A note on first passages in branching Brownian motions

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Abstract: Consider a realization of a one-dimensional branching Brownian motion generated by a single particle at the origin. Fix $x > 0$ and count any particle visiting x as the first among the members of its line of descent from the initial particle. The resulting first-passage process indexed by x is a Markov branching process which can be characterized in terms of the original branching mechanism. In this note we study the first-passage process with emphasis on supercritical properties.

Introduction We consider a Markov branching Brownian motion on the real line with an initial particle located at the origin at time zero. Each particle, independently, is assigned an exponentially distributed lifelength with common rate $\alpha > 0$ and performs a Brownian motion starting at the position where its parent died. At time of death a particle is removed and replaced with a random number of new particles according to the offspring distribution $\{p_k : k \geq 0\}$.

To this type of branching tree we associate an integer-valued random process $Z = \{Z_x : x \geq 0\}$, $Z_0 = 1$, indexed by the space variable x restricted to the positive halfline. This process counts for each $x > 0$ the number of particles in the tree with the property that they have *no ancestor who ever visited the space location x but during their own life a first visit to x indeed occured.*

In Kaj and Salminen (1992) the process Z is defined rigorously as a functional of a branching Brownian motion $X = \{X_t : t > 0\}$, $X_0 = 0$, and it is shown that under the assumption of subcritical or critical branching Z is a continuous time Galton-Watson process. Its branching rate is $\sqrt{2\alpha}$ and the offspring distribution $\{r_k : k \geq 0\}$ is determined in terms of the given probabilities p_k. The object here is to extend this result to cover also supercritical branching and to demonstrate that features of X relevant for the supercritical case nicely carry over to Z. This includes extinction sets, Harris transformation and martingale convergence.

The first-passage process We will only repeat the construction of X and Z briefly and refer the reader to Kaj and Salminen (1992), which is intended to be selfcontained but to a large extent is based on the work of Neveu (1987) and Chauvin (1991).

A *particle* $u = (j_1, \ldots, j_n)$ is represented by a sequence of reproduction numbers $j_i \geq 1$ which signify that u is the j_nth child of $(j_1, \ldots, j_{n-1})$, who is the j_{n-1}th child of ... the j_1th child of the initial particle, denoted $\{0\}$. A *tree* ω is a set of particles which contains $\{0\}$ and all their ancestors and progeny. Particles are partially ordered by $u < v$ if u is an ancestor of v. To each particle u associate a lifelength ζ^u and a spatial motion γ^u. Let Ω^o denote the set of all *marked trees* ω^o defined as

$$\omega^o = \{\omega \, , \, \{(\zeta^u, \gamma^u)\} \, : \, u \in \omega\} \, ,$$

where ω runs over all trees. Now we let $\mathcal{F}^o$ denote the smallest σ-algebra on Ω^o which makes all marked trees measurable and P be the probability measure on $(\Omega^o, \mathcal{F}^o)$ such that all ζ^u are exponential with parameter α, all γ^u are standard Brownian motions starting at the location where the parent of u died, the branching is governed by the offspring generating function $F(u) := \sum_k p_k u^k$ and all particles evolve independently. We assume that the offspring distribution is proper ($F(1) = 1$) and impose the *non-explosion hypothesis*: For every $\varepsilon > 0$

$$\int_{1-\varepsilon}^{1} (u - F(u))^{-1} \, du = \infty \, . \tag{1}$$

The branching Brownian motion X is the canonical process on the triple $(\Omega^o, \mathcal{F}^o, P)$. In addition we let $N = \{N_t : t > 0\}$ denote the process which counts the total number of particles present at time t irrespective of their position in space.

