}{2}\int_0^T \left\langle Q(\xi_t)\,,(h_t^\xi-\phi_t)\otimes(h_t^\xi-\phi_t)\right\rangle dt=\frac{1}{2}\left\|h^\xi-\phi\right\|^2_{Q(\xi)}\,.$$

We close this section with the observation that both S and $\widetilde{S}_\phi$ are lower semicontinuous. This follows from A1, which implies that $J(\cdot,f)$ is continuous for any $f\in\Phi_I^\infty$, and from relation (4.4) of Lemma 4.3.

6. Exponential tightness

Throughout this section we work with a fixed initial measure μ and the family of probability measures $P^\varepsilon=P_\mu^\varepsilon$. We study relatively compact subsets of $\mathcal{C}(I,E)$ and consider the basic exponential tightness property in the following form.

Proposition 6.1. *For each $s>0$ there exists a compact subset $\mathcal{K}\subset\mathcal{C}(I,E)$ such that*

$$\limsup_{\varepsilon\to 0}\varepsilon\log P^\varepsilon(X\notin\mathcal{K})\le -s.$$

Recall the cases a) to d) of Section 1. For a proof of the proposition in the case b) we may refer to Schied (1995), where specific properties of Super-Brownian motion are used in order to establish the necessary tail estimates.

In case d), where $E=\mathcal{S}'(R^d)$, the Skorokhod topology on $D(I,E)$ was introduced by Mitoma (1983) who gave criteria for tightness and for a set to be relatively compact. It follows from Mitoma (1983), Theorem 3.1, that exponential tightness of X under P^ε is equivalent to exponential tightness of the real valued projections $\langle X,f\rangle$ for each $f\in\mathcal{S}(R^d)$. Expanding on Mitoma's method Jakubowski (1986) proved the corresponding result for situations like in the cases a) and c). We recall this construction.

The family of linear functionals

$$\mathbb{F}=\left\{\langle\cdot,f\rangle : f\in\Phi_I^\infty\right\},$$

is additive and separates points on $\mathcal{C}(I,E)$. Fix a countable sequence $\{f_1,f_2,\dots\}$ of functions in Φ_I^∞ and build up $\mathcal{K}$ as an intersection of sets $\mathcal{K}_{f_k}$, $k\ge 1$, defined as follows. For given f in Φ_I^∞, suppose for now we have fixed two real zero sequences ϱ_n and δ_n. These will be choosen at a later stage. Denote by K_f the set of real continuous functions

$$K_f=\left\{x\in\mathcal{C}(I,R):|x(0)|\le\langle\mu,\widetilde{\varphi}\rangle\|f\|\right\}\cap\bigcap_{n\ge 1}D_n,$$

where

$$D_n=\left\{x\in\mathcal{C}(I,R):\sup_{0\le s<t\le T,t-s<\delta_n}|x(s)-x(t)|\le\varrho_n\right\}. \qquad (6.2)$$

The set K_f is compact in $\mathcal{C}(I,R)$ according to the Arzéla-Ascoli Theorem. For fixed $L>0$, let $\mathcal{K}_f$ denote the set of paths in $\mathcal{C}(I,E)$ defined by

$$\mathcal{K}_f = \big\{\nu \in \mathcal{C}(I,E):\ \langle \nu_t, \widetilde{\varphi}\rangle \le L, t\in I,\ t\mapsto \langle \nu_t, f\rangle \in K_f\big\}.$$

It is a consequence of Jakubowski's criterion that

$$\mathcal{K} = \bigcap_k \mathcal{K}_{f_k}$$

is compact in $\mathcal{C}(I,E)$. Furthermore,

$$P^\varepsilon(X\in\mathcal{K}^c) \le P^\varepsilon\big(\sup_{t\in I}\langle X_t,\widetilde{\varphi}\rangle \ge L\big) + \sum_{k=1}^\infty P^\varepsilon(\langle X_\cdot, f\rangle \in K^c_{f_k}). \tag{6.3}$$

The next two lemmas are devoted to estimating the terms in (6.3) towards the end of proving Proposition 6.1. The statements and their proofs are partly parallell to the Lemmas 5.5 and 5.6 in Dawson and Gärtner (1987), and adapted to the situation under study. Note that in the case a) the first lemma is trivial.

