On scaling limits of arrival processes with long-range dependence

Ingemar Kaj
Department of Mathematics, Uppsala University

Extended abstract

Various classes of arrival processes in telecommunication traffic modeling based on heavy-tailed interarrival time distributions exhibit long-range dependence. This includes arrival rate processes of Anick-Mitra-Sondhi (AMS) type where the rate process is an on/off-process with heavy-tailed on-period distribution and/or off-period distribution, as well as generalized Kosten type models (infinite source Poisson) with rate process given by the M/G/∞ queueing model with heavy-tailed service time distribution.

The nature of such arrival processes can be studied by investigating rescaling limit processes that arise from using appropriate space-time scaling schemes. The typical behavior is the following dichotomy, which is described below: If the connection rate per time unit is fast then the generic limit process is fractional Brownian motion, whereas if the number of connections increases slowly relative to time then a stable Lévy process appears as the approximating limit process. Such results have been obtained by Taqqu *et al.* (1997) and Levy and Taqqu (2000), for cases where the rescaling of time and space variables are performed in two separate steps. Similar results are obtained for joint scaling in time and space in Mikosh *et al.* (2002) for on-off processes and infinite source Poisson models and in Pipiras *et al.* (2002) for renewal rate processes. Gaigalas and Kaj (2002) consider an arrival process built from stationary renewal processes with heavy tailed interrenewal times. In addition to fast and slow connection rates an intermediate critical scaling regime is studied and a new limit process is established. This limiting process is neither Gaussian nor stable. It has continuous paths and stationary increments but is not self-similar.

In this note we extend the results of Gaigalas and Kaj (2002) to a class of renewal reward processes and discuss the interpretation of the scaling limit process.

Consider a sequence of independent, non-negative random variables $U_1, U_2, \ldots$ with distribution functions $F_1(t) = P(U_1 \le t)$ and $F(t) = P(U_k \le t)$, $k \ge 2$. It is assumed that the distribution given by $F(t)$ has finite expected value $\mu = E(U_2)$ and that T_1 is equipped with the equilibrium distribution function $F_1(t) = F_{\mathrm{eq}}(t) = \frac{1}{\mu}\int_0^t (1 - F(s))\, ds$. Let N_t, $t \ge 0$, denote the stationary renewal counting process associated with the sequence of interrenewal times $(U_k)_{k\ge1}$ and, in addition, let $(X_k)_{k\ge1}$ be an i.i.d. sequence of random variables, independent of N_t, signifying rewards associated with each renewal event:

$$N_t = \sum_{n=1}^{\infty} 1_{\{S_n \le t\}}, \quad S_n = \sum_{k=1}^{n} U_k, \qquad A_t = \sum_{k=1}^{N_t} X_k,$$

where A_t is the total accumulated reward in the interval $[0, t]$. The mean reward $0 < \nu = E(X_1) < \infty$ is supposed to be positive and exist finitely and hence the mean accumulated reward is $E(A_t) = \nu\, t/\mu$, by Walds lemma using the stationarity of the renewal process.

The basic assumption of heavy tails in the model is that the interrenewal time distribution $F(t)$ belongs to the domain of attraction of a stable law with index $1+\beta$, $0<\beta<1$, namely

$$1-F(t)\sim t^{-(1+\beta)}\,L_1(t)\qquad t\to\infty \tag{1}$$

($x_n\sim y_n$ iff $x_n/y_n\to 1$ as $n\to\infty$), where $L_1(t)$ is a function which is slowly varying at infinity, i.e. $L_1(xt)/L_1(t)\to 1$ for each $x>0$ as $t\to\infty$. For the reward distribution $G(t)=P(X_k\le t)$ we assume either finite variance, $\operatorname{Var}X<\infty$, or that $G(t)$ has a regularly varying tail with exponent $1+\gamma$, $\beta<\gamma<1$ (lighter tail than the one of the interrenewal distribution), i.e. there is a slowly varying function L_2 such that

$$1-G(t)\sim t^{-(1+\gamma)}L_2(t),\quad t\to\infty.$$

Let $(\{N_t^i,A_t^i\})_{i\ge 1}$ be a sequence of independent copies of the renewal process. We are interested in the limit behavior of the superposition process $W_t^m=\sum_{i=1}^m A_t^i$, suitably scaled. The primary application we have in mind is that W_t^m counts the accumulated workload generated from m independent connections or users sharing a common medium, such as a local area network, when the arrival stream of a single connection is characterized by a heavy tailed interarrival distribution and the work generated at each arrival (e.g. packet size) either has finite variance or has a lighter tail than the interarrival times.

Gaigalas and Kaj (2002) studies the limit of the centralized and scaled arrival process (rewards $X_k\equiv 1$)

$$Y^{(m)}(t)=\frac{1}{b_m}\sum_{i=1}^m\left(N^i_{a_m t}-\frac{a_m t}{\mu}\right),\quad m\to\infty, \tag{2}$$

where a_m is such that $a_m\to\infty$ and satisfies one of the following conditions:

$$\frac{a_m^\beta}{mL_1(a_m)}\to\begin{cases}0, & \text{Fast connection rate (FCR)}\\ 1, & \text{Intermediate connection rate (ICR)}\\ \infty. & \text{Slow connection rate (SCR)}\end{cases}$$

The sequence b_m is an admissible sequence for (2) to converge in distribution to a non-degenerate limit. For the case ICR the norming sequence is simply $b_m=a_m$. Under the condition FCR the limit is fractional Brownian motion with Hurst index $H=1-\beta/2$ and under SCR a stable Lévy process with index $\alpha=1+\beta$ arises. In the intermediate regime ICR, however, weak convergence is established towards a limit process $Y_{\beta,\mu}$, which in a sense provides a link between fractional Brownian motion and stable Lévy processes. In this note we extend this result to renewal reward processes as follows.