Further, in order to define Z, let σ_y^u denote the time of the first visit of a particle u to the point $y \geq 0$ and set $\sigma_y^u = \infty$ if u never visits y. Then

$$\tau_y^u := \begin{cases} \sigma_y^u \, , & \text{if } \sigma_y^u < \infty \text{ and there is no } v < u \text{ with } \sigma_y^v < \infty \\ +\infty \, , & \text{otherwise} \end{cases}$$

is the time of the first visit to y for those particles u which do not have a predecessor v that already did visit y. Now for a marked tree ω^o let

$$L_x(\omega^o) = \{u \in \omega : \; \tau_x^u(\omega^o) < \infty\}$$

be the set of particles with this first passage property in common and define the (rightcontinuous) *first passage process* associated with X as

$$Z = \{Z_x = |L_{x+}| : \, x \geq 0\} \, ,$$

where $|\{\cdot\}|$ denotes the number of elements in $\{\cdot\}$.

Supercritical branching trees In this section we collect some well-known properties, emphasizing supercritical aspects, of a continuous time Galton-Watson process $Y = \{Y_t : t \geq 0\}$, with branching rate η and offspring generating function H, say (see e.g. Athreya and Ney 1972, Ch. III). The basic example in our context is the total size process N with $N_0 = 1$, branching rate α and offspring generating function F. Our aim is to show that the first-passage process constitutes a further example and then to compare the properties of the two.

We include the defective case, $H(1) \leq 1$, we suppose that $H(u)/H(1)$ satisfies (1) and interpret $p_\infty := 1 - H(1) \geq 0$ as the probability of a jump to an additional state $\{+\infty\}$ (hence properties referring to $H(1) < 1$ does not apply to the given offspring generating function F). Define

$$q = \inf\{u \geq 0 : H(u) = u\}.$$

Of course, if Y is (sub)critical then $q = 1$. Consider the initial-value problems

$$\varphi' = \eta(H(\varphi) - \varphi), \qquad \varphi(-t_0) = 1, \tag{2}$$

where the parameter $t_0 > 0$. If $p_\infty > 0$ there is a unique decreasing solution φ on $[-t_0, \infty[$ with $\varphi'(-t_0) = -\eta(1 - H(1)) < 0$ and values in $]q, 1]$, $\varphi(\infty) = q$. If H is proper ($H(1) = 1$) the modified system

$$\varphi' = \eta(H(\varphi) - \varphi), \qquad \varphi(-\infty) = 1, \tag{2'}$$

has a solution $\varphi \in]q, 1[$, $\varphi(\infty) = q$, defined on the whole real line. Moreover, in both the defective and the proper case the related system

$$\varphi' = \eta(H(\varphi) - \varphi), \qquad \varphi(-t_0) = 0, \tag{3}$$

has an increasing unique solution taking values in $[0, q[$, $\varphi(\infty) = q$, with $\varphi'(-t_0) = \eta p_0$ ($p_0 = 0$ if and only if $q = 0$ and then $\varphi \equiv 0$).

Put $\overline{\varphi}(t, u) = \mathbf{E}[u^{Y_t}]$, $0 \leq u \leq 1$. The solutions of (2), (2′) and (3) determine $\overline{\varphi}(t, u)$ by the relation

$$\overline{\varphi}(t, \varphi(s)) = \varphi(t + s), \quad s \geq -t_0, \; t > 0.$$

(For this characterization of Y see also Neveu (1987).) For $q < u \leq 1$ and $t_0 < \infty$ the solutions of (2) apply and then $\overline{\varphi}(t, 1) = P(Y_t < \infty)$ and $\varphi(0) = P(Y_{t_0} < \infty)$. On the other hand, (2′) yields $\overline{\varphi}(t, 1) = 1$, $t \geq 0$. Hence this is the relevant φ for a proper supercritical branching process, i.e., $P(Y_t < \infty)$ for all t and $H(1) = 1$. Taking $\varphi = q$ results in $\overline{\varphi}(t, q) = q$ for all $t \geq 0$. Lastly, if $0 \leq u < q$ the solutions of (3) prescribe $\overline{\varphi}(t, u)$. Here $\overline{\varphi}(t, 0) = P(Y_t = 0)$ and the choice of t_0 is reflected in $\varphi(0) = P(Y_{t_0} = 0)$.