Lemma 6.4. *For each $L>0$,*

$$P^\varepsilon\Big(\sup_{t\in I}\langle X_t,\widetilde{\varphi}\rangle \ge L\Big) \le \exp -\frac{1}{\varepsilon}R_L,$$

where $R_L = Le^{-(k_1+k_2/2)T} - \langle \mu,\widetilde{\varphi}\rangle$.

Proof. In this proof let $\lambda = k_1 + k_2/2$. We obtain from the martingale problem (2,2–3) applied to the function $f_t(x) = e^{-\lambda t}\widetilde{\varphi}(x)$ that

$$M_t(f) = e^{-\lambda t}\langle X_t,\widetilde{\varphi}\rangle - \langle X_0,\widetilde{\varphi}\rangle - \int_0^t e^{-\lambda s}\langle X_s, \mathcal{A}\widetilde{\varphi} - \lambda\widetilde{\varphi}\rangle\, ds$$

is a local martingale with

$$\langle\!\langle M(f)\rangle\!\rangle_t = \varepsilon\int_0^t e^{-2\lambda s}\big\langle Q(X_s)\,, \widetilde{\varphi}\otimes\widetilde{\varphi}\big\rangle\, ds.$$

Hence

$$\begin{aligned} e^{-\lambda t}\langle X_t,\widetilde{\varphi}\rangle =& \langle X_0,\widetilde{\varphi}\rangle + M_t(f) - \frac{1}{2\varepsilon}\langle\!\langle M(f)\rangle\!\rangle_t \\ &+ \int_0^t e^{-\lambda s}\langle X_s,\mathcal{A}\widetilde{\varphi} - \lambda\widetilde{\varphi}\rangle\, ds + \frac12\int_0^t e^{-2\lambda s}\big\langle Q(X_s)\,,\widetilde{\varphi}\otimes\widetilde{\varphi}\big\rangle\, ds.\end{aligned}$$

By A2,

$$\langle X_s,\mathcal{A}\widetilde{\varphi} - \lambda\widetilde{\varphi}\rangle + \frac12 e^{-\lambda s}\big\langle Q(X_s)\,,\widetilde{\varphi}\otimes\widetilde{\varphi}\big\rangle \le \big\langle X_s\,, (k_1 + \frac{k_2}{2} - \lambda)\widetilde{\varphi}\big\rangle = 0,$$

therefore

$$\begin{aligned} e^{-\lambda t}\langle X_t,\widetilde{\varphi}\rangle &\le \langle X_0,\widetilde{\varphi}\rangle + \varepsilon\Big\{\frac{1}{\varepsilon}M_t(f) - \frac{1}{2\varepsilon^2}\langle\!\langle M(f)\rangle\!\rangle_t\Big\} \\ &= \langle X_0,\widetilde{\varphi}\rangle + \varepsilon\log Z^\varepsilon_t(f)\end{aligned}$$

in terms of the exponential martingale of Lemma 5.2. Since Z_t is a supermartingale and $Z_0 = 1$, the stated estimate

$$P^\varepsilon\big(\sup_{t\in I}\langle X_t,\widetilde{\varphi}\rangle \ge L\big) \le P^\varepsilon\big(\sup_{t\in I} Z^\varepsilon_t(f) \ge \exp R_L/\varepsilon\big) \le \exp -R_L/\varepsilon,$$

readily follows as a consequence of Doob's maximal inequality. $\square$

Lemma 6.5. *Fix $f \in \Phi_I^\infty$. For any $\gamma > 0$*

$$P^\varepsilon(\langle X_\cdot, f\rangle \in K_f^c) \leq \exp -\gamma/\varepsilon.$$

