Asymptotic fluctuations of a particle system with singular interaction

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Abstract

A particle system on the real line with singular interaction consisting of electrostatic repulsion and a linear restoring force is considered. The empirical measure process is known to converge weakly in a space of continuous measure valued functions. In this work we show that the fluctuations around the limiting process, appropriately scaled, converge weakly to a Gaussian distribution-valued process.

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0. Introduction.

We are going to study a sequence of particle systems where there are n particles and the location of particle i at time t is at $\lambda_t^{i,n} = \lambda_t^i$. The motion of the system is governed by the system of Itô equations

$$d\lambda_t^i = \frac{\sigma}{\sqrt{n}}\, d\beta_t^i - \theta\lambda_t^i\, dt + \frac{\alpha}{n}\sum_{j\neq i}^{n} \frac{dt}{\lambda_t^i - \lambda_t^j}, \qquad i = 1, \ldots, n, \tag{1}$$

where $\{\beta_t^i,\, i = 1, \ldots, n\}$ are independent standard Brownian motions. This system has been studied by Rogers and Shi (1993) and Chan (1992). Rogers and Shi (1993) found that if $2\alpha \geq \sigma^2$ then almost surely a unique solution exists in which particles never collide. If $0 < 2\alpha < \sigma^2$ boundary conditions can be imposed so that the order is still well defined, see Cépa and Lépingle (1997). A main object of study is the empirical measure process

$$X_t^n = \sum_{i=1}^{n} \frac{1}{n}\delta_{\lambda_t^i}, \qquad t \geq 0, \tag{2}$$

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which asymptotic behavior as $n \to \infty$ was studied in Rogers and Shi (1993) and Chan (1992). They found that X_t^n converges weakly in the space of continuous probability measure-valued processes to a process X_t, $t \geq 0$, which follows a deterministic law. The limit has a unique stationary measure X_∞ which has the density of a scaled semi-circle around zero.

This particle system has interesting applications in the field of random matrices. A symmetric matrix process S_t in which all elements of the upper triangle are independent Ornstein-Uhlenbeck processes will have eigenvalues governed by equations (1), see Dyson (1962) or Chan (1992). In this example the stationary measure of the Ornstein-Uhlenbeck process is normal and hence the stationary measure S_∞ on the symmetric matrices is equivalent to a symmetric matrix consisting of independent normal random variables. Also this kind of matrices appear at finite time if we start each element in the matrix at zero. Now this kind of matrices satisfies the Wigner semi-circle law, see Mehta (1967), which states that the density $\pi(x)$ of the asymptotic distribution of the normalized eigenvalues is given by $\pi(x) = c\sqrt{r^2 - x^2}$, $-r \leq x \leq r$, i.e. a semicircle. We shall consider the scaled fluctuations $Y_t^n = n(X_t^n - X_t)$. Convergence of such fluctuations has been studied for other models. Examples of this kind of studies are superprocesses, Dawson, Fleischmann and Gorostiza (1989), and mean field interaction models, Dawson (1983). Related work on fluctuations in similar models are Spohn (1986), (1998) who studied the stationary fluctuation limit of a sequence of interacting Brownian particle systems of the same type on the line and on the circle, Duvillard [7], who considered polynomial test-functions of the model above and Johansson (1998) who considered the fluctuations of eigenvalues of random Hermitian matrices. The aim of this paper is to prove weak convergence of Y_t^n to a distribution valued Gaussian process. The measure associated with this process can be characterized as the probability measure that satisfies the requirement of Definition 1 in Section 1. Especially it is a solution to the martingale problem in Section 1. The uniqueness argument in Section 6 gives, as a byproduct, a method to calculate the Gaussian finite dimensional distributions.

1. Notation.

Fix $T > 0$ and let for each $n \geq 1$ (Ω^n, P^n) be a probability space such that there exists measurable functions $\beta_t^1(\omega), \dots, \beta_t^n(\omega) : \Omega^n \to C([0,T], R)$ which are independent Brownian motions. Let $\alpha, \theta > 0$ and define the particle positions $\lambda_t^1, \dots \lambda_t^n$, $0 \leq t \leq T$, by the Itô equations (1). The results in Rogers and Shi (1993) imply that the particles are almost surely uniquely defined. The restriction $2\alpha \geq \sigma^2$ may be removed, see Cépa and Lépingle (1997).

By Itô's formula, the empirical measure, X_t^n, acting on a C^2-function f almost surely satisfies

$$\begin{aligned}\langle X_t^n, f\rangle &= \langle X_0^n, f\rangle + \frac{\alpha}{2}\int_0^t \iint \frac{f'(x) - f'(y)}{x - y} X_s^n(dx) X_s^n(dy)\, ds \\ &\quad - \theta \int_0^t \int x f'(x) X_s^n(dx)\, ds + \frac{\gamma}{n}\int_0^t \langle X_s^n, f''\rangle\, ds + \frac{\sigma}{n\sqrt{n}} \int_0^t \sum_{i=1}^n f'(\lambda_s^i)\, d\beta_s^i,\end{aligned} \tag{3}$$

where $\gamma = (\sigma^2 - \alpha)/2$.

Remark. Note that the case $\theta = 0$ deals with the dynamics of a matrix with Brownian motion elements. Excluding this possibility is not necessary and the arguments below can be modified to include this. Letting $z_t \to z_t e^{\theta t}$ in eq. (10) in Lemma 2 below, we see that we also could allow for negative θ.

Let $\mathcal{D} = \{f \in C^2(R) : f, xf'(x), f''(x) \text{ are bounded}\}$. The results in Rogers and Shi (1993) or Chan (1992) allow us to define X_t as the unique nonrandom continuous measure-valued process which solves

$$\begin{aligned}\langle X_t, f\rangle =&\langle X_0, f\rangle + \frac{\alpha}{2}\int_0^t \iint \frac{f'(x)-f'(y)}{x-y} X_s(dx)X_s(dy)\,ds \\ &- \theta\int_0^t \int xf'(x)\,X_s(dx)\,ds, \qquad f\in\mathcal{D}.\end{aligned} \tag{4}$$

By making the transformation $\theta\lambda \to \lambda$, $\theta t \to t$, $\theta\alpha \to \alpha$, $\sqrt{\theta}\sigma \to \sigma$ and $\sqrt{\theta}\beta_t \to \beta_t$ we can take $\theta = 1$. Let $\mathcal{M}$ denote the space of probability measures on the real line endowed with the weak topology. Rogers and Shi (1993) and Chan (1992) prove that X_t^n, $t \geq 0$, converge weakly in $C([0,\infty),\mathcal{M})$ to X_t, $t\geq 0$, if X_0^n converge weakly to X_0 in $\mathcal{M}$. Weak convergence of measures implies that each functional,

$$\langle X_t^n, f\rangle = \int f(x) X_t^n(dx),$$

where f is a bounded continuous function, converge weakly in $C[0,T]$.

Consider the fluctuation process $Y_t^n = n(X_t^n - X_t)$. This process satisfies for $f\in\mathcal{D}$ almost surely the equation

$$\begin{aligned}\langle Y_t^n, f\rangle = \langle Y_0^n, f\rangle &- \int_0^t \int xf'(x)\,Y_s^n(dx)\,ds \\ &+ \frac{\alpha}{2}\int_0^t \iint \frac{f'(x)-f'(y)}{x-y}[X_s^n(dx)Y_s^n(dy) + X_s(dx)Y_s^n(dy)]\,ds \\ &+ \gamma\int_0^t \langle X_s^n, f''\rangle\,ds + \frac{\sigma}{\sqrt{n}}\int_0^t \sum_{i=1}^n f'(\lambda_s^i)\,d\beta_s^i.\end{aligned} \tag{5}$$

Let $C^{\infty\#}$ be the space of periodic C^∞-functions with period π. Denote the restriction of $C^{\infty\#}$-functions to the interval $(-\pi/2,\pi/2)$ by $C^{\infty\#}(-\pi/2,\pi/2)$. We have that $C^{\infty\#}$ is a nuclear Fréchet space, see Treves (1967), hence $C^{\infty\#}(-\pi/2,\pi/2)$ is also a nuclear Fréchet space isomorphic to $C^{\infty\#}$. Consider the linear map $v_{\tan} : C^{\infty\#}(-\pi/2,\pi/2) \to C^\infty(-\infty,\infty)$ by

$$v_{\tan}(f) = f\circ\arctan\,.$$

The map $v_{\tan}$ is one-to-one. Put $\widetilde{\mathcal{S}} = v_{\tan}(C^{\infty\#}(-\pi/2,\pi/2))$ and endow $\widetilde{\mathcal{S}}$ with the topology carried by $v_{\tan}^{-1}$, making $v_{\tan}$ an isomorphism. Because of this isomorphism $\widetilde{\mathcal{S}}$ is a nuclear Frechét space. Denote by $\widetilde{\mathcal{S}}'$ the dual of $\widetilde{\mathcal{S}}$ with its strong topology. The strong dual is a locally convex space and hence we have a set of semi-norms $\{p_\alpha, \alpha\in\mathcal{I}\}$ defining the topology. Let $C([0,T],\widetilde{\mathcal{S}}')$ be the space of continuous functions from $[0,T]$ to $\widetilde{\mathcal{S}}'$. For $Z_t\in C([0,T],\widetilde{\mathcal{S}}')$ define

$$|||Z_t|||_\alpha = \sup_{0\leq t\leq T}|p_\alpha(Z_t)|.$$

Let the topology on $C([0,T],\widetilde{\mathcal{S}}')$ be the projective limit topology of $\{|||\cdot|||_\alpha, \alpha\in\mathcal{I}\}$. This makes $C([0,T],\widetilde{\mathcal{S}}')$ a completely regular topological space. Also associate with this space the Borel algebra $\mathcal{B}$ and let the filtration $\mathcal{F}_t$, $0\le t\le T$ be the one generated by $(C([0,T],\widetilde{\mathcal{S}}'),\mathcal{B})$. For fixed $k\ge 0$ let $\bar{H}_k$ be the Hilbert space of k times differentiable periodic functions of period π with the usual scalar product and denote the norms by

$$||f||_k^{\#}=\sqrt{\int_{-\pi/2}^{\pi/2}\sum_{i=0}^{k}|f^{(i)}(x)|^2\,dx}.$$

Using $\tan(x)$, we now define H_k as $v_{\tan}(\bar{H}_k)$. The induced scalar product results in the norms

$$||f||_k=\sqrt{\int_{-\pi/2}^{\pi/2}\sum_{i=0}^{k}\left|\frac{d^i}{dx^i}f(\tan(x))\right|^2\,dx}.$$

The sequence of spaces $\{H_k\}_{p=1}^{\infty}$ satisfies $H_k\supset H_{k+1}$, $||\cdot||_k\le||\cdot||_{k+1}$ and $\widetilde{\mathcal{S}}$ is the projective limit of this sequence. The dual space of H_k is H_{-k}, which is a Hilbert space with norm

$$||Z||_{-k}=\sup_{||f||_k\le 1}|\langle Z,f\rangle|.$$

Define as usual $C([0,T],H_{-k})$ to be the Banach space with norm

$$|||Z_t|||_{-k}=\sup_{0\le t\le T}||Z_t||_{-k}.$$

On $C([0,T],H_{-k})$ we shall consider the norm topology and the Borel algebra $\mathcal{B}_k$. Also define the filtration $\mathcal{F}_t^k$, $0\le t\le T$ as the filtration generated by the pair $(C([0,T],H_{-k}),\mathcal{B}_k)$. The aim of the paper is to prove Theorem 4 which shows that the sequence Y_t^n, $0\le t\le T$, converges weakly in $(C([0,T],H_{-6}),\mathcal{B}_6)$ under appropriate initial conditions.