Let

$$A = \{Y_t \to \infty \text{ as } t \to \infty\}, \quad B = \{Y_t = 0 \text{ for some } t\}.$$

Then $P(A \cup B) = 1$ and the probability of the *extinction set* B is $P(B) = q$. If $p_\infty > 0$ then $A = \{Y_t = +\infty \text{ for some } t\}$ *a.s.* The supercritical branching process can be decomposed in two parts corresponding to the sets A and B, see Athreya and Ney (1972), Ch. I.12. We recall this idea briefly:

$\{i\}$ For $t \geq 0$ define a new process $Y_t^{(1)}$ as follows. If $\omega^o \in A$ let $Y_t^{(1)}(\omega^o)$ be the number of particles among the $Y_t(\omega^o)$ existing at time t which has an infinite number of descendents, if $\omega^o \in B$ let $Y_t^{(1)}(\omega^o) = 0$. Then the law of $\{Y_t^{(1)}, t \geq 0\}$ conditioned on A is the law of a continuous time Galton-Watson process with the offspring generating function

$$\widetilde{H}(u) := \frac{H((1-q)u + q) - q}{1-q}, \quad 0 \leq u \leq 1 .$$

This process is called the *Harris-transformation* of Y. Note that $Y^{(1)}$ is obtained from Y by "erasing all the branches which become extinct". This point of view is adopted in Neveu (1986). In Salminen (1992) the construction is done for (rather) general Markov branching processes and so called terminal lines. It is quite obvious that the above described erasure procedure can also be performed for the branching Brownian motion X. The resulting process, called the Harris-transformation of X, is again a branching Brownian motion but now the generating function of the offspring distribution is given by $\widetilde{H}$.

$\{ii\}$ The law of the process $Y_t^{(2)} = Y_t - Y_t^{(1)}$ can be obtained from the law of Y by conditioning on extinction. Here, the offspring generating function of $Y^{(2)}$ is

$$\widehat{H}(u) := \frac{H(qu)}{q}, \quad 0 \leq u \leq 1 .$$

A similar transformation in the case of X leads again to a branching Brownian motion with $\widehat{H}$ as the generating function of the offspring distribution. Note that $\widetilde{q} = 0$ and $\widehat{q} = 1$, using self-explanatory notation.

We shall denote by Y_t^* the branching process with the proper offspring generating function $H(u)/H(1)$. It is obtained from Y_t by conditioning on the event that jumps to $\{+\infty\}$ do not occur. For ease of notation we may keep the star and identify Y and Y^* in the case $p_\infty = 0$. For example, we have

$$\mathbf{E}[Y_t^*] = \exp\{\eta(\frac{H'(1)}{H(1)} - 1)t\} . \tag{4}$$

Also note that the extinction probability increases from q to the smallest q^* with $H(q^*) = q^* H(1)$ and that $\widetilde{H}$ and $\widehat{H}$ change accordingly if we shift from Y to Y^*. If Y^* is supercritical ($H'(1) > H(1)$) then $Y_t^* / \mathbf{E}[Y_t^*]$, $t \geq 0$, is a nonnegative martingale such that the limit $W = \lim_{t\to\infty} Y_t^* / \mathbf{E}[Y_t^*]$ exists *a.s.* and either $W = 0$ *a.s.* or $\mathbf{E}[W] = 1$, the latter holding if and only if

$$\sum_k p_k \, k \log k < \infty . \tag{5}$$

Properties of the first passage process We are now prepared to formulate a somewhat more general result than Theorem 1 in Kaj and Salminen (1992). The part of the proof which is covered there is only sketched below.