Proof. Since $\langle X_0, f\rangle = \langle \mu, f\rangle \leq \langle \mu, \widetilde{\varphi}\rangle \|f\|$, the summands in the last term of (6.3) are probabilities of the form

$$P^\varepsilon(\langle X_\cdot, f\rangle \notin K_f) = P^\varepsilon\Big(\langle X_\cdot, f\rangle \in \bigcup_{n\geq 1} D_n^c\Big) \leq \sum_{n=1}^\infty P^\varepsilon\big(\langle X_\cdot, f\rangle \in D_n^c\big). \tag{6.6}$$

Fix $f \in \Phi_I^\infty$ and take $\gamma > 0$ arbitrarily. Set

$$\kappa = \|\mathcal{A}f\| + k_2 \|f\|^2/2 < \infty.$$

We show first that for each n, assuming without loss of generality that $\delta_n \in (0, T/2]$ and $L = L_n > 1$, such that $L < \varrho_n/4\kappa\delta_n$

$$P^\varepsilon(\langle X_\cdot, f\rangle \in D_n^c) \leq 2T\delta_n^{-1}\Big[\exp -\frac{1}{\varepsilon}\frac{(\varrho_n - 4L\kappa\delta_n)^2}{32\delta_n\kappa L} + \exp -\frac{1}{\varepsilon}R_L\Big], \tag{6.7}$$

where $R_L = Le^{-(k_1+k_2/2)T} - \langle \mu, \widetilde{\varphi}\rangle$.

For each $\delta_n \in (0, T/2]$ and $\varrho_n > 0$,

$$\begin{aligned}
P^\varepsilon(\langle X_\cdot, f\rangle \in D_n^c) &= P\big(\sup_{0\leq s<t\leq T,\, t-s<\delta_n} \big|\langle X_t, f\rangle - \langle X_s, f\rangle\big| > \varrho_n\big)\\
&\leq \sum_{k=0}^{[T/\delta_n]-1} P\big(\sup_{0\leq k\delta_n<t<(k+2)\delta_n\wedge T} \big|\langle X_t, f\rangle - \langle X_{k\delta_n}, f\rangle\big| > \varrho_n/2\big)\\
&\leq \frac{T}{\delta_n} P\big(\sup_{t<2\delta_n} \big|\langle X_t, f\rangle - \langle \mu, f\rangle\big| > \varrho_n/2\big).
\end{aligned}$$

Recall again the exponential martingale $Z_t^\varepsilon(f)$ of Lemma 5.2 and observe that for each $\beta > 0$,

$$\begin{aligned}
\langle X_t, f\rangle - \langle \mu, f\rangle &= \varepsilon \log Z_t^\varepsilon(f) + \int_0^t \langle X_s, \mathcal{A}f\rangle\, ds + \frac{1}{2}\int_0^t \big\langle Q(X_s)\,, f\otimes f\big\rangle\\
&= \frac{\varepsilon}{\beta}\log Z_t^\varepsilon(\beta f) + \int_0^t \langle X_s, \mathcal{A}f\rangle\, ds + \frac{\beta}{2}\int_0^t \big\langle Q(X_s)\,, f\otimes f\big\rangle.
\end{aligned}$$

However,

$$\begin{aligned}
\langle X_s, \mathcal{A}f\rangle + \frac{\beta}{2}\big\langle Q(X_s)\,, f\otimes f\big\rangle &\leq \big(\|\mathcal{A}f\| + \beta\, k_2\|f\|^2/2\big)\, \langle X_s, \widetilde{\varphi}\rangle\\
&\leq \kappa(1+\beta)\, \langle X_s, \widetilde{\varphi}\rangle.
\end{aligned}$$