Next we define notation used in the formulation of the martingale problem. Let $Z\in\widetilde{\mathcal{S}}'$. For a sequence $\bar{f}=\{f_j\}_{j=1}^{k}\subset\widetilde{\mathcal{S}}$ and a function $F\in C_b^2(R^k,R)$ define $F_{\bar{f}}(Z)=F(\langle Z,f_1\rangle,\ldots,\langle Z,f_k\rangle)$. Also define $L_t^nF_{\bar{f}}(Z,w)=L_t^nF_{\bar{f}}(Z)$ as

$$\begin{aligned}L_t^nF_{\bar{f}}(Z)=&\sum_{j=1}^{k}\frac{\partial F_{\bar{f}}}{\partial x_j}(Z)\Big(\langle Z,\frac{\alpha}{2}(A_t^nf_j+A_tf_j)-Bf_j(x)\rangle+\gamma\langle X_t^n,f_j''\rangle\Big)\\&+\sum_{j,l=1}^{k}\frac{\sigma^2}{2}\frac{\partial^2F_{\bar{f}}}{\partial x_j\partial x_l}(Z)\langle X_t^n,f_j'f_l'\rangle,\end{aligned}$$

where

$$\begin{aligned}A_t^nf(x)&=\int\frac{f'(x)-f'(y)}{x-y}X_t^n(dy),\\A_tf(x)&=\int\frac{f'(x)-f'(y)}{x-y}X_t(dy),\\Bf(x)&=xf'(x).\end{aligned}$$

Also define

$$\begin{aligned}L_tF_{\bar{f}}(Z)=&\sum_{j=1}^{k}\frac{\partial F_{\bar{f}}}{\partial x_j}(Z)\Big(\langle Z,\alpha A_tf_j-Bf_j\rangle+\gamma\langle X_t,f_j''\rangle\Big)\\&+\sum_{j,l=1}^{k}\frac{\sigma^2}{2}\frac{\partial^2F_{\bar{f}}}{\partial x_j\partial x_l}(Z)\langle X_t,f_j'f_l'\rangle.\end{aligned}$$

As we will see in Section 2.2, A_t and A_t^n maps $\widetilde{\mathcal{S}}$ into $\widetilde{\mathcal{S}}$ so L_t^n is well defined. From (5) we have for every $s \geq 0$

$$\Phi_s^{t,n}(Y^n) = F_{\bar{f}}(Y_t^n) - F_{\bar{f}}(Y_s^n) - \int_s^t L_\tau^n F_{\bar{f}}(Y_\tau^n) d\tau, \qquad s \leq t$$

is a $\mathcal{F}_t$-martingale. We prove in Section 2.5 that if a convergent subsequence of Y_t^n converges weakly to some process Y_t then

$$\Phi_s^t(Y) = F_{\bar{f}}(Y_t) - F_{\bar{f}}(Y_s) - \int_s^t L_\tau F_{\bar{f}}(Y_\tau)\, d\tau, \qquad s \leq t$$

is a martingale in t for every s.

Define $D_t(z)$ as a generic nonrandom function which is nondecreasing in t and locally bounded in z. Consider approximations of a function $f \in \widetilde{S}$ by finite linear combinations of resolvents $f_z = 1/(x-z)$ and the identity function. Hence consider approximating functions g^j of the form

$$g^j(x) = c_j + d_j \mathrm{Re}(\frac{1}{x-i}) + \sum_{m=1}^{n(j)} a_{jm} \mathrm{Im}(\frac{1}{x - z_{jm}}).$$

Fix a set of probability measures $\mathcal{A}$ on $(C([0,T], \widetilde{S}'), \mathcal{B})$. Consider maps $\bar{g}$ from $\widetilde{S}$ to the space of sequences of functions, each function g^j being a finite linear combination of resolvents and the identity. We say that such a map is a *weak approximation of $\widetilde{S}$ with resolvents relative $\mathcal{A}$* if for each $P \in \mathcal{A}$ and process Y_t associated with P, it is possible to approximate each $f \in \widetilde{S}$ with a sequence of approximating functions $\bar{g}(f) = \{g^j\}$ of resolvents such that the weak limit of

$$\langle Y_t, f - g^j \rangle$$

is zero for all $t \in [0, T]$.

Definition 1. *Let μ be an initial probability measure on $\widetilde{S}'$, i.e. $\mu(Y; \langle Y, 1\rangle = 0) = 1$. Define the set $\mathcal{A}(\bar{g}, \mu)$ of probability measures on $(C([0,\infty), \widetilde{S}'), \mathcal{B})$ as those probability measures satisfying*

(i) The law of Y_0 is equal to μ.

(ii) $\bar{g}$ is a weak approximation of $\widetilde{S}$ with resolvents relative $\mathcal{A}(\bar{g}, \mu)$.

(iii) The process $\Phi_0^t(Y)$ is a $\mathcal{F}_t$-martingale.

(iv) $E|\langle Y_t, f_z''\rangle| \leq D_t(z)$, $E|\langle Y_t, f_z\rangle|^2 \leq D_t(z)$ and $E|\langle Y_t, f_z'\rangle|^2 \leq D_t(z)$.

(v) $E|\langle Y_{t+h} - Y_t, f_z\rangle|^2 = o(1)$ and $E|\langle Y_{t+h} - Y_t, f_z'\rangle| = o(1)$ as $h \to 0$.

We will work with Cauchy transforms

$$M_t^n(z) = \langle X_t^n, f_z \rangle$$

and

$$M_t(z) = \langle X_t, f_z \rangle,$$

where $f_z = 1/(x - z)$. The Cauchy transform was, in this setting, introduced by Rogers and Shi (1993) to solve (4). We now state the main theorem (Theorem 4 proved in section 2.7).

Theorem. *Let $z = a + ib$. Suppose that Y_0^n converges weakly in $\widetilde{\mathcal{S}}'$ to Y_0 and that there is a constant C such that for all n*

$$E|M_0^n(z) - M_0(z)|^2 = E\Big|\int \frac{1}{x-z}[X_0^n(dx) - X_0(dx)]\Big|^2 \leq \frac{C}{n^2 b^2}.$$

Then the sequence of processes Y_t^n converges weakly, not only in $(C([0,T], \widetilde{\mathcal{S}}'), \mathcal{B})$ but also in $C([0,T], H_{-6})$ to a Gaussian process Y_t, i.e. all the finite dimensional distributions are Gaussian. There also exists a $\bar{g}$ such that properties (i)-(v) in Definition 1 holds, in particular the quantity $\Phi_s^t(Y)$ in the martingale problem is indeed a $\mathcal{F}_t^6$-martingale.

Throughout the paper we use a generic constant C which mean that C stands for any finite value and may be implicitly changed. The same holds for $D(z)$ with the exception that $D(z)$ is bounded on the upper half plane. Likewise C_t and $D_t(z)$ are generically bounded in the parameters and nondecreasing in t.

2. Prerequisites and proof of the theorem.

The outline of the prerequisites and the proof is as follows. In the first subsection a key estimate for the Cauchy transforms is proven which results in Proposition 1. The second subsection takes care of facts about the spaces $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}'$. In the third subsection estimates later on used in the proof of the theorems are gathered. Section 2.4 is the proof of a tightness theorem and Section 2.5 proves that the martingale problem converges. In the sixth subsection a uniqueness result is proven and finally the seventh subsection contains the final argument proving Theorem 4.

2.1. Properties of the Cauchy transform.

We shall prove an estimate of $E|M_t^n(z) - M_t(z)|$. All formulas below involving z will be defined only on the set $\{z; \mathrm{Im}(z) \neq 0\}$. Taking the differential of $M_t(z)$ and using (3) and (4) lead to

$$dM_t(z) = \frac{\partial}{\partial z}\Big(\frac{\alpha}{2}M_t(z)^2 + zM_t(z)\Big)\,dt \tag{6}$$

and

$$\begin{aligned} dM_t^n(z) = {} & \frac{\partial}{\partial z}\Big(\frac{\alpha}{2}(M_t^n(z))^2 + zM_t^n(z)\Big)\,dt \\ & + \frac{\gamma}{n}\frac{\partial^2}{\partial z^2}M_t^n(z)\,dt - \frac{\sigma}{n\sqrt{n}}\sum_{i=1}^n \frac{1}{(\lambda_t^i - z)^2}d\beta_t^i, \qquad P^n\text{-a.s.} \end{aligned} \tag{7}$$

The first step towards the estimate is to note

Lemma 1. *If we have a function valued stochastic process $f_t = f_t(x)$ that satisfies*

$$\frac{\partial f_t}{\partial t} = xf_t'(x) - \frac{\alpha}{2}\int \frac{f_t'(x) - f_t'(y)}{x - y}(X_t^n(dy) + X_t(dy)), \quad t \geq 0 \tag{8}$$

then

$$|\langle Y_t^n, f_t\rangle| \le |\langle Y_0^n, f_0\rangle| + \gamma\int_0^t |\langle X_s^n, f_s''\rangle|\, ds + \frac{\sigma}{\sqrt{n}}\Big|\int_0^t \sum_{i=1}^n f_s'(\lambda_s^i)\, d\beta_s^i\Big|. \tag{9}$$

Proof: The process in (8) applied to formula (4) yields

$$\begin{aligned} d\langle Y_t^n, f_t\rangle = &\int x f_t'(x)\, Y_t^n(dx)\, dt \\ &+ \frac{\alpha}{2}\iint \frac{f_t'(x) - f_t'(y)}{x-y}[X_t^n(dx)Y_t^n(dy) + X_t(dx)Y_t^n(dy)]\, dt \\ &+ \gamma\langle X_t^n, f_t''\rangle\, dt + \frac{\sigma}{\sqrt{n}}\sum_{i=1}^n f_t'(\lambda_t^i)\, d\beta_t^i \\ &+ \int x f_t'(x) Y_t^n(dx)\, dt - \frac{\alpha}{2}\iint \frac{f_t'(x) - f_t(y)}{x-y}[X_t^n(dy) + X_t(dy)]Y^n(dx)\, dt. \end{aligned}$$

After cancellation of terms we end up with

$$d\langle Y_t^n, f_t\rangle = \gamma\langle X_s^n, f_t''\rangle\, dt + \frac{\sigma}{\sqrt{n}}\sum_{i=1}^n f_t'(\lambda_t^i)\, d\beta_t^i$$

which in integrated form proves the lemma. □

Our next goal is to find solutions to the linear equation (8).

Lemma 2. *Equation (8) with terminal condition*

$$f_T(x) = \frac{c}{x-z}$$

has the solution

$$f_t(x) = \frac{c_t}{x - z_t},$$

where c_t and z_t satisfies

$$\frac{dz_t}{dt} = -(z_t + \frac{\alpha}{2}(M_t^n(z_t) + M_t(z_t))), \qquad z_T = z \tag{10}$$

and

$$\frac{dc_t}{dt} = -[1 + \frac{\alpha}{2}((M_t^n)'(z_t) + M_t'(z_t))]\, c_t, \qquad c_T = c. \tag{11}$$

Writing $z_t = a_t + ib_t$, $|b_t|$ is decreasing in t and

$$|c_t| \le |c_T|\frac{|b_t|}{|b_T|}.$$

Proof: By Lemma 3 below (10) has a unique solution and $|b_t|$ is decreasing. Hence we have the unique solution

$$c_t = c\exp(\int_t^T (1 + \frac{\alpha}{2}((M_s^n)'(z_s) + M_s'(z_s)))\, ds \tag{12}$$

to (11). To see that $f_t = c_t/(x - z_t)$ is a solution of (8) we first differentiate with respect to t and get

$$\frac{\partial f_t}{\partial t} = \frac{c'_t}{x - z_t} + \frac{c_t z'_t}{(x - z_t)^2}.$$

The right hand side of (8) becomes

$$-\frac{c_t}{x - z_t} - \frac{c_t z_t}{(x - z_t)^2} - \frac{\alpha}{2}\frac{c_t}{(x - z_t)^2}(M^n_t + M_t) - \frac{\alpha}{2}\frac{c_t}{x - z_t}((M^n_t)' + M'_t)$$

which by (10) and (11) equals the left hand side. To estimate $|c_t|$ we use (12) to get

$$|c_t| \le |c| \exp\Big(\int_t^T \Big(1 + \frac{\alpha}{2}\int \frac{1}{(x - a_s)^2 + b_s^2}(X^n_s(dx) + X_s(dx))\,ds\Big)\Big).$$

Using the imaginary part of (10),

$$\frac{d}{dt} b_t = -\Big(b_t + \frac{\alpha}{2}\int \frac{b_t}{(x - a_t)^2 + b_t^2}(X^n_t(dx) + X_t(dx))\Big),$$

we conclude

$$|c_t| \le |c| \exp(-\int_{|b_t|}^{|b_T|} \frac{db}{b}) = |c|\frac{|b_t|}{|b_T|}.$$

□

The following lemma yields existence and uniqueness for solutions of (10).