THEOREM. *The first passage process Z_x associated with a branching Brownian motion X with creation rate α and offspring generating function F with extinction probability $0 \le q \le 1$ is a continuous time Galton-Watson process, such that the creation rate of Z equals $\sqrt{2\alpha}$, the extinction probability of Z is q and the extinction sets of Z and N are a.s. equal and the offspring generating function $G(v) = \sum_k r_k v^k$ of Z is given by*

$$G(v) = v + sgn\{q - v\}\sqrt{2\int_v^q (F(u) - u)\,du}\,, \quad 0 \le v \le 1\,. \tag{6}$$

In particular, in the supercritical case G is always defective with

$$r_\infty := 1 - G(1) = \sqrt{2\int_q^1 (u - F(u))\,du} > 0\,.$$

Proof The Markov and branching properties and hence the conclusion that Z is a continuous time Galton-Watson process follow from the strong Markov property at the stopping line $\tau_x = \{\tau_x^u\}$. The proof of the fact that the branching rate equals $\sqrt{2\alpha}$ is based on the observation that the first jump of Z occurs after a time which has the same distribution as $\sup_{t \le \tau} B_t$, where B is a standard Brownian motion and τ an exponential random variable with mean $1/\alpha$. In the rest of the proof we put $\eta_\alpha := \sqrt{2\alpha}$.

Consider now the extinction sets of Z and N. Since the branching Brownian motion, if it does not become extinct, visits all the points in $\mathbf{R}$ with probability one it follows that

$$\{N_t = 0 \text{ for some } t\} \subseteq \{Z_x = 0 \text{ for some } x\} \quad a.s.$$

The reverse inclusion is obviously true and, hence, the extinction sets are *a.s.* equal.

Put $\overline{\psi}(x, u) = \mathbf{E}[u^{Z_x}]$, $x \ge 0$, $0 \le u \le 1$. From the above it follows for $0 < u < 1$ that $\lim_{x\to\infty} \overline{\psi}(x, u) = q$. Without presuming the condition $F'(1) \le 1$ we can still follow the derivations in Kaj and Salminen (1992) to the point of showing that $\overline{\psi}(x, u)$, $x \ge 0$, satisfies

$$\psi''(x) = -\eta_\alpha^2 (F(\psi(x)) - \psi(x))\,. \tag{7}$$

Using $\lim_{x\to\infty} \psi'(x) = 0$, we obtain

$$\psi'(x)^2 = -2\int_x^\infty \psi'(y)\psi''(y)\,dy = 2\eta_a^2 \int_x^\infty \psi'(y)(F(\psi(y)) - \psi(y))\,dy\,.$$

By a change-of-variable,

$$\psi'(x)^2 = 2\eta_a^2 \int_{\psi(x)}^q (F(u) - u)\,du\,.$$

(Here we use $\psi(\infty) = q$ and allow integration from the right to the left of an interval.) Thus, since ψ is monotone,

$$\psi'(x) = \eta_\alpha(G(\psi(x)) - \psi(x)), \tag{8}$$

where G is as given in (6). It is easily seen that $v \mapsto G(v)$ is a moment generating function, which is defect in the case $q < 1$. Hence equation (8) is of the type (2) or (3), whereas the type (2′) does not apply. In fact, it is known that there are no decreasing (travelling wave) solutions of equation (7) on $\mathbf{R}$ with $\psi(-\infty) = 1$ and $\psi(\infty) = q < 1$, see Chauvin (1991), Thm 3.1.

According to the previous section we then obtain solutions of (8) which uniquely determine $\overline{\psi}(x, u)$, $x \geq 0$, by the relation

$$\overline{\psi}(x, \psi(y)) = \psi(x + y), \quad x \geq 0,$$

where y is allowed to vary over an interval $[-x_0, \infty)$, $x_0 > 0$. □

We proceed stating some further properties of Z and of the conditioned process Z^* with offspring generating function $G(u)/G(1)$. The proof of the first corollary is straightforward from (6).

COROLLARY 1. *$\{i\}$ The process Z conditioned on extinction has the offspring generating function*

$$\widehat{G}(v) = v + \sqrt{2\int_v^1 (\widehat{F}(u) - u)\, du}, \quad 0 \leq v \leq 1,$$

i.e., it is also obtained from X conditioned on extinction.
$\{ii\}$ The Harris-transformation of Z has the offspring generating function

$$\widetilde{G}(v) = v - \sqrt{2\int_0^v (\widetilde{F}(u) - u)\, du}, \quad 0 \leq v \leq 1,$$

i.e., it is also obtained from the Harris-transformation of X.