Hence,

$$\langle X_t, f\rangle - \langle \mu, f\rangle \leq \frac{\varepsilon}{\beta}\log Z_t^\varepsilon(\beta f) + t\kappa(1+\beta)\sup_{0\leq s\leq t}\langle X_s, \widetilde{\varphi}\rangle,$$

and therefore

$$
\begin{aligned}
&P\big(\sup_{t<2\delta_n}\big[\langle X_t,f\rangle-\langle\mu,f\rangle\big]>\varrho_n/2\big)\\
&\quad\le P\Big(\frac{\varepsilon}{\beta}\sup_{t<2\delta_n}\log Z_t^\varepsilon(\beta f)>\varrho_n/2-2\delta\kappa(1+\beta)\sup_{0\le s\le T}\langle X_s,\widetilde{\varphi}\rangle\Big)\\
&\quad\le P\Big(\frac{\varepsilon}{\beta}\sup_{t<2\delta_n}\log Z_t^\varepsilon(\beta f)>\varrho_n/2-2\delta_n\kappa(1+\beta)L\Big)+P\big(\sup_{s\in I}\langle X_s,\widetilde{\varphi}\rangle>L\big).
\end{aligned}
$$

In order to continue estimating this last bound, apply Doob's maximal inequality on the first term and Lemma 6.4 on the second. We find

$$
\begin{aligned}
&P\big(\sup_{t<2\delta_n}\big[\langle X_t,f\rangle-\langle\mu,f\rangle\big]>\varrho_n/2\big)\\
&\qquad\le\exp-\frac{\beta}{\varepsilon}\big[\varrho_n/2-2\delta_n\kappa(1+\beta)L\big]+\exp-\frac{1}{\varepsilon}R_L.
\end{aligned}
$$

Minimize with respect to β. Then, provided $\varrho_n>4L\delta_n\kappa$,

$$
\begin{aligned}
P\big(\sup_{t<2\delta_n}\big[\langle X_t,f\rangle-\langle\mu,f\rangle\big]>\varrho_n/2\big)\le&\exp-\frac{1}{\varepsilon}(\varrho_n-4L\kappa\delta_n)^2/32L\kappa\delta_n\\
&+\exp-\frac{1}{\varepsilon}R_L.
\end{aligned}
$$

The same upper bound applies if we replace f by $-f$ and repeat the steps above. Hence, for all $\delta_n\in(0,T/2]$ and provided $\varrho_n>4L\delta_n\kappa$

$$
P^\varepsilon(\langle X.,f\rangle\in D_n^c)\le\frac{2T}{\delta_n}\Big[\exp-\frac{1}{\varepsilon}\frac{(\varrho_n-4L\delta_n\kappa)^2}{32L\delta_n\kappa}+\exp-\frac{1}{\varepsilon}R_L\Big],
$$

which is (6.7).

Now we make the following choice of parameters. For each $n\ge1$, let

$$
\begin{aligned}
\delta_n&=T/2n^3\\
L_n&=e^{(k_1+k_2/2)T}(4n\gamma+\langle\mu,\widetilde{\varphi}\rangle)\\
\varrho_n&=4L_n\delta_n\kappa+(128nL_n\delta_n\gamma)^{1/2}\sim1/\sqrt{n},
\end{aligned}
$$

and assume without loss of generality that γ/ε is sufficiently large. Then

$$
\begin{aligned}
2T\delta_n^{-1}&=4n^3\\
(\varrho_n-4L_n\delta_n\kappa)^2/32L_n\delta_n\kappa&=4n\gamma\\
R_{L_n}&=4n\gamma
\end{aligned}
$$

and, by (6.6),

$$
P^\varepsilon(\langle X.,f\rangle\in K_f^c)\le\sum_{n=1}^\infty8n^3e^{-4n\gamma/\varepsilon}\le\sum_{n=1}^\infty e^{-2n\gamma/\varepsilon}\le e^{-\gamma/\varepsilon}.
$$