Lemma 3. *For those $\omega \in \Omega^n$ such that $X^n_t(dx, \omega)$, $0 \le t \le T$, is continuous we have for each T and z with nonzero imaginary part a unique solution $z_t = z_t(z, T, \omega)$, $0 \le t \le T$, of (10) with terminal condition $z_T = z$. Moreover, the absolute value of the imaginary part, $|b_t|$, is nonincreasing.*

Proof (sketch): Fix ω such that $X^n_t(dx, \omega)$ is continuous and suppose, without loss of generality, that $b_T > 0$. Then existence and uniqueness of a solution of (10) are consequences of the following three facts. First we have that

$$\frac{d}{dt} b_t < 0 \text{ for } b_t > 0$$

and so the imaginary part of any Picard approximation decreases monotonically. Secondly we have the Lipschitz condition

$$\begin{aligned} &|\alpha M^n_\tau(\xi) + \xi - \alpha M^n_\tau(w) - w| \le |\xi - w|(\alpha/\mathrm{Im}(z)^2 + 1), \\ &\qquad \forall \xi, w \text{ such that } |\mathrm{Im}(\xi)|, \mathrm{Im}(w) > b_T. \end{aligned}$$

Thirdly, $M^n_t(z)$ is continuous in $[0, T] \times \{z : b_T > 0\}$ since $X^n_t(dx, \omega)$ is continuous. □

Lemma 4. *For each $0 \le t \le T$,*

$$e^{T-t}|b_T| \le |b_t| \le \sqrt{b_T^2 + \alpha}\, e^{T-t} \tag{13}$$

and

$$\frac{\alpha}{2}\int \frac{e^{-t} b_{T-t}}{(a_{T-t} + x)^2 + b_{T-t}^2}\,[X^n_{T-t}(dx) + X_{T-t}(dx)] = \frac{d}{dt} e^{-t} b_{T-t}. \tag{14}$$

Proof: Using the imaginary part of (10) and assuming $b_T > 0$ we get

$$0 \leq \frac{d(e^{-t}b_{T-t})}{dt} = \frac{\alpha}{2}\int \frac{e^{-t}b_{T-t}}{(a_{T-t}+x)^2 + b_{T-t}^2}\,[X_{T-t}^n(dx) + X_{T-t}(dx)]$$
$$\leq \alpha \frac{e^{-2t}}{e^{-t}b_{T-t}}.$$

The equality shows (14). Property (13) follows from the left and right estimates. If $b_T < 0$ the lemma follows similarly. □

Proposition 1. *Let $z = a + ib$ and suppose there exists a C such that*

$$E|M_0^n(z) - M_0(z)| \leq \frac{C}{n|b|},$$

then there is a new constant C such that

$$E|M_T^n(z) - M_T(z)| \leq \frac{C\ln(1+\frac{1}{|b|})}{|b|n}.$$

If instead the condition

$$E|M_0^n(z) - M_0(z)|^2 \leq \frac{C}{n^2|b|^2}$$

is satisfied, then

$$E|M_T^n(z) - M_T(z)|^2 \leq \frac{C\ln^2(1+\frac{1}{|b|})}{b^2n^2}.$$

Remark. The author is not aware of any sharper estimate. Comparing with Johansson (1998), the above proposition imply that we need $1+\epsilon$ differentiations in stead of $2+\epsilon$. Also, referring to Johansson (1998) we conclude that the minimum condition on the number of derivations is somewhere between $1/2$, the exact order derived for the circular ensemble, and $1+\epsilon$.

Proof: Recall that $z_T(z) = z$, $a_t + ib_t = z_t = z_t(z)$ is the stochastic process defined by (10) and c_t, s.t. $|c_t| \leq b_t/b_T$, is defined by (11). Lemma 1 and Lemma 2 yield

$$n|M_T^n(z) - M_T(z)| \leq \frac{|b_0|}{|b_T|}n|M_0^n(z_0) - M_0(z_0)| + \gamma\int_0^T \frac{|b_t|}{|b_T|}|(M_t^n)''(z_t)|dt$$
$$+\Big|\frac{\sigma}{\sqrt{n}}\int_0^T\sum_{i=1}^n c_t\frac{1}{(\lambda_t^i - z_t)^2}d\beta_t^i\Big|.$$

Taking expectation the first term is majorized by $C/|b_T|$ and the second one can be majorized with the help of (13), (14) in Lemma 4 and a possibly larger C to yield

$$\gamma\int_0^T \frac{|b_t|}{|b_T|}|(M_t^n)''(z_t)|dt \leq C\frac{1}{|b_T|}\int_0^T\Big(1+\frac{\alpha}{2}\int\frac{1}{|x-z_t|^2}(X^n(dx)+X(dx)\Big)\,dt$$
$$\leq -C\int_0^T \frac{1}{b_t}\frac{db_t}{dt}\,dt$$
$$\leq C|\ln(\frac{b_0}{b_T})| \leq C\ln(1+\frac{1}{|b_T|}).$$

Finally the last term can be estimated with the help of Jensen's inequality and basic properties of the stochastic integral by

$$E\Big|\dots\Big| \leq \Big(E\Big|\dots\Big|^2\Big)^{1/2} \leq \Big(\frac{|c_T|^2\sigma^2}{|b_T|^2} E\int_0^T \int \frac{b_t^2}{((a_t+x)^2+b_t^2)^2} X_t^n(dx)\,dt\Big)^{1/2},$$

and, again, using (13), (14) in Lemma 4 and $b = b_T$ we get the majorization

$$C\ln(1+\frac{1}{|b|})/|b|$$

for some constant C. The first statement of the proposition follows immediately and the second follows from a similar argument. □

2.2 Properties of $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{S}}'$. We begin with two simple lemmas.

Lemma 5. *If $f \in \widetilde{\mathcal{S}}$ then $xf'(x) \in \widetilde{\mathcal{S}}$, $f'(x) \in \widetilde{\mathcal{S}}$ and $xf'(x) \in L^2$*

Proof: Let $f_{\#}(t) = f(\tan(t))$. Then

$$f'(x) = \frac{d}{dx} f_{\#}(\arctan(x)) = \frac{1}{1+x^2} f'_{\#}(\arctan(x))$$

which implies that $xf'(x) \in L^2$. Moreover,

$$f'(\tan(t)) = \frac{1}{1+\tan^2(t)} f'_{\#}(t) = \cos^2(t) f'_{\#}(t)$$

and if $g(x) = xf'(x)$ then

$$g(\tan(t)) = \tan(t) f'(\tan(t)) = \cos(t)\sin(t) f'_{\#}(t).$$

Hence $xf'(x)$ and $f'(x)$ with their arguments replaced by $\tan(t)$ are infinitely differentiable and periodic with period π, i.e. both $xf'(x)$ and $f'(x)$ are in $\widetilde{\mathcal{S}}$. □

Lemma 6. *For each z, $Im(z) \neq 0$,*

$$Im(f_z), Re(f_z) \in \widetilde{\mathcal{S}}.$$

Proof: It is enough to show that $f_z(\tan(x))$ is an infinitely differentiable periodic function of period π. But this is immediate since

$$f_z(\tan(x)) = \frac{1}{\tan x - z} = \frac{\cos(x)}{\sin(x) - z\cos(x)}.$$

□

Next, we show that the derivative operators and functional operators commute.

Lemma 7. *For all $Z \in \widetilde{\mathcal{S}}'$ we have*

$$\frac{\partial^l}{\partial z^l}\langle Z, f_z\rangle = \langle Z, \frac{\partial^l}{\partial z^l} f_z\rangle.$$

Proof: To show this it is enough to prove that

$$\frac{\partial^l}{\partial z^l}\frac{f_{z+h}(x)-f_z(x)}{h}$$

converges in $\widetilde{S}$ to the right limit, or to show that

$$\|\frac{d^k}{dx^k}\frac{\partial^l}{\partial z^l}(\frac{f_{z+h}-f_z}{h}-\frac{\partial}{\partial z}f_z)\circ(\tan(x))\|_\infty\to 0$$

for all $k\geq 0$. But

$$\frac{f_{z+h}-f_z}{h}=f_{z+h}f_z$$

and using $\frac{\partial}{\partial z}f_z=f_zf_z$ we get

$$\frac{f_{z+h}-f_z}{h}-\frac{\partial}{\partial z}f_z=h\frac{\partial^l}{\partial z^l}f_{z+h}f_z^2.$$

Now

$$f_{z+h}f_z^2\circ(\tan x)=\frac{\cos^3(x)}{(\sin(x)-\cos(x)(z+h))(\sin(x)-z\cos(x))^2},$$

and we see that if we take the k:th derivative w.r.t. x and the l:th derivative with respect to z, we get a bounded periodic function in x and h for h in a neighborhood of zero. $\square$

Let $\mathcal{F}(f)(s)=\int_{-\infty}^{\infty}f(x)e^{-ixs}\,dx$ and $\mathcal{F}^{-1}(f)(s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{ixs}\,dx$. We will need the following integrability result.

Lemma 8. *Suppose q is odd, $q\geq 1$. For all $q+1$ times differentiable functions f such that $f^{(q+1)}$ is integrable*

$$\int|\mathcal{F}^{-1}[(1+|s|)^{q+1}\mathcal{F}(f)](x)|\,dx$$
$$\leq C\int(|f(x)|+|f^{(q+1)}(x)|)\,dx.$$

Moreover if $f\in H_{q+1}$ and $f\in L^1$ the right hand side is finite. If q is even the lemma is true for $q+1$ replaced by $q+2$

Proof: Suppose f,f'' are integrable. We have the identity

$$\begin{aligned}\mathcal{F}^{-1}[|s|\mathcal{F}(f)](x)&=-\lim_{b\to 0}\frac{d}{db}\mathcal{F}^{-1}[e^{-b|s|}\mathcal{F}(f)](x)\\&=-\lim_{b\to 0}\frac{1}{\pi}\frac{d}{db}\int\mathrm{Im}\Big(\frac{1}{x-a+ib}\Big)f(a)\,da\\&=-\lim_{b\to 0}\frac{1}{\pi}\int\frac{d}{da}\mathrm{Re}\Big(\frac{1}{x-a+ib}\Big)f(a)\,da\end{aligned}$$

Now, by splitting the integral in the last line and using partial integration it follows that the last line is the sum of the two terms

$$\frac{1}{\pi}\int 1_{\{|x-a|>1\}}\frac{1}{(x-a)^2}f'(a)\,da$$

and

$$\lim_{b\to 0}\frac{1}{\pi}\int 1_{\{|x-a|\le 1\}}\mathrm{Re}\Big(\frac{1}{x-a+ib}\Big)f'(a)\,da = \\ -\frac{1}{\pi}E(f''(x+UV))+f''(x-UV)),$$

where U,V are i.i.d. random variables with a uniform distribution on $[0,1]$. Hence

$$\int |\mathcal{F}^{-1}[|s|s^{2j}\mathcal{F}(f)](a)|\,da \le C\int |f^{(2j+2)}(x)|\,dx$$

Now, q is odd implies

$$(1+|s|)^{q+1} = \Big(s^{q+1}+\sum_{j=0}^{(q-1)/2} s^{2j}(C_j+D_j|s|)\Big)$$

for some constants $C_j, D_j \le C_q$. Therefore

$$\int |\mathcal{F}^{-1}[(1+|s|)^{q+1}\mathcal{F}(f)](a)|\,da \le C\sum_{j=0}^{(q+1)/2}\int |f^{2j}(x)|\,dx.$$

If $\int |f^{(2j)}(x)|\,dx \le C\int(|f(x)|+|f^{(q+1)}|)\,dx$, $2j\le q+1$ the first statement in the lemma follows. Also if $f=g(\arctan(x))\in H_{q+1}$ then

$$|f^{(q+1)}(x)| = \Big|\frac{d^{q+1}}{dx^{q+1}}g(\arctan(x))\Big| \le \frac{C}{1+x^2}|g^{(q+1)}(\arctan(x))|$$

Hence

$$\int |f^{(q+1)}(x)|\,dx \le C\int_{-\pi/2}^{\pi/2}|g^{(q+1)}(s)|\,ds \le C\|f\|_{q+1},$$

and the second statement follows. It remains to prove

$$\int |f^{(2j)}(x)|\,dx \le C\int(|f(x)|+|f^{(q+1)}(x)|)\,dx,$$

where $2j\le q+1$. To see this use the identity

$$\frac{s^{2j}}{(1+s^2)^k} = \frac{s^{2j+2}}{(1+s^2)^{k+1}}+\frac{s^{2j}}{(1+s^2)^{k+1}}$$

repeatedly to get

$$s^{2j} = \frac{s^{2j}}{(1+s^2)^j}+s^{2j+2}\sum_{k=0}^{j}\frac{1}{(1+s^2)^{k+1}}.$$

Also

$$s^{2j}=(s^2+1-1)^j=\sum_{k=0}^{j}\binom{j}{k}(s^2+1)^k(-1)^{j-k}.$$

The Fourier transform of $1/(1+x^2)^k$ is a probability distribution related to a stochastic variable X_k. Therefore

$$\int |f^{(2j)}(x)|\,dx = \int |\mathcal{F}s^{2j}\mathcal{F}^{-1}f|\,dx$$

is estimated by

$$\sum_{k=0}^{j}\int|\mathcal{F}\frac{s^{2j+2}}{(1+s^2)^k}\mathcal{F}^{-1}f|\,dx+\sum_{k=0}^{j}\binom{j}{k}\int|\mathcal{F}\frac{1}{(1+s^2)^k}\mathcal{F}^{-1}f|\,dx,$$

which is not larger than

$$j\int|f^{2q+2}(x)|\,dx+2^j\int|f(x)|\,dx.$$

Using this inductively yields the estimate

$$\int|f^{(2j)}(x)|\,dx\le((q+1)/2)!\,2^{(q+1)/2+1}\Big(\int|f^{(q+1)}(x)|\,dx+\int|f(x)|\,dx\Big).$$

□

Next we prove a useful representation of functions in $\widetilde{\mathcal{S}}$.