Theorem. *Under the intermediate growth condition (ICR), as $m\to\infty$ the weak convergence of processes*

$$\left\{\frac{1}{a_m}\sum_{i=1}^m\left(A^i_{a_m t}-\frac{\nu a_m t}{\mu}\right)\right\}\ \Rightarrow\ \{\nu\,Y_{\beta,\mu}(t)\} \tag{3}$$

holds, where $Y_{\beta,\mu}(t)$ is a zero mean, non-Gaussian and non-stable process with stationary increments. The limit process is not self-similar and the trajectories are Hölder continuous of order γ for any $0<\gamma<1$. The finite-dimensional distributions of the increments of $Y_{\beta,\mu}(t)$ are characterized by the following cumulant generating function:

$$\log E\exp\Big\{\sum_{i=1}^n\theta_i\big(Y_{\beta,\mu}(t_i)-Y_{\beta,\mu}(t_{i-1})\big)\Big\}=\frac{1}{\mu^3\beta}\sum_{i=1}^n\theta_i^2\int_0^{t_i-t_{i-1}}\int_0^v e^{-\theta_i u/\mu}u^{-\beta}\,du dv$$

$$+\frac{1}{\mu^3\beta}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\theta_i\theta_j e^{-\sum_{k=i+1}^{j-1}\theta_k(t_k-t_{k-1})/\mu}$$
$$\times\int_0^{t_i-t_{i-1}}\int_0^{t_j-t_{j-1}} e^{-\theta_j u/\mu}e^{-\theta_i v/\mu}(t_{j-1}-t_i+u+v)^{-\beta}\,dudv, \qquad (4)$$

where $0=t_0\le t_1\le\ldots\le t_n$.

The relationship of the limit process to fractional Brownian motion is clarified by noting that the process $Y_{\beta,\mu}(t)$ has the same covariance function as the (multiple of) fractional Brownian motion $\sigma_{\beta,\mu}\,B_H(t)$, while higher order cumulants and moments are different, and obeys the scaling limit relation

$$c^H\,Y_{\beta,\mu}(t/c)\;\Rightarrow\;\sigma_{\beta,\mu}\,B_H(t),\quad H=1-\beta/2,\qquad c\to\infty, \qquad (5)$$

in the sense of weak convergence of continuous processes.

Regarding the interpretation of the limit process as a packet arrival model, recall that the sum $W_t^m=\sum_{i=1}^m A_t^i$ in (2) is the accumulated packet load generated by m connections sharing a common medium, when the arrival stream from each source is characterized by a heavy-tailed interarrival distribution. It follows from the theorem that for large m

$$W_t^m\approx\frac{1}{\mu}\nu mt+a_m\,\nu Y_{\beta,\mu}(t/a_m)\sim\frac{1}{\mu}\nu mt+\nu\sqrt{mL_1(a_m)}\,a_m^{1-\beta/2}\,Y_{\beta,\mu}(t/a_m).$$

Invoking (5) gives the coarser approximative representation

$$W_t^m\approx\frac{1}{\mu}\nu mt+\nu\sqrt{mL_1(a_m)}\,B_{1-\beta/2}(t).$$

This provides a verification of the model for Ethernet traffic proposed by Norros (1995). A more comprehensive discussion of arrival process modeling with long-range dependence and further references can be found in Kaj (2002).

References

R. Gaigalas, I. Kaj (2002), Convergence of scaled renewal processes and a packet arrival model, U.U.D.M. Report 2002:2, Uppsala University. Submitted.

I. Kaj (2002), *Stochastic modeling in broadband communications systems*, to appear in SIAM Monographs in Mathematical Modeling and Computation.

J. B. Levy & M. S. Taqqu (2000), Renewal reward processes with heavy-tailed inter-renewal times and heavy-tailed rewards, *Bernoulli* **6**(1), 23-44.

Th. Mikosch, S. Resnick, H. Rootzén, A. Stegeman (2002), Is network traffic approximated by stable Lévy motion or fractional Brownian motion?, to appear in *Ann. Appl. Probab.*

I. Norros (1995), On the use of fractional Brownian motion in the theory of connectionless networks, IEEE J. Select. Areas Commun. **13** 953-962.

V. Pipiras, M. S. Taqqu & J. B. Levy (2002), Slow, fast and arbitrary growth conditions for renewal reward processes when the renewals and the rewards are heavy-tailed, Boston University, Preprint, to appear in *Probab. Theory Relat. Fields.*

M. S. Taqqu, W. Willinger & R. Sherman (1997), Proof of a fundamental result in self-similar traffic modeling, *Computer Communications Review* **27**(2) 5-23.