□

Proof of Proposition 6.1. Let $s>1$ and recall (6.3). Apply Lemma 6.4 with $L=(s+\langle\mu,\widetilde{\varphi}\rangle)\exp(k_1+k_2/2)T$ hence $R_L=s$, and Lemma 6.5 with $\gamma=sk$. Then

$$
P^\varepsilon(X\in\mathcal{K}^c)\le\exp-s/\varepsilon+\sum_{k=1}^\infty\exp\{-sk/\varepsilon\}\le\exp\{-s/\varepsilon\}\frac{2-\exp\{-s/\varepsilon\}}{1-\exp\{-s/\varepsilon\}}.
$$

Hence

$$
\limsup_{\varepsilon\to0}\varepsilon\log P^\varepsilon(X\notin\mathcal{K})\le-s.
$$

□

Corollary 6.7. *For each $\phi \in \Phi_I^\infty$ the sequence $\widetilde{P}_\phi^\varepsilon$ is exponentially tight as $\varepsilon \to 0$.*

Proof. We mimic the proof given above in all detail noting that the drift term for the local martingale $M_t(f)$ is replaced under $\widetilde{P}_\phi^\varepsilon$ by the drift for $N_t(f)$ with an additional term of the form $\int_0^t \big\langle Q(X_s)\,, \phi_s \otimes f\big\rangle\, ds$. Hence, in the proof of Lemma 6.4 replace λ by

$$\lambda[\phi] = \lambda + k_2\|\phi\|_I = k_1 + k_2(1/2 + \|\phi\|_I),$$

to get the upper bound

$$\widetilde{P}_\phi^\varepsilon\Big(\sup_{t\in I}\langle X_t, \widetilde{\varphi}\rangle \geq L\Big) \leq \exp -\frac{1}{\varepsilon} R_L[\phi],$$

where $R_L[\phi] = L\exp\{-\lambda[\phi]T\} - \langle \mu, \widetilde{\varphi}\rangle$. To obtain the counterpart of Lemma 6.5 we use

$$\kappa[\phi] = \kappa + k_2\|f\|\,\|\phi\|_I = \|\mathcal{A}f\| + k_2\|f\|\,\big(\|f\|/2 + \|\phi\|_I\big)$$

instead of κ and make the obvious alterations in the proof. Finally, the concluding parts of the proof of Proposition 6.1 remain the same in order to finish the proof of the corollary. □

7. Local lower and upper bounds

To complete the proof of the theorem it remains first of all to establish a local upper bound for compact sets. This will be the content of Lemma 7.1 which together with the exponential tightness estimate Proposition 6.1 yield the upper bound for closed sets in Theorem 3.1.

To prove the lower bound it is enough to establish a local large deviation lower bound for a fixed ν such that $h = h^\nu \in \Phi_I^\infty$. For this we apply the change-of-measure of Section 5 with Radon-Nikodym derivative $Z_t^\varepsilon(h)$. For the corresponding sequence of measures $\widetilde{P}_h^\varepsilon$, $\varepsilon > 0$, we then prove in Lemma 7.2 a law of large numbers in the limit $\varepsilon \to 0$; here again the exponential tightness estimate is crucial as well as the already proven upper bound Lemma 7.1. The local lower bound Lemma 7.3 concludes the proof of Theorem 3.1.

Then it still remains to discuss the example of cylindrical Brownian motion, for which we have claimed that the lower bound is true without any restriction to regular paths. In connection with this we make further comments on the general problem of avoiding the set H_μ^{reg}.

As announced we are now able to complete the proof of the upper bound.

Lemma 7.1. *For every compact set K in $\mathcal{C}(I,E)$,*

$$\limsup_{\varepsilon\to 0} \varepsilon \log P^\varepsilon\big(X \in K\big) \leq -\inf_{\nu\in K} S_\mu(\nu)\,.$$

Proof. For any $f \in \Phi_I^\infty$,

$$P^\varepsilon(X \in K) \leq \exp\Big\{ -\varepsilon^{-1}\inf_{\nu\in K} J_{0,T}(\nu, f)\Big\}\, E^\varepsilon\big[\exp \varepsilon^{-1} J_{0,T}(X,f)\,; X \in K\big]\,.$$