Lemma 9. *For all $f\in H_{q+1}$, $q+1\ge 2$, we can find constants A,B such that if*

$$g=f-A-BRe(1/x-i),$$

then we have $\|g\|_{L^1}\le C\|g\|_{q+1}$. Moreover, f can be decomposed as

$$f(x)=A+BRe(1/x-i)+\iint Im(f_z(x))\,h(z)dadb,$$

where $z=a+bi$ and

$$h(a,b)=\frac{1}{\Gamma(q+1)}b^qe^{-b}\mathcal{F}^{-1}[(1+|s|)^{q+1}\mathcal{F}(g)](a)$$

if $b>0$ and zero elsewhere. Furthermore if $f(x)\in L^1$ we choose $g=f$.

Remark. The requirement of at least H_2 is not optimal. Actually we could probably sharpen the lemma to include the spaces $f\in H_{1+\epsilon}$. Here we have generalized our notation to mean the isomorfism of the standard spaces $\hat{H}_r$, i.e. the Hilbert spaces with the scalar product $(f,g)_r=\sum_k(1+k^2)^r\hat{f}_k\hat{g}_k^*$, where $\hat{f}_k,\hat{g}_k$ are the Fourier coefficients of f,g.

Proof: To be able to decompose f as a linear combination of $\mathrm{Im}(f_z)$ we must also have $f\in L^1$. This is not always the case and hence we subtract a slowly decreasing part $A+B\mathrm{Re}(1/(x-i))$ from f before decomposing f such that if $g=f-A-B\mathrm{Re}(1/(x-i))$ then $g(x)\in L^1$. To prove this we show first that such A and B can be found. From Lemma 6 we get

$$v_{\tan}^{-1}(\mathrm{Re}(f_z))\in C^{\#}(-\pi/2,\pi/2).$$

Now consider $g_{\#}=v_{\tan}^{-1}(g)=g(\tan)$, $f_{\#}=v_{\tan}^{-1}(f)=f(\tan)$. Note that $g_{\#}$ is a function in $C^{\infty\#}(-\pi/2,\pi/2)$. By choosing $A=f_{\#}(-\pi/2)$ and $B=-f'_{\#}(-\pi/2)$ we get $g_{\#}(\pi/2)=g_{\#}(-\pi/2)=g'_{\#}(\pi/2)=g'_{\#}(-\pi/2)=0$ and $|A|,|B|<\infty$. Writing

$$g_{\#}(t)=\int_{-\pi/2}^{t}\int_{-\pi/2}^{t_1}g''_{\#}(t_2)\,dt_2dt_1.$$

We see

$$|g_{\#}(t)| \le \frac{3}{2}(t+\pi/2)^{3/2}\|g_{\#}\|_2^{1/2}.$$

Therefore, for some constant C,

$$|g(x)| \le C/(1+|x|^{3/2}),$$

and hence $g \in L^1$.

The next step is to prove that

$$I(x) = \iint h(z)\mathrm{Im}(f_z(x))\, dadb$$

is a decomposition of g and hence equal to g. By Lemma 6, $g \in H_{q+1}$ and by using Lemma 5 repeatedly on g we see that Lemma 8 implies that $h(z)$ is integrable, i.e.

$$\iint |h(z)|\, dadb = \frac{1}{\Gamma(q+1)}\int_0^\infty b^q e^{-b}\, db \int |\mathcal{F}^{-1}[(1+|s|)^{q+1}\mathcal{F}(g)](a)|\, da < \infty.$$

Hence the candidate $I(x)$ for the decomposition of g is integrable, i.e.

$$\begin{aligned}&\iint |h(z)| \int |\mathrm{Im}(f_z(x))|\, dxdadb \\ &\qquad = \iint |h(z)| \int \frac{|b|}{(x-a)^2+b^2}\, dxdadb = \pi \iint |h(z)|\, dadb < \infty.\end{aligned}$$

By Fubini,

$$\begin{aligned}\mathcal{F}(I(x))(s) &= \iiint h(z)\mathrm{Im}(f_z(x))e^{-ixs}\, dadbdx \\ &= \iint h(z)\int \mathrm{Im}(f_z(x))e^{-ixs}\, dxdadb \\ &= \iint h(z)e^{-b|s|}e^{-isa}\, dadb \\ &= \int_0^\infty \frac{1}{\Gamma(q+1)} b^q e^{-b(1+|s|)}\, db \int \mathcal{F}^{-1}[(1+|s|)^{q+1}\mathcal{F}(g)](a)e^{-isa}\, da \\ &= \frac{1}{(1+|s|)^{q+1}}(1+|s|)^{q+1}\mathcal{F}(g) = \mathcal{F}(g)(s).\end{aligned}$$

This together with the one-to-one property of $\mathcal{F} : L^1 \to C(R, \mathbf{C})$ shows that $I(x) = g$. □

Lemma 11. *For $q \ge 1$ the operators A_t and A_t^n introduced in Section 1, map H_{q+2} continuously into H_q and hence $\widetilde{S}$ continuously into $\widetilde{S}$.*

Proof: Let μ be a probability measure and let $\hat{\mu}(t : t \le a) = \mu(x : x \le \tan(a))$, $f \in H_{q+2}$ and $f^{\#}(s) = f'(\tan(s))$. Recall that $s, t \in [-\pi/2, \pi/2]$. Note that $f' \in H_{q+1}$ and by a change of variables

$$\int \frac{f'(x)-f'(y)}{x-y}\mu(dy) = \int \frac{f'(\tan(s))-f'(\tan(t))}{\tan(s)-\tan(t)}\hat{\mu}(dt).$$

We replace $f'(\tan)$ with $f^{\#}$ and reformulate

$$\int \frac{f'(\tan(s)) - f'(\tan(t))}{\tan(s) - \tan(t)} \hat{\mu}(dt) = \int \frac{f^{\#}(s) - f^{\#}(t)}{s - t} \frac{s - t}{\tan(s) - \tan(t)} \hat{\mu}(dt).$$

Using $f^{\#}(s) - f^{\#}(t) = (s-t)\int_0^1 f^{\#'}(sv + (1-v)t)\,dv$ and rearranging we conclude

$$\int \frac{f^{\#}(s) - f^{\#}(t)}{s - t} \frac{s - t}{\tan(s) - \tan(t)} \hat{\mu}(dt) = \\ \int_0^1 \int f^{\#'}(sv + (1-v)t) \frac{(s-t)\cos(s)\cos(t)}{\sin(s-t)} \hat{\mu}(dt)dv.$$

The singularities at $s = t$ and $s - t = \pm\pi$ of the kernel

$$\frac{(s-t)\cos(s)\cos(t)}{\sin(s-t)} \hat{\mu}(dt)$$

are removable and hence this kernel may be replaced with a kernel $K(s,t)$ such that $\frac{\partial^l}{\partial s^l} K(s,t)$ is bounded in both s,t for all l. To show the desired regularity it is enough to prove the finiteness of

$$\int_0^1 \iint v^q \Big(f^{\#(q+1)}(sv + t(1-v)) \Big)^2 K(s,t) \hat{\mu}(dt)\, dvds.$$

But since $q > 0$ we can change variables $s' = vs + t(1-v)$ and use Fubini to first integrate over s',

$$\Big\| \int \frac{f'(x) - f'(y)}{x - y} \mu(dx) \Big\|_q \leq C \|f\|_{q+2}$$

and the lemma follows. □

The next goal is to prove continuity of the map $Z_t \to \langle Z_t, A_t f\rangle$. The first step is to prove

Lemma 12. *Fix $k \geq 0$ and let $f \in H_{k+5}$. Then there is a constant C_k such that for all $t, \delta \geq 0$,*

$$\sup_{s \in [-\pi/2, \pi/2]} \Big| \frac{d^k}{ds^k} A_{t+\delta} f(\tan(s)) - \frac{d^k}{ds^k} A_t f(\tan(s)) \Big| < C_k \|f\|_{k+5} \delta.$$

Proof: Let

$$g_s(x) = \frac{d^k}{ds^k} \frac{f'(\tan(s)) - f'(x)}{\tan(s) - x}.$$

As we shall see $\sup_{s \in [-\pi/2, \pi/2]} \|g_s\|_\infty < \infty$, hence

$$\frac{d^k}{ds^k} A_t f(\tan(s)) = \langle X_t, g_s \rangle$$

and it is enough to prove that

$$|\langle X_{t+\delta}, g_s \rangle - \langle X_t, g_s \rangle| \leq \delta C.$$

Recall that we have defined $Bf(x) = xf'(x)$. Putting g_s in equation (4) leads to

$$\langle X_{t+\delta}, g_s\rangle - \langle X_t, g_s\rangle$$
$$= -\int_t^{t+\delta} \langle X_\tau, Bg_s\rangle\, d\tau + \frac{\alpha}{2}\int_t^{t+\delta} \iint \frac{g_s'(x) - g_s'(y)}{x-y} X_\tau(dx)X_\tau(dy)\, d\tau.$$

Since

$$\Big|\frac{f(x)-f(y)}{x-y}\Big| \le \|f'\|_\infty$$

it follows that

$$|\langle X_{t+\delta}, g_s\rangle - \langle X_t, g_s\rangle| \le \delta(\|Bg_s\|_\infty + \frac{\alpha}{2}\|g_s''\|_\infty).$$

Note that

$$\frac{d}{dx} = \frac{1}{\tan'(\eta)}\frac{d}{d\eta}, \qquad \frac{d^2}{dx^2} = -\frac{\tan''}{(\tan')^3}\frac{d}{d\eta} + \frac{1}{(\tan')^2}\frac{d^2}{d\eta^2}$$

and

$$\max\Big(\Big|\frac{\tan}{\tan'}\Big|, \Big|\frac{\tan''}{(\tan')^3}\Big|, \Big|\frac{1}{(\tan')^2}\Big|\Big) \le C.$$

This implies

$$\|Bg_s\|_\infty + \frac{\alpha}{2}\|g_s''\|_\infty \le C\Big(\Big\|\frac{d}{d\eta}g_s(\tan(\eta))\Big\|_\infty + \Big\|\frac{d^2}{d\eta^2}g_s(\tan(\eta))\Big\|_\infty\Big).$$

A slight modification of Lemma 11 together with the estimate $\|f^{(q)}\|_\infty \le C\|f\|_q$ yields for all natural numbers k, l the bound

$$\sup_{s,\eta\in[-\pi/2,\pi/2]} \Big|\frac{d^k}{ds^k}\frac{d^l}{d\eta^l}\frac{f'(\tan(s)) - f'(\tan(\eta))}{\tan(s)-\tan(\eta)}\Big| < C_{k,l}\|f\|_{k+l+3}.$$

Hence $\|\frac{d^l}{d\eta^l}g_s(\tan(\eta))\|_\infty < \infty$ and the proof is complete. □

Using this lemma we prove

Lemma 13. *For every $q \ge 1$, $f \in H_{q+5}$ and $Z_t \in C([0,T], H_{-q})$, we have $\langle Z_t, A_t f\rangle \in C[0,T]$ and $Z_t \to \langle Z_t, A_t f\rangle$ is continuous.*

Proof: Lemma 11 yields $A_t f \in H_q$ and

$$|\langle Z_t, A_t f\rangle - \langle Z_s, A_s f\rangle| \le |\langle Z_t, A_t f - A_s f\rangle| + |\langle Z_t - Z_s, A_s f\rangle| \to 0, \qquad s \to t,$$

by Lemma 12 and the continuity of Z_t. Moreover, by another application of Lemma 11, for every sequence $Z_t^i \to Z_t$ in $C([0,T], H_{-q})$ we have

$$\lim_i \sup_{t\le T} |\langle Z_t^i - Z_t, A_t f\rangle| \le C \lim_i \sup_{t \le T}\|Z_t^i - Z_t\|_{-q}\|f\|_q = 0.$$

□

2.3 Estimates related to Y_t^n. The following estimates are consequences of Proposition 1 and the representation lemma, Lemma 9. Recall that $z = a + ib$ and that the formulas below hold for all z such that $b \neq 0$. (Actually we only need these equations for $b > 0$.)