COROLLARY 2. *If X (or, equivalently, N) is supercritical ($q < 1$) then Z^* is supercritical with finite mean*

$$\mathbf{E}[Z_x^*] = \exp\{\sqrt{2\alpha}\,(\frac{r_\infty}{1 - r_\infty})\, x\}.$$

The martingale limit $W = \lim_{x\to\infty} Z_x^* / \mathbf{E}[Z_x^*]$ exists a.s. and is nontrivial.

Proof Since $G'(1) = 1 > G(1)$, Z is supercritical with finite mean obtained from (4).

To prove the last statement we need to establish (5) for the probabilities $\{r_k/G(1) : k \geq 1\}$. However, this instance of (5) is equivalent to (Athreya and Ney, 1972, Ch I.10)

$$\int_0^1 v^{-2}(G(1 - v) - G(1) + G'(1)v)\, dv < \infty. \tag{8}$$

By a Taylor expansion (recall that we have assumed $F(1) = 1$ throughout)

$$G(1-v) - G(1) + G'(1)v \sim \int_0^v (1 - F(1-u) + u)\, du\,, \quad \text{as } v \to 0\,.$$

Hence (8) is equivalent to

$$\int_0^1 \frac{1}{u}(1 - F(1-u))\, du < \infty\,.$$

Let U be a random variable with the distribution $\{p_k\}$. Changing variable we see that (8) is equivalent to

$$\int_0^c \frac{1}{u}(1 - F(e^{-u}))\, du = \mathbf{E}[\int_0^{cU} \frac{1}{u}(1 - e^{-u})\, du] < \infty\,,$$

for some positive constant c. Clearly this is the same as $\mathbf{E}[\log_+ U] < \infty$ ($\log_+ x := \log(x \vee 1)$), which in its turn is the same as

$$\sum_k p_k \log_+ k < \infty\,. \tag{9}$$

We finish the proof by showing that (9) follows from (1). Actually, the argument below shows that a non-explosive branching process has more than the logarithmic moment required here. We use a reformulation of the non-explosion hypothesis due to Doney (1984). He proves that (1) is equivalent to

$$\sum_{n=1}^{\infty} (nM(n))^{-1} = \infty\,, \quad M(n) := \sum_{k=0}^{n} \sum_{i=k+1}^{\infty} p_i\,.$$

Let b_n be a sequence such that $1/b_n$ is convex, decreasing and $\sum_n b_n^{-1} < \infty$. Since the divergent sequence $(nM(n))^{-1}$ is convex and decreasing to zero there is a n_0 such that $b_n > nM(n)$, all $n > n_0$. Hence

$$\sum_{k=n_0}^{\infty} P(U > k) \sum_{n=k}^{\infty} n b_n^{-2} \le \sum_{n=n_0}^{\infty} n b_n^{-2} M(n) < \sum_{n=n_0}^{\infty} b_n^{-1} < \infty,.$$

In particular, take $b_n = n^{3/2}$ to see that (9) holds. $\square$

References

Athreya, K.B. and Ney, P.E. (1972). Branching processes. Springer, Berlin-Heidelberg-New York.

Chauvin, B. (1991). Product martingales and stopping lines for branching Brownian motion. *Ann. Probab.* **19**, 1195-1295.

Doney, R.A. (1984). A note on some results of Schuh. *J. Appl. Prob.* **21**, 192-196.

Kaj, I. and Salminen, P. (1992). On a first passage problem for branching Brownian motions. To appear: *Ann. Appl. Probab.* 14pp.

Neveu, J. (1986). Erasing a branching tree. *Adv. Appl. Prob. Spec. Suppl. 1986*, 101-108.

Neveu, J. (1987). Multiplicative martingales for spatial branching processes. In: *Seminar in Stochastic Processes*, Çinlar, Chung, Getoor, (Eds.). Birkhuser, Boston-Basel, pp. 223-242.

Salminen, P. (1992). Cutting Markovian trees. To appear: *Proceedings of the 2nd Finnish-Soviet Symposium on Prob. Theory and Math. Statistics, Annales Academiae Fennicae, Series A.*

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