By Lemma 5.2

$$E^{\varepsilon}\big[\exp\varepsilon^{-1}J_{0,T}(X,f)\,;X\in K\big]\le E^{\varepsilon}\big[Z_t^{\varepsilon}(f)\big]=1\,.$$

Hence

$$\limsup_{\varepsilon\to 0}\varepsilon\log P^{\varepsilon}\big(X\in K\big)\le-\inf_{\nu\in K}J_{0,T}(\nu,f)\,,\qquad f\in\Phi_I^{\infty}\,,$$

therefore

$$\limsup_{\varepsilon\to 0}\varepsilon\log P^{\varepsilon}\big(X\in K\big)\le-\sup_{f\in\Phi_I^{\infty}}\inf_{\nu\in K}J_{0,T}(\nu,f)\,.$$

To finish the proof we will show that

$$\sup_{f\in\mathcal{C}_I^{\infty}}\inf_{\nu\in K}J_{0,T}(\nu,f)\ge\inf_{\nu\in K}S_{\mu}(\nu)\,.$$

Indeed, since it follows from A1 that for every $f\in\Phi_I^{\infty}$ the map $\nu\mapsto J_{0,T}(\nu,f)$ on $\mathcal{C}(I,E)$ is continuous and since K is compact, there exists an element $\nu_0\in K$ such that

$$\inf_{\nu\in K}J_{0,T}(\nu,f)=J_{0,T}(\nu_0,f)\,.$$

Thus, by (4.4),

$$\sup_{f\in\Phi_I^{\infty}}\inf_{\nu\in K}J_{0,T}(\nu,f)=\sup_{f\in\Phi_I^{\infty}}J_{0,T}(\nu_0,f)=S_{\mu}(\nu_0)\ge\inf_{\nu\in K}S_{\mu}(\nu)\,.$$

□

Next follows our final preparation for the proof of the lower bound, the law of large numbers for the perturbed measures $\widetilde{P}_h^{\varepsilon}$.

Lemma 7.2. *Fix ν such that $h\in\Phi_I^{\infty}$. For any open ball B_ν in $\mathcal{C}(I,E)$ which contains ν*

$$\widetilde{P}_h^{\varepsilon}\big(X\in B_\nu\big)\to 1\,,\quad\varepsilon\to 0\,.$$

Proof. By Corollary 6.7, $\widetilde{P}_h^{\varepsilon}$ is exponentially tight in ε. Hence, for every $\alpha>0$ there exists a compact set K_α in $\mathcal{C}(I,E)$ such that

$$\limsup_{\varepsilon\to 0}\varepsilon\log\widetilde{P}_h^{\varepsilon}\big(X\in K_\alpha^c\big)<-\alpha<0\,.$$

The set $B_\nu^c\cap K_\alpha$ being compact, by an application of the large deviation upper bound Lemma 7.1 to the probability measure $\widetilde{P}_h^{\varepsilon}$ with associated rate function $\widetilde{S}_h$,

$$\limsup_{\varepsilon\to 0}\varepsilon\log\widetilde{P}_h^{\varepsilon}\big(X\in B_\nu^c\cap K_\alpha\big)\le-\inf_{\xi\in B_\nu^c\cap K_\alpha}\widetilde{S}_h(\xi)\,.$$

Moreover, since $\widetilde{S}_h(\cdot)$ is lower semicontinuous there exists an element $\xi_0\in B_\nu^c\cap K_\alpha$ such that

$$\inf_{\xi\in B_\nu^c\cap K_\alpha}\widetilde{S}_h(\xi)=\widetilde{S}_h(\xi_0)>0\,.$$