Lemma 14. *If $E|M_0^n(z) - M_0(z)| \leq C/n|b|$ then for all $f \in \widetilde{\mathcal{S}}$ there is a C and a $D(z)$ locally bounded in z such that*

$$\begin{aligned} E|\langle Y_t^n, f\rangle| &\leq C\|f\|_2, \\ E|\langle Y_t^n, f_z'\rangle| &\leq D(z), \\ E|\langle Y_t^n, f_z''\rangle| &\leq D(z). \end{aligned}$$

If the second moment condition $E|M_0^n(z) - M_0(z)|^2 \leq C/n^2b^2$ holds we get for new $C, D(z)$ that

$$\begin{aligned} E\langle Y_t^n, f\rangle^2 &\leq C\|f\|_2^2, \\ E|\langle Y_t^n, f_z\rangle|^2 &\leq D(z). \end{aligned}$$

Proof: From the proof of Lemma 9 we obtain that the constants A, B in the decomposition of f are bounded by $C\|f\|_2$. Hence we can assume that $f(x) \in L^1$ and use $q = 1$ in Lemma 9 to decompose f. An application of Fubini yields

$$E|\langle Y_t^n, f\rangle| \leq \iint |h(z)| nE|M_t^n(z) - M_t(z)|\, dadb.$$

Therefore Proposition 1 implies

$$E|\langle Y_t^n, f\rangle| \leq C \iint |h(z)| \log(1 + 1/b)/b\, dadb < \infty.$$

Because, by Lemma 8 we have

$$\iint |h(z)| log(1 + 1/b)/b\, dadb \leq C\Big(\int |f(x)| + |f''(x)|\, dx\Big) < C\|f\|_2.$$

If $f = f_z'$ or $f = f_z''$ the right hand side above is continuous on $\mathrm{Im}(z) \neq 0$ and the first statement in the lemma hold. The second statement follows similarly from Proposition 1 and the decomposition with $q = 2$. Again assume that the slowly decreasing parts of f are removed, and that the constants A, B are less than $C\|f\|_2^2$. We have $h(a, b) = be^{-b}h_1(a)$ and

$$\Big(\int be^{-b}h_1(a)(M_t(z) - M_t^n(z))\, dadb\Big)^2 \leq C_a \int e^{-b}|h_1(a)|b^2|M_t(z) - M_t^n(z)|^2\, dadb,$$

where

$$C_a = \int |h_1(a)\, da| \leq C\|f\|_2.$$

Proposition 1 yields

$$E|\langle Y_t^n, f\rangle| \leq C\|f\|_2^2 \int b^2 e^{-b}\Big(\frac{\log(1 + \frac{1}{b})}{b}\Big)^2 db \leq C\|f\|_2^2.$$

Hence the last two inequalities follow and the lemma is proved. □

As a consequence we have the following lemma.

Lemma 15. *If $f \in H_4$ and the weaker assumptions of Lemma 14 hold, then*

$$E \sup_{s \le T} |\langle Y_s^n, f\rangle| \le C_T \|f\|_4.$$

For $f \in \widetilde{\mathcal{S}}$

$$E \sup_{s \le T} |\langle Y_s^n, Bf\rangle| \le C_T,$$
$$E \sup_{s \le T} |\langle Y_s^n, A_s f\rangle| \le C_T$$

and

$$E \sup_{s \le T} |\langle Y_s^n, A_s^n f\rangle| \le C_T.$$

Also

$$E|\langle Y_{t+h}^n - Y_t^n, f_z'\rangle| = o(1), \qquad h \to 0.$$

Proof: The proof is based on the estimate in Lemma 14. Consider expression (5) for $\langle Y_t^n, f\rangle$, and let

$$U_s^1 = \frac{\alpha}{2} \iint \frac{f'(x) - f'(y)}{x - y} [X_s^n(dx) Y_s^n(dy) + X_s(dx) Y_s^n(dy)],$$
$$U_s^2 = \int x f'(x)\, Y_s^n(dx)$$

and

$$M_t = \frac{\sigma}{\sqrt{n}} \int_0^t \sum_{i=1}^n f'(\lambda_s^i)\, d\beta_s^i.$$

Taking supremum and expectation in (5), we get

$$E \sup_{t \le T} |\langle Y_t^n, f\rangle| \le E|\langle Y_0^n, f\rangle| + \int_0^T E|U_s^2|\, ds + \int_0^T E|U_s^1|\, ds$$
$$+ \gamma \int_0^T E|\langle X_s^n, f''\rangle|\, ds + E \sup_{t \le T} |M_t|.$$

Now the assumptions imply

$$E|\langle Y_0^n, f\rangle| \le C\|f\|_2$$

and

$$\gamma E \int_0^t |\langle X_s^n, f''\rangle|\, ds \le C_T \|f\|_3.$$

Lemma 14 and $xf' \in H_2$ imply $E|U_s^2| \le C\|f\|_3$. Using $f' \in H_3 \cap L^1$ and

$$\left(\frac{1}{x - z} - \frac{1}{y - z}\right)/(x - y) = \frac{1}{x - z} \frac{1}{y - z}$$

we get from Lemma 9

$$\langle Y_t^n, A_t f\rangle = \int h(a,b) n(M_t^n(z) - M_t(z)) M_t(z)\, da db.$$

and

$$\langle Y_t^n, A_t^n f\rangle = \int h(a,b)n(M_t^n(z) - M_t(z))M_t^n(z)\, dadb.$$

Using $\max(|M_t(z)|, |M_t^n(z)|) \le 1/b$ and Lemma 14, the expectation of the absolute value of the integrand is bounded by

$$C|h(a,b)|\log(1+1/b)/b^2.$$

Therefore by Lemma 8,

$$E|U_s^1| = \frac{\alpha}{2}E|\langle Y_t^n, A_t f\rangle + \langle Y_t^n, A_t^n f\rangle| \le C\|f\|_4.$$

If the first inequality in the lemma is true we may use this inequality on the above expresions for $\langle Y_t^n, A_t f\rangle$ and $\langle Y_t^n, A_t^n f\rangle$ to get the third and fourth bounds in the lemma. It remains to bound the Martingale. From the properties of the stochastic integral and Doobs maximal inequality we have

$$E\sup_{t\le T}|M_t| \le C_T\Big(\int_0^T E|\langle X_t^n, (f')^2\rangle|\, dt\Big)^{1/2} \le C_T\|f\|_2$$

and the the first inequality. The second inequality follows from Lemma 5 and the previous argument. Finally, the last inequality also follows from the previous argument with the slight modification of taking the integral representation for $\langle Y_{t+h}^n - Y_t^n, f_z\rangle$ instead of $\langle Y_t^n, f\rangle$. □

We also have

Lemma 16. *Suppose that $f \in H_4$ and the stronger assumption of Lemma 14 hold. Then,*

$$E\sup_{s\le T}|\langle Y_s^n, f\rangle|^2 \le C_T\|f\|_4^2.$$

For $f \in \widetilde{\mathcal{S}}$ we have

$$E\sup_{s\le T}|\langle Y_s^n, A_s f\rangle|^2 \le C_T$$

and

$$E\sup_{s\le T}|\langle Y_s^n, A_s^n f\rangle|^2 \le C_T.$$

Also

$$E|\langle Y_{t+h}^n - Y_t^n, f_z\rangle|^2 = o(1), \qquad h \to 0.$$

Proof: Taking supremum and expectation of the square of (5), we get

$$E\sup_{t\le T}|\langle Y_t^n, f\rangle|^2 \le C_T\Big(E|\langle Y_0^n, F\rangle|^2 + \int_0^T E|U_s^2|^2\, ds + \int_0^T E|U_s^1|^2\, ds$$
$$+ \gamma\int_0^T E|\langle X_s^n, f''\rangle|^2\, ds + E\sup_{t\le T}|M_t|^2\Big).$$

The proof of the lemma now follows as in Lemma 15.

In Section 2.5 we will need

Lemma 17.

If $f \in H_4$ and the stronger assumptions of Lemma 14 hold then

$$E|\langle Y_t^n, A_t f - A_t^n f\rangle| \le \|f\|_4 C_t/n.$$

Proof: Using a decomposition of $f' \in H_3 \cap L^1$ with $q = 2$ in Lemma 9, we get

$$\langle Y_t^n, A_t f - A_t^n f\rangle = \iint \frac{1}{n}\frac{f'(x) - f'(y)}{x - y} Y_t^n(dx) Y_t^n(dy) = \int h(z) n(M_t(z) - M_t^n(z))^2 \, dadb.$$

By Proposition 1 the expectation of the absolute value of the integrand is less than $C|h(a,b)|\log(1+1/b)^2/nb^2$ and the lemma follows from Lemma 8 and Lemma 9. □

Remark. It seems plausible that we could restrict ourselves to a family of test-functions of the type $f(a+bx)$, and let $V_t(a,b) = \langle Y_t, f(a+bxe^{-t})\rangle$. Hence obtain the differential representation

$$dV_t = \iint V_t(a', b')\mu(a, b, da', db')\, dadb + h(a,b) + dB_t(a,b),$$

where μ is a distribution. The use of this representation can perhaps yield sharper estimates and theorems.

2.4 Tightness. Recall that $Y_t^n = n(X_t^n - X_t)$. We are going to prove tightness of the measures $\{P^n\}_{n=1}^{\infty}$ associated with the processes $\{Y_t^n\}_{n=1}^{\infty}$ under some conditions on the initial measures. The method we shall adopt to prove tightness is the so called Mitomas Lemma. This Lemma is actually a remark in the end of his paper, Mitoma (1983). The statement in our setting translates as follows:

Mitomas Lemma. *If the sequence of probability measures $\{P^n\}$ is k-continuous, i.e. for all $\epsilon, \rho > 0$ there exists a $\delta > 0$ such that*

$$P^n\Big(x \in C(\widetilde{S}', [0,T]); \sup_t |\langle x_t, \xi\rangle| > \epsilon\Big) \le \rho \text{ if } \|\xi\|_k \le \delta, \quad n \ge 1,$$

then if there exist a nuclear mapping from H_p to H_k, the tightness of the stochastic processes $\langle Y_t^n, f\rangle$ in $C[0,T]$ for all $f \in H_p$ imply tightness of $\{P^n\}$ in $C(H_{-p}, [0,T])$. Moreover if the finite dimensional distributions associated with $\widetilde{S}'$ converge weakly, the same is true for $\{P^n\}$ in $C(H_{-p}, [0,T])$.

In our case we show that with $k = 4$, k-continuity implies convergence in H_{-p} for $p = 6$. The 4-continuity follows from Lemma 15 and Chebychev's inequality, i.e.

Lemma 18. *If there is a C such that*

$$E|M_0^n(z) - M_0(z)| \le \frac{C}{n|b|}$$

then for all $\epsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that

$$P\Big(Y_t^n \in C([0,T], \widetilde{S}')\,;\, \sup_{t \le T} |\langle Y_t^n, f\rangle| > \epsilon\Big) \le \rho \text{ if } \|f\|_4 \le \delta.$$

Theorem 1. *Suppose we have a constant C such that for all n*

$$E|M_0^n(z) - M_0(z)| \leq \frac{C}{n|b|}.$$

Then the sequence $\{P^n\}_{n=1}^{\infty}$, or equivalently $\{Y_t^n\}_{n=1}^{\infty}$, is tight in $(C([0,T], H_{-6}), \mathcal{B}_6)$.

Proof of Theorem 1: Note that $\widetilde{\mathcal{S}}'$ satisfies the requirements in Mitoma (1983). By Mitomas Lemma it is enough to prove

(a) $\langle Y_t^n, f\rangle$ is tight for all $f \in \widetilde{\mathcal{S}}$,

(b) There is a nuclear map from $\bar{H}_6$ to $\bar{H}_4$.