Therefore, as $\varepsilon\to 0$

$$\widetilde{P}_h^{\varepsilon}\big(X\in B_\nu^c\big)\le\widetilde{P}_h^{\varepsilon}\big(X\in K_\alpha^c\big)+\widetilde{P}_h^{\varepsilon}\big(X\in B_\nu^c\cap K_\alpha\big)\to 0,$$

which completes the proof of the lemma. □

Lemma 7.3. *Fix $\mu \in E$ with $\nu_0 = \mu$ and $\nu \in \mathcal{C}(I, E) \cap H_\mu^{\mathrm{reg}}$. Let V be an open neighborhood of ν. Then*

$$\liminf_{\varepsilon \to 0} \ \varepsilon \log P^\varepsilon(X \in V) \geq -S(\nu)\,.$$

Proof. By Lemma 4.3 $S(\nu) < \infty$ since $\nu \in H_\mu$. By (4.10) we have $J_{0,T}(\nu, h) = S(\nu)$. Hence

$$\begin{aligned} P^\varepsilon(X \in V) &= \widetilde{E}_h^\varepsilon\Big[\exp -\varepsilon^{-1} J_{0,T}(X,h)\,; X \in V\Big] \\ &= \widetilde{E}_h^\varepsilon\Big[\exp -\varepsilon^{-1}\big[J_{0,T}(X,h) - J_{0,T}(\nu,h)\big]\,; X \in V\Big] \ \exp -\varepsilon^{-1} J_{0,T}(\nu,h) \\ &\geq \widetilde{E}_h^\varepsilon\Big[\exp -\varepsilon^{-1}\big|J_{0,T}(X,h) - J_{0,T}(\nu,h)\big|\,; X \in V\Big] \ \exp -S(\nu)/\varepsilon. \end{aligned}$$

For $\delta > 0$ let

$$G_\nu^\delta = \Big\{\xi \in \mathcal{C}(I,E) : \big|J_{0,T}(\xi,h) - J_{0,T}(\nu,h)\big| < \delta\Big\}.$$

Then

$$P^\varepsilon(X \in V) \geq \widetilde{P}_h(X \in V \cap G_\nu^\delta) \ \exp -(S(\nu) + \delta)/\varepsilon.$$

Since $V \cap G_\nu^\delta$ is an open neighborhood of ν we conclude using Lemma 7.2 that

$$\liminf_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(X \in V) \geq -S(\nu) - \delta,$$

δ being an arbitrary number the result follows. $\square$

Finally we discuss the cylindrical Brownian motion. Here $\mathcal{A} = 0$ and

$$Q(\mu\,; dx, dy) = \sigma^2 \delta_x(dy)\, dx.$$

In this case the set of functions Φ_I^∞ may be identified with the Schwartz functions $\mathcal{C}_0^\infty(I \times R^d)$ and the space $L^2(\nu, Q)$ reduces to the completion of $\mathcal{C}_0^\infty(I \times R^d)$ into the ordinary space $L^2(I \times R^d)$ with squared norm

$$\|f\|_0^2 := \|f\|_{Q(\nu)}^2 = \sigma^2 \iint_{I \times R^d} |f(t,x)|^2 \, dx dt.$$

Proposition 7.4. *For the example of cylindrical Brownian motion the large deviation lower bound*

$$\liminf_{\varepsilon \to 0} \varepsilon \log P^\varepsilon(X \in G) \geq -\inf_{\nu \in G} S(\nu)$$

holds, where in the case $\nu \in H_\mu$

$$S(\nu) = \frac{1}{2\sigma^2} \iint_{I \times R^d} \big(d\dot{\nu}_t/dx\big)^2 \, dx\, dt\,.$$

Proof. Note that if $S(\nu) = \infty$ there is nothing to prove. It is therefore enough to prove the local lower bound for each $\nu \in \mathcal{C}(I, E)$ such that $S(\nu) < \infty$, that is, for each $\nu \in H_\mu$. Let ν be fixed in H_μ, hence $h = h^\nu$ is fixed in $L^2(I \times R^d)$. Consider a sequence of functions ϕ^n in $\mathcal{C}_0^\infty(I \times R^d)$ which approximates h in the sense that