The existence of a nuclear map from $H_6 \to H_4$ follows from Treves (1967) p 527. Hence we can use Lemma 18 to conclude 4-continuity. By Mitomas Lemma, it is enough to prove the tightness of the stochastic processes $\langle Y_t^n, f\rangle$, $f \in \widetilde{\mathcal{S}}$. We prove tightness of $\langle Y_t^n, f\rangle$ in the manner of Aldous (1978). That is, we prove

(i) for each fixed t, $\{\langle Y_t^n, f\rangle\}$ is tight in R,

(ii) For any given stopping times τ_n bounded by T and a sequence $\delta_n \to 0$ as $n \to \infty$, then $\langle Y_{\tau_n+\delta_n}^n, f\rangle - \langle Y_{\tau_n}^n, f\rangle \to 0$ in probability as $n \to \infty$.

Property (i) follows from Chebychev's inequality and Lemma 14. Let

$$\|f_{\cdot}\|_t = \sup_{0\leq\tau\leq t} |f_\tau|.$$

To prove (ii) we use equation (5). This yields

$$\begin{aligned}
E|\langle Y_{\tau_n+\delta_n}^n, f\rangle - \langle Y_{\tau_n}^n, f\rangle| \leq{}& \frac{\alpha\delta_n}{2} E\Big\| \iint \frac{f'(x)-f'(y)}{x-y}\, X_{\cdot}^n(dx) Y_{\cdot}^n(dy)\Big\|_{T+\delta_n} \\
&+ \frac{\alpha\delta_n}{2} E\Big\| \iint \frac{f'(x)-f'(y)}{x-y}\, X_{\cdot}(dx) Y_{\cdot}^n(dy)\Big\|_{T+\delta_n} \\
&+ \delta_n E\Big\| \int x f'(x)\, Y_{\cdot}^n(dx)\Big\|_{T+\delta_n} \\
&+ \gamma\delta_n E\|\langle X_{\cdot}^n, f''\rangle\|_{T+\delta_n} \\
&+ \sigma E\Big| \int_{\tau_n}^{\tau_n+\delta_n} \frac{1}{\sqrt{n}} \sum_{i=1}^n f'(\lambda_s^i) d\beta_s^i\Big|.
\end{aligned}$$

The first four expectations are all bounded, by Lemma 15. In view of Jensen's inequality and

$$E\Big[\Big(\int_{T\wedge(\tau_n+\delta_n)}^{T\wedge\tau_n} f'(\lambda_s^i)\, d\beta_s^i\Big)^2\Big] = E\Big[\int_{T\wedge(\tau_n+\delta_n)}^{T\wedge\tau_n} (f_s'(\lambda_s^i))^2 ds\Big]$$

(see Karatzas and Shreve, p.139), the final term also tends to zero. $\square$

Remark. It seems plausible that after imposing a moment condition on $M_0^n(z) - M_0(z)$ and using an argument as in the proof of Theorem 12.3, Billingsley (1968), we could prove tightness in H_{-4}. To see this note that $\|Y_t^n - Y_s^n\| = \sup_{\|f\|\le 1} |\langle Y_t^n - Y_s^n, f\rangle|$. Hence it is enough to check whether

$$E(\sup_{\|f\|\le 1} |\langle Y_t^n - Y_s^n, f\rangle|)^\alpha \le C|s-t|^\beta$$

for suitable α, β. By writing this difference in integral form, and using (5) and

$$|\langle Y_t^n, f\rangle| \le \int \Big(\int (1+|a|)^2 |h(a,b)|^2\, da\Big)^{1/2} \Big(\int \Big(\frac{n|M_t^n(z) - M_t(z)|}{1+|a|}\Big)^2 da\Big)^{1/2} db,$$

we see that a moment condition on $n(M_t(z) - M^n(z))$ is enough. But this follows from a moment condition on $n(M_0(z) - M_0^n(z))$ by an argument similar to the proof of the main result in section 2.1, Proposition 1.

2.5 Convergence of martingales. Recall the definitions $\bar f = \{f_j\}_{j=1}^k$, $F \in C_b^2(R^k, R)$ and $F_{\bar f}(Z) = F(\langle Z, f_1\rangle, \ldots, \langle Z, f_k\rangle)$. Also recall the rest of the notation of the martingale problem in Section 1. We proved in the previous section that the sequence $\{Y_t^n\}_{n=1}^\infty$ is tight in $C([0,T], H_{-6})$. We shall in this section prove that every convergent subsequence converges to a limiting distribution process Y_t such that

$$\Phi_s^t(Y) = F_{\bar f}(Y_t) - F_{\bar f}(Y_s) - \int_s^t L_\tau F_{\bar f}(Y_\tau)\, d\tau, \qquad t \ge s,$$

is a martingale. Let P^n be the measure associated with the process Y_t^n.

Theorem 2. *Suppose that*

$$E|M_0^n(z) - M_0(z)|^2 \le \frac{C}{b^2 n^2}.$$

Then for any $l \ge 4$ and limiting measure P of the sequence $\{P^n\}_{i=1}^\infty$ in the topology of weak convergence in $(C([0,t], H_{-l}), \mathcal{B}_l)$, $\Phi_s^t(Y)$ is a $\mathcal{F}_t^l$-martingale.

Note that $\Phi_s^t(Y)$ is a martingale if and only if

$$E\Phi_s^t(Y)\chi(Y) = 0$$

for all bounded continuous $\mathcal{F}_s^l-$ measurable functions $\chi : C([0,T], H_{-l}) \to R$. We begin with a mean square estimate.

Lemma 19.

Let $f \in \widetilde{S}$. Under the condition in Theorem 2 we have

$$E\Phi_s^t(Y^n)^2 \le C_t \qquad 0 \le s \le t.$$

Proof: We have

$$\Phi_s^t(Y^n)^2 \le C\Big(F_{\bar f}(Y_t^n)^2 + F_{\bar f}(Y_s^n)^2 + \Big(\int_s^t L_\tau F_{\bar f}(Y_\tau^n)\,d\tau\Big)^2\Big).$$

The boundedness of F_f and Jensen's inequality imply

$$\Phi_s^t(Y^n)^2 \le C_t\Big(1 + \int_s^t L_\tau F_{\bar f}(Y_\tau^n)^2\,d\tau\Big).$$

Taking expectation of this and using Fatou lead to

$$E\Phi_s^t(Y^n)^2 \le C_t\Big(1 + \int_s^t EL_\tau F_{\bar f}(Y_\tau^n)^2\,d\tau\Big).$$

The integrand is bounded by

$$\begin{aligned} EL_t F_{\bar f}(Y_t^n)^2 \le C_t\Big(&\sum_{i=1}^k (E\langle Y_t^n, Bf_i\rangle^2 + E\langle Y_t^n, A_t f_i\rangle^2 + \langle X_t, f_i''\rangle^2) \\ &+ \sum_{i,j=1}^k \langle X_t, f_i' f_j'\rangle^2\Big), \end{aligned}$$

where we used that the partial derivatives up to second order of F_f are bounded. From this we see that the terms containing f_i'' and $f_i' f_j'$ are bounded, hence it is enough to prove that $E\langle Y_t^n, Bf\rangle^2$ and $E\langle Y_t^n, A_t f\rangle^2$ are uniformly bounded in n for fixed t. But this has been shown in Lemma 16. □

Lemma 20. *Let $f \in \widetilde{S}$. Under the condition in Theorem 2 we have*

$$E|\Phi_s^t(Y^n) - \Phi_s^{t,n}(Y^n)| < C_t/n.$$

Proof: By definition of Φ_s^t

$$\begin{aligned} E|\Phi_s^t(Y^n) - \Phi_s^{t,n}(Y^n)| &= E\Big|\int_s^t L_\tau F_{\bar f}(Y_\tau^n) - L_\tau^n F_{\bar f}(Y_\tau^n)\,d\tau\Big| \\ &\le \int_s^t E|L_\tau F_{\bar f}(Y_\tau^n) - L_\tau^n F_{\bar f}(Y_\tau^n)|\,d\tau. \end{aligned}$$

Since all derivatives up to second order of F_f are bounded, we have

$$\begin{aligned} |L_t F_{\bar f}(Y_t^n) - L_t^n F_{\bar f}(Y_t^n)| \le C\Big(&\sum_{j=1}^k (|\langle Y_t^n, A_t f_j - A_t^n f_j\rangle| + \frac{1}{n}|\langle Y^n, f_j''\rangle|) \\ &+ \sum_{j,l=1}^k \frac{1}{n}|\langle Y_t^n, f_j' f_l'\rangle|\Big). \end{aligned}$$

Furthermore, using the bounds in Lemma 14 and 17, we get

$$E|L_t F_{\bar f}(Y_t^n) - L_t^n F_{\bar f}(Y_t^n)| \le C_t/n$$

and hence

$$E|\Phi_s^t(Y^n) - \Phi_s^{t,n}(Y^n)| \le \frac{C_t}{n}\int_s^t d\tau \le \frac{tC_t}{n}.$$

□

Proof of Theorem 2: Suppose that we have a subsequence $\{Y_t^n\}_{n=1}^\infty$ converging weakly to Y_t in $C([0,T],H_{-l},\mathcal{B}_l)$.

We note that $\Phi_s^t(\cdot)$ is a continuous function. To see this, suppose $Z_t^n \to Z_t$ is a convergent sequence in $C([0,T],H_{-l})$. Then it is enough to show that $L_tF_{\bar{f}}(Z^n)$ converges in $C[0,T]$, because F_f is continuous and the integral over a domain in $[0,T]$ is a continuous map from $C([0,T],R)$ to R. Now, all the derivatives up to second order of F_f converge in $C([0,T],R)$ and the same is true for $-\langle Z_t^n, Bf_i\rangle$. Moreover $\langle Z_t^n, A_tf_i\rangle$ converges in $C([0,T],R)$ by Lemma 13. Hence the continuity of $\Phi_s^t(\cdot)$ is demonstrated.

As a consequence $\Phi_s^t(Y^n)\chi(Y^n)$ converges weakly. Because of the uniform bound in Lemma 19 and the boundedness of χ we get $\lim_n E\Phi_s^t(Y^n)\chi(Y^n) = E\Phi_s^t(Y)\chi(Y)$. Hence for all $\epsilon > 0$ we have an N such that for all $n > N$

$$|E\Phi_s^t(Y^n)\chi(Y^n) - E\Phi_s^t(Y)\chi(Y)| < \epsilon.$$

By choosing N possibly larger we derive from Lemma 20,

$$E|\Phi_s^t(Y^n) - \Phi_s^{t,n}(Y^n)||\chi(Y^n)| \le \frac{C_t\|\chi\|_\infty}{n} < \epsilon.$$

Finally noting that χ is also a continuous map $\chi : C([0,T],R^n) \to R$ where we interpret $\chi(x_t^1,\ldots,x_t^n)$ as

$$\chi(x_t^1,\ldots,x_t^n) = \chi(\sum_{i=1}^n \frac{1}{n}\delta_{x_t^i}),$$

we conclude

$$\begin{aligned}|E\Phi_s^t(Y)\chi(Y)| &\le |E\Phi_s^t(Y)\chi(Y) - E\Phi_s^t(Y^n)\chi(Y^n)| \\ &\quad + E|\Phi_s^t(Y^n) - \Phi_s^{t,n}(Y^n)||\chi(Y^n)| + |E\Phi_s^{t,n}(Y^n)\chi(Y^n)| \\ &\le 2\epsilon.\end{aligned}$$

□

Corollary 1. *Under the conditions in Theorem 2 let P be the limit measure in Theorem 2 and let $\widetilde{P}$ be the restriction of P to the Borel algebra $\mathcal{B}$. Then under $(\widetilde{P}, C([0,T],\widetilde{\mathcal{S}}'),\mathcal{F}_t)$ we have that Φ_s^t is a martingale.*

Proof: By Theorem 2 there is a l such that Φ_s^t is a $(P, C([0,T],H_{-l}),\mathcal{F}_t^l)$-martingale. Now if Φ_s^t were measurable with respect to $\mathcal{B}$, we would have for all $A \in \mathcal{F}_s$ that

$$\begin{aligned}\int_A E(\Phi_s^t|\mathcal{F}_s)d\widetilde{P} &= \int_A \Phi_s^t\, d\widetilde{P} \\ &= \int_{A\cap C([0,T],H_{-q})} \Phi_s^t\, dP \\ &= \int_{A\cap C([0,T],H_{-q})} E(\Phi_s^t|\mathcal{F}_s^q)\, dP = 0,\end{aligned}$$

since $\mathcal{F}_s \cap C([0,T],H_{-l}) \subset \mathcal{F}_s^q$. It remains to prove the measurability of Φ_s^t. Φ_s^t is a continuous function of elements of the type $\langle Y_t, f\rangle$, $0 \le t \le T$ and $\langle Y_t, A_tf\rangle$, $0 \le t \le T$.