$$\|\phi^n - h\|_0 \to 0\,, \quad n \to \infty.$$

Let ν^n denote the unique solution of the McKean-Vlasov equation corresponding to ϕ^n according to Lemma 4.12. For each f in $\mathcal{C}_0^\infty(R^d)$ we then have

$$\langle \nu_t, f\rangle = \langle \mu, f\rangle + \sigma^2 \int_0^t \int_{R^d} f(x) h(s,x)\, dx ds,$$

$$\langle \nu_t^n, f\rangle = \langle \mu, f\rangle + \sigma^2 \int_0^t \int_{R^d} f(x) \phi^n(s,x)\, dx ds,$$

hence

$$\begin{aligned} \big|\langle \nu_t^n - \nu_t, f\rangle\big| &\le \sigma^2 \iint_{I\times R^d} |f(x)|\, |\phi^n(s,x) - h(s,x)|\, dx ds \\ &\le \sigma^2\, \|f\|_0\, \|\phi^n - h\|_0 \to 0\,, \quad n \to \infty. \end{aligned}$$

Therefore, as $n \to \infty$

$$\nu^n \Longrightarrow \nu \qquad \text{(weakly)}. \tag{7.5}$$

With this additional structure of weak convergence we have the following modification of the law of large numbers stated in Lemma 7.2: Let B_ν denote an open ball in $\mathcal{C}(I,E)$ centered at ν. By (7.5), there exists n_0 and an open ball $\widehat{B}_{\nu^n}$ centered at ν^n such that for all $n \ge n_0$ we have $\widehat{B}_{\nu^n} \subset B_\nu$. By Lemma 7.2,

$$\lim_{\varepsilon\to 0} \widetilde{P}^\varepsilon_{\phi^n}(X \in B_\nu) \ge \lim_{\varepsilon\to 0} \widetilde{P}^\varepsilon_{\phi^n}(X \in \widehat{B}_{\nu^n}) = 1,$$

hence

$$\lim_{n\to\infty} \lim_{\varepsilon\to 0} \widetilde{P}^\varepsilon_{\phi^n}(X \in B_\nu) = 1. \tag{7.6}$$

Now the proof of Lemma 7.3 is easily modified. Indeed, by (4.4)

$$\begin{aligned} P^\varepsilon(X \in V) &= \widetilde{E}^\varepsilon_{\phi^n}\Big[\exp -\varepsilon^{-1} J_{0,T}(X,\phi^n)\,;\, X \in V\Big] \\ &\ge \widetilde{E}^\varepsilon_{\phi^n}\Big[\exp -\varepsilon^{-1}\big|J_{0,T}(X,\phi^n) - J_{0,T}(\nu,\phi^n)\big|\,;\, X \in V\Big]\, \exp -S(\nu)/\varepsilon. \end{aligned}$$

Thus

$$P^\varepsilon(X \in V) \ge \widetilde{P}^\varepsilon_{\phi^n}(X \in V \cap G_\nu^{\delta,n})\, \exp -(S(\nu) + \delta)/\varepsilon.$$

where

$$G_\nu^{\delta,n} = \Big\{\xi \in \mathcal{C}(I,E) : \big|J_{0,T}(\xi,\phi^n) - J_{0,T}(\nu,\phi^n)\big| < \delta\Big\}.$$

But in view of (7.6), for n sufficiently large

$$\liminf_{\varepsilon\to 0} \varepsilon \log \widetilde{P}^\varepsilon_{\phi^n}(X \in V \cap G_\nu^{\delta,n}) = 0,$$

which completes the proof. $\square$

Remark 7.7. The proof of Proposition 7.4 outlines a general approach to the problem of lifting in the lower bound the restriction to the set of regular paths H_μ^{reg}. The weak convergence (7.5) amounts to showing that H_μ^{reg} is a dense subset of H_μ in the topology of weak convergence. However, for any other choice of covariance measure Q and an approximating sequence ϕ^n in $L^2(\nu, Q)$ we are unable to obtain the appropriate estimate of $\langle \nu_t - \nu_t^n, f\rangle$ in order to conclude (7.5).

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