It is not hard to see that $\langle Y_t.f\rangle$ is measurable. To see that $\langle Y_t, A_t f\rangle$ is measurable note that $A_t f(\arctan(x)) = \sum_{i=-\infty}^{\infty} a_i(t)\exp(inx)$ and that $\exp(inx)$ is an orthogonal basis in $\bar{H}^q$ for all $q \geq 0$. From Lemma 12 we get constants C_q such that $\|A_t f\|_q^{\#} \leq C_q$. Hence there are new constants C_q such that

$$\sum_{j=-\infty}^{\infty} |j|^{2q}|a_j(t)|^2 \leq C_q < \infty.$$

Moreover, since $Y_t \in C([0,T], H_{-l})$ we have

$$\sup_{0\leq t\leq T} |\langle Y_t, \exp(ij\arctan(x)\rangle| \leq C|j|^l$$

so that

$$\langle Y_t, A_t f\rangle = \lim_{m\to\infty}\sum_{j=-m}^{m} a_j(t)\langle Y_t, \exp(ij\arctan(x))\rangle,$$

where the limit is in $C([0,T],R)$. Hence $\langle Y_t, A_t f\rangle$ is measurable and the corollary follows. $\square$

2.6 Uniqueness. We showed in the previous section that the limiting measure must satisfy a martingale problem. Recall Definition 1 stated in Section 1. In this section a corresponding uniqueness theorem is proved.

Theorem 3. *The set $\mathcal{A}(\bar{g},\mu)$ contains at most one measure. If μ is a point mass then a measure in $\mathcal{A}(\bar{g},\mu)$ has Gaussian finite dimensional distributions.*

By Mitomas Lemma stated in Section 2.4 we only need to verify the convergence of the finite dimensional distributions. To prove the theorem it is enough to show that for all $P_1, P_2 \in \mathcal{A}(\bar{g},\mu)$, the finite dimensional distributions of Y_t, $0\leq t\leq T$, are equal under P_1 and P_2.

Let $\{t_i\}_{i=1}^k$ be such that $0\leq t_k < t_{k-1} < \ldots < t_1 \leq T$ and take $\{f^j\}_{j=1}^k$ such that $f^j \in \widetilde{\mathcal{S}}$. Using property (ii) we conclude that for all $\epsilon > 0$ we can take $l(\epsilon)$ so large such that for $g_j = g_j^{l(\epsilon)}$, $j = 1\ldots k$, where $\{g_j^k\}_{k=1}^{\infty} = \bar{g}(f^j)$, we have

$$\begin{aligned}|E_j(&\exp(i\langle Y_{t_1}, f^1\rangle + \cdots + i\langle Y_{t_k}, f^k\rangle)) \\ &- E_j(\exp(i\langle Y_{t_1}, g_1\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle))| < \epsilon \qquad j = 1,2.\end{aligned}$$

Hence it is enough to show

$$E_1(\exp(i\langle Y_{t_1}, g_1\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle)) = E_2(\exp(i\langle Y_{t_1}, g_1\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle)),$$

where E_i is expectation with respect to measure P^i. The proof of the theorem is preceded by two lemmas. To simplify notations, let

$$g_1 = c + d\mathrm{Re}\frac{1}{x-i} + \sum_{i=1}^{l_1}\frac{a_i}{x-z_i},$$

$$H = \exp(i\langle Y_{t_2}, g_2\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle),$$

$$K_t = K_t(a_1, \ldots a_{l_1}, z_1, \ldots, z_{l_1}) = \exp(i\langle Y_t, g_1\rangle), \qquad t_1 \geq t \geq t_2$$

and note that $|H| = |K_t| = 1$.

Lemma 21. *We have that EHK_t, $EHK_t\langle Y_t, f_z\rangle$ and $EHK_t\langle Y_t, f'_z\rangle$ are continuous in t.*

Proof: Recall the notation $f_z(x) = 1/(x-z)$. By property (v) of Definition 1

$$\begin{aligned}|EHK_{t+h} - EHK_t| &\leq E|\langle Y_{t+h} - Y_t, g_1\rangle| \\ &\leq C\sum_j^{l_1} E|\langle Y_{t+h} - Y_t, f_{z_j}\rangle| = o(1).\end{aligned}$$

By properties (iv) and (v) and by using $(\sum^N x_i)^2 \leq C_N \sum^N x_i^2$,

$$\begin{aligned}&|EHK_{t+h}\langle Y_{t+h}, f_z\rangle - EHK_t\langle Y_t, f_z\rangle| \\ &\quad\leq |EH(K_{t+h} - K_t)\langle Y_{t+h}, f_z\rangle| + |EHK_t\langle Y_{t+h} - Y_t, f_z\rangle| \\ &\quad\leq (E|K_{t+h} - K_t|^2)^{1/2}(E|\langle Y_{t+h}, f_z\rangle|^2)^{1/2} + E|\langle Y_{t+h} - Y_t, f_z\rangle| = o(1).\end{aligned}$$

The continuity of the third function follows from a similar argument. □

Noting that $\mathrm{Re}(f_z) = \frac{1}{2}(f_z + f_{\bar{z}}(x))$ and $\mathrm{Im}(f_z) = -\frac{1}{2}(if_z + \bar{i}f_{\bar{z}})$ we get

$$g_1 = c + \sum_{j=1}^{l_1}(a_j f_{z_j} + \bar{a}_j f_{\bar{z}_j}),$$

where $l_1 \to l_1 + 1$, $a_{l_1} = d/2$ and $-ia_j/2 \to a_j$. In what follows suppose that g_j is of the extended form above. Also, properties (i) and (iii) yield $\langle Y_t, 1\rangle = 0$ a.s. and hence we can drop the constant c in the representation of g_j.

Let $\eta = (a_1, \ldots, a_{l_1}, z_1, \ldots, z_{l_1})$ and call a function $G(t,\eta)$ *locally uniformly η-continuous* if to every point (t,η) there exists a neighborhood U such that for all $\epsilon > 0$, there exists a $\delta > 0$, such that whenever $(t',\xi) \in U$, $(t',\eta) \in U$ and $|\xi - \eta| \leq \delta$, we have $|G(t',\xi) - G(t',\eta)| \leq \epsilon$.

Lemma 22. *The functions EHK_t, $EHK_t\langle Y_t, f_z\rangle$ and $EHK_t\langle Y_t, f'_z\rangle$ are all locally uniformly η-continuous.*

Proof: For random variables $Z(w,t,x,y)$, such that $\frac{\partial Z}{\partial x}(w,t,x,y)$ and $\frac{\partial Z}{\partial y}(w,t,x,y)$ exist and are locally integrable in x respectively in y, we have

$$\begin{aligned}&Z(w,t,x+h,y+k) - Z(w,t,x,y) \\ &\quad= \int_0^h \frac{\partial Z}{\partial x}(w,t,x+s,y+k)\,ds + \int_0^k \frac{\partial Z}{\partial y}(w,t,x,y+s)\,ds\end{aligned}$$

and hence

$$|EZ(t',x+h,x+k) - EZ(t',x,y)| \leq h \sup_{(t',x,y)\in U} E|\frac{\partial Z}{\partial x}| + k \sup_{(t',x,y\in U)} E|\frac{\partial Z}{\partial y}|,$$

where U is a convex neighborhood of (t, x, y), $(t', x, y) \in U$, $(t', x, y+k) \in U$ and $(t', x+h, y+k) \in U$. Hence locally uniform η-continuity will follow from local integrability of the partial derivatives and local boundedness of the expectation of the partial derivatives. Lemma 7 implies that the derivatives of the three terms are all continuous and hence locally integrable. To take derivatives, we note that if $z = a + ib$ then

$$\begin{aligned}\frac{\partial}{\partial a} &= \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial b} &= i\Big(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\Big),\end{aligned}$$

and it is enough to verify that the expectation of the absolute value of the differential with respect to $a_j, \bar{a}_j, z_j, \bar{z}_j, j = 1 \ldots l_1$ is locally bounded. Now taking the absolute value of the derivative of HK_t, $HK_t\langle Y_t, f_z\rangle$ respectively $HK_t\langle Y_t, f'_z\rangle$, we see that local boundedness will follow from local boundedness of the terms

$$\begin{aligned}&E|\langle Y_t, f_z\rangle|, \quad E|\langle Y_t, f'_z\rangle|, \quad E|\langle Y_t, f''_z\rangle| \\ &E|\langle Y_t, f_z\rangle||\langle Y_t, f_{z'}\rangle| \le (E|\langle Y_t, f_z\rangle|^2)^{1/2}(E|\langle Y_t, f_{z'}\rangle|^2)^{1/2} \\ &E|\langle Y_t, f_z\rangle||\langle Y_t, f'_{z'}\rangle| \le (E|\langle Y_t, f_z\rangle|^2)^{1/2}(E|\langle Y_t, f'_{z'}\rangle|^2)^{1/2} \\ &E|\langle Y_t, f'_z\rangle||\langle Y_t, f'_{z'}\rangle| \le (E|\langle Y_t, f'_z\rangle|^2)^{1/2}(E|\langle Y_t, f'_{z'}\rangle|^2)^{1/2}.\end{aligned}$$

But this follows from property (iv) of Definition 1, and the lemma is proved. □

Proof of Theorem 3: Recall that $0 \le t_k < \ldots < t_2 < t_1 \le T$ and assumne $t_2 \le t \le t_1$. Fix one of the measures, then conditioning on $\mathcal{F}_{t_2}$ will give us

$$\begin{aligned}E\exp(i\langle Y_t, g_1\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle) =& E\exp(i\langle Y_{t_2}, g_2\rangle, + \cdots + i\langle Y_{t_k}, g_k\rangle) \\ &\times E(\exp(i\langle Y_t, g_1\rangle)|\mathcal{F}_{t_2}).\end{aligned}$$

Use the martingale property (iii) in Definition 1 to get

$$\begin{aligned}&E\exp(i\langle Y_t, g_1\rangle, + \cdots + i\langle Y_{t_k}, g_k\rangle) \\ &\qquad = E\exp(i\langle Y_{t_2}, g_2\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle)E(\exp(i\langle Y_t, g_1\rangle)|\mathcal{F}_{t_2}) \\ &\qquad = E\exp(i\langle Y_{t_2}, g_1 + g_2\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle) \\ &\qquad\quad + E\exp(i\langle Y_{t_2}, g_2\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle)\int_{t_2}^t Q_s\, ds,\end{aligned}$$

where

$$\begin{aligned}Q_s = \exp(i\langle Y_s, g_1\rangle)\Big(\sum_{j=1}^{l_1}\Big(&(\frac{\alpha}{2}M_s(z_j) + z_j)ia_j\langle Y_s, -f'_{z_j}\rangle \\ &+ (\frac{\alpha}{2}M_s(\bar{z}_j) + \bar{z}_j)i\bar{a}_j\langle Y_s, -f'_{\bar{z}_j}\rangle \\ &+ a_j(\frac{\alpha}{2}\frac{\partial}{\partial z_j}M_s(z_j) + 1)i\langle Y_s, f_{z_j}\rangle \\ &+ \bar{a}_j(\frac{\alpha}{2}\frac{\partial}{\partial \bar{z}_j}M_s(\bar{z}_j) + 1)i\langle Y_s, f_{\bar{z}_j}\rangle\Big) \\ + i\gamma\langle X_s, g''_1\rangle - \frac{\sigma^2}{2}\langle X_s, (g'_1)^2\rangle\Big)\end{aligned}$$

Now using property (iv) we see that

$$
\begin{aligned}
&E\exp(i\langle Y_{t_2}, g_2\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle)\int_{t_2}^{t} Q_s\,ds\\
&\qquad = \int_{t_2}^{t} E\exp(i\langle Y_{t_2}, g_2\rangle, + \cdots + i\langle Y_{t_k}, g_k\rangle)Q_s\,ds.
\end{aligned}
$$

We shall prove that the integrand is continuous. For a function $G(t,x,y)$ to be continuous, we may use

$$
\begin{aligned}
|G(t+h,x+k,y+l) - G(t,x,y)| \le{}& |G(t+h,x+k,y+l) - G(t+h,x,y)|\\
&+ |G(t+h,x,y) - G(t,x,y)|,
\end{aligned}
$$

and it is enough to show continuity in time for fixed η and locally uniform η-continuity. Because all the nonrandom elements in the expression for Q_s are continuous, we only need the terms of the form EHK_t, $EHK_t\langle Y_t, f_z\rangle$ and $EHK_t\langle Y_t, f_z'\rangle$ to be continuous. By Lemma 21, they are continuous in time and by Lemma 22 they are locally uniformly η-continuous. Hence we have proved the continuity of not only EQ_s but also each of the terms in EQ_s. Define $\phi_t = EHK_t$ and take the time derivative of ϕ_t to get

$$
\begin{aligned}
\frac{\partial\phi_t}{\partial t} = EH\Big(\sum_{j=1}^{l_1}\Big(&(\frac{\alpha}{2}M_s(z_j)+z_j)\frac{\partial K_t}{\partial z_j} + (\frac{\alpha}{2}M_s(\bar z_j)+\bar z_j)\frac{\partial K_t}{\partial \bar z_j}\\
&+ a_j(\frac{\alpha}{2}\frac{\partial}{\partial z_j}M_s(z_j)+1)\frac{\partial K_t}{\partial a_j} + \bar a_j(\frac{\alpha}{2}\frac{\partial}{\partial \bar z_j}M_s(\bar z_j)+1)\frac{\partial K_t}{\partial \bar a_j}\Big)\\
&+ i\gamma\langle X_s, g_1''\rangle K_t - \frac{\sigma^2}{2}\langle X_s, (g_1')^2\rangle K_t\Big)
\end{aligned}
$$

The expected values of the terms involving a differentiation are locally bounded and continuous, thus we can interchange expectation and differentiation to obtain the differential equation

$$
\begin{aligned}
\frac{\partial\phi_t}{\partial t} = \sum_{j=1}^{l_1}\Big(&(\frac{\alpha}{2}M_t(z_j)+z_j)\frac{\partial \phi_t}{\partial z_j} + (\frac{\alpha}{2}M_t(\bar z_j)+\bar z_j)\frac{\partial \phi_t}{\partial \bar z_j}\\
&+ a_j(\frac{\alpha}{2}\frac{\partial}{\partial z_j}M_t(z_j)+1)\frac{\partial \phi_t}{\partial a_j} + \bar a_j(\frac{\alpha}{2}\frac{\partial}{\partial \bar z_j}M_t(\bar z_j)+1)\frac{\partial \phi_t}{\partial \bar a_j}\Big)\\
&+ (i\gamma\langle X_t, g_1''\rangle - \frac{\sigma^2}{2}\langle X_t, (g_1')^2\rangle)\phi_t,
\end{aligned}
$$

where $\frac{\partial\phi_t}{\partial t}, \frac{\partial\phi_t}{\partial z_j}, \frac{\partial\phi_t}{\partial \bar z_j}, \frac{\partial\phi_t}{\partial a_j}$ and $\frac{\partial\phi_t}{\partial \bar a_j}$ all are continuous. This equation has, as we shall show, a solution. We solve first for the characteristics and then for a solution, i.e. solve the equations

$$
\begin{aligned}
\frac{dz_j}{dt} &= -\frac{\alpha}{2}M_t(z_j) - z_j, && j = 1,2,\ldots,l_1,\\
\frac{da_j}{dt} &= -a_j(\frac{\alpha}{2}\frac{\partial}{\partial z_j}M_t(z_j)+1), && j = 1,2,\ldots,l_1,\\
\frac{d\phi_t}{dt} &= (i\gamma\langle X_t, g_1''\rangle - \frac{\sigma^2}{2}\langle X_t, (g_1')^2\rangle)\phi_t.
\end{aligned}
$$

From Lemma 3, we first get the solutions $z_j^1(t)$, $j = 1,\dots l_1$ such that $z_j^1(t_1) = z_j$, $j = 1,\dots,l_1$. Secondly we solve for a_j, $j = 1,\dots l_1$. This will give us $a_j^1(t)$, $j = 1,\dots,l_1$ given by

$$a_j^1(t) = a_j \exp\Big(\int_t^{t_1} (\frac{\alpha}{2}\frac{\partial}{\partial z_j^1} M_s(z_j^1(s)) + 1)\, ds\Big) \qquad j = 1,\dots,l_1.$$

Finally

$$\phi_{t_1} = \phi_{t_2} \exp\Big(\int_{t_2}^{t_1} (i\gamma\langle X_s, g_1''\rangle - \frac{\sigma^2}{2}\langle X_s, (g_1')^2\rangle)\, ds\Big),$$

where

$$\phi_{t_2} = E\exp(i\langle Y_{t_2}, g_1 + g_2\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle),$$

and

$$g_1 = g_1(a_1^1(t),\dots,a_{l_1}^1(t_2), z_1^1(t_2),\dots,z_{l_1}^1(t_2)).$$

To see that we have a unique solution among those which are continuously differentiable we introduce the notation ϕ_t^j for a generic solution and $\phi_t^{j,1}$ and $\phi_t^{j,2}$ for two different solutions between t_j and t_{j+1}. Define the difference $\phi_t^{j,\Delta} = \phi_t^{j,1} - \phi_t^{j,2}$. They will satisfy the same equation as ϕ_t^j does by linearity. Along the characteristics for $j = 1$ we have

$$\frac{d\phi_t^{1,\Delta}}{dt} = (i\gamma\langle X_t, g_1''\rangle - \frac{\sigma^2}{2}\langle X_t, (g_1')^2\rangle)\phi_t^{1,\Delta}.$$

The factor in front of $\phi_t^{1,\Delta}$ is bounded along every characteristic and hence by Grönwall's lemma

$$|\phi_t^{1,\Delta}| \le C_t|\phi_{t_2}^{1,\Delta}|.$$

Proceed as above for $j = 2$ to get $\phi_{t_2}^2$ as

$$\phi_{t_2}^2 = \phi_{t_3}^2 \exp\Big(\int_{t_3}^{t_2} (i\gamma\langle X_s, (g_1 + g_2)''\rangle - \frac{\sigma^2}{2}\langle X_s, ((g_1 + g_2)')^2\rangle)\, ds\Big),$$

where

$$g_2 = g_2(a_1^2(t),\dots,a_{l_2}^2(t), z_1^2(t),\dots,z_{l_2}^2(t))$$

and

$$\phi_{t_3} = E\exp(i\langle Y_{t_3}, g_1 + g_2 + g_3\rangle + \cdots + i\langle Y_{t_k}, g_k\rangle)$$

and so on up to

$$\phi_{t_k}^k = \phi_0^k \exp\Big(\int_0^{t_k} (i\gamma\langle X_s, (g_1 + \cdots + g_k)''\rangle - \frac{\sigma^2}{2}\langle X_s, ((g_1 + \cdots + g_k)')^2\rangle)\, ds\Big),$$

where

$$g_k = g_k(a_1^k(t),\dots,a_{l_k}^k(t), z_1^k(t),\dots,z_{l_k}^k(t))$$

and

$$\phi_0^k = E\exp(i\langle Y_0, g_1 + \cdots + g_k\rangle).$$

The difference of two solutions, $\phi_t^{\Delta,j} = \phi_t^{j,1} - \phi_t^{j,2}$, satisfies by Grönwall's lemma

$$|\phi_{t_j}^{\Delta,j}| \le C_t|\phi_{t_{j-1}}^{\Delta,j}|.$$

Hence uniqueness follows by induction since two measures in $\mathcal{A}(\bar{g},\mu)$ have the same initial measure implying that $\phi^k_{t_k}$ is unique, implies $\phi^{k-1}_{t_{k-1}}$ is unique, etc. Hence

$$E_1(\exp(i\langle Y_{t_1},g_1\rangle+\cdots+i\langle Y_{t_k},g_k\rangle))=E_2(\exp(i\langle Y_{t_1},g_1\rangle+\cdots+i\langle Y_{t_k},g_k\rangle))$$

which implies that $\mathcal{A}(\bar{g},\mu)$ contains at most one measure. To see that a measure in $\mathcal{A}(\mu,\bar{g})$ is Gaussian if μ has a point mass at μ_0, we note that if in the expression derived for ϕ_t we replace g_j by $s_j g_j$ then $\phi_{t_1}=\varphi(s_1,\dots,s_k)$, where φ is the characteristic function of a Gaussian distribution. Since sequences of Gaussian random vectors that converge in distribution converge to a Gaussian vector, we conclude that the measure is Gaussian. $\square$

2.7 Weak convergence. In this final section we put everything together and prove the following convergence result

Theorem 4. *Let $z=a+ib$ and suppose that Y^n_0 converges weakly in $\widetilde{\mathcal{S}}'$ to Y_0. Also suppose that there is a constant C such that*

$$E|M^n_0(z)-M_0(z)|^2\le\frac{C}{b^2n^2}.$$

Then the sequence of processes Y^n_t converges weakly in $(C([0,T],\widetilde{\mathcal{S}}'),\mathcal{B})$ to a Gaussian process Y_t in $C([0,T],H_{-6})$ and there exists a $\bar{g}$ such that properties (i)-(v) in Definition 1 holds, in particular Φ^t_0 is a $\mathcal{F}_t$-martingale.

Proof: By Theorem 1 the sequence Y^n_t is tight in $(C([0,T],H_{-6}),\mathcal{B}_6)$. If we find a $\bar{g}$ such that every convergent subsequence belongs to $\mathcal{A}(Y_0,\bar{g})$ the theorem will follow from the uniqueness result in Theorem 3. Hence it is enough to verify properties (i)-(v) in Definition 1.

Property (i), $P(\langle Y_0,1\rangle=0)=1$ follows from the same property of the converging sequence $\langle Y^n_0,1\rangle$.

To prove (ii) we construct a map $\bar{g}$ with the desired property. Take $f\in\widetilde{\mathcal{S}}$ and fix $\epsilon=1/j$. From Lemma 9 we know that there exists c,d such that if $g=f-c-d\,\mathrm{Re}(1/x-i)$, then $g(x)\in L^1$ and we can decompose g as

$$g(x)=\int h(z)\mathrm{Im}(f_z(x))\,dadb,$$

where

$$h(z)=1_{\{b\ge 0\}}\frac{1}{\Gamma(q+1)}b^q e^{-b}\mathcal{F}^{-1}[(1+|s|)^{q+1}\mathcal{F}(g)](a)$$

and $q=5$. By Lemma 15 there exists C_T such that

$$E\|\langle Y^n,f_z\rangle\|_t\le C_T(1+\frac{1}{b^q}).$$

Take a compact set $K=[a_1,a_2]\times[b_1,b_2]\subset(-\infty,\infty)\times(0,\infty)$ such that

$$\int|h(z)|(1+\frac{1}{b^q})1_{K^c}(a,b)\,dadb\le\epsilon. \tag{17}$$

Since $h(z)$ is continuously differentiable and K compact there exists a constant C' such that

$$\sup_{(a,b)\in K} \max\Big(|\frac{\partial h}{\partial a}|, |\frac{\partial h}{\partial b}|\Big) = C' < \infty$$

and hence

$$|h(a+h,b+k) - h(a,b)| \le C'(|h|+|k|) \le \sqrt{2}C'\sqrt{h^2+k^2}.$$

For all $\eta > 0$ we can decompose K as a disjoint union of rectangles $K = \cup_{m=1}^{k_\eta} A_m$ such that $|z_1 - z_2| \le \eta$ for all z_1, z_2 such that $z_1, z_2 \in A_m$ for some A_m. Define

$$g^\eta(x) = \sum_{m=1}^{k_\eta} a_m^\eta h(z_m)\mathrm{Im}(f_{z_m}(x)),$$

where $z_m \in A_m$ and $a_m^\eta = \int_{A_m} dadb$. We now choose η. First let η be so small that

$$\sqrt{2}C'(1+\frac{1}{b_1^q})\int_K dadb\, \eta \le \epsilon. \tag{18}$$

Secondly choose η, possibly smaller, such that

$$\int_K |h(z)|\, dadb\, \eta \sup_{(z_1,z_2)\in K\times K} E|\langle Y_t^n, \mathrm{Im}(f_{z_1} f_{z_2})\rangle| \le C_T\epsilon. \tag{19}$$

To see that this can be done, note that by Lemma 14

$$\sup_{(z_1,z_2)\in K\times K} E|\langle Y_t^n, \mathrm{Im}(f_{z_1} f_{z_2})\rangle| < C. \tag{20}$$

Letting $g^j = \bar{g}(f)^j = c + d\mathrm{Re}(1/x - i) + g^\eta$ we conclude

$$\begin{aligned}
E|\langle Y_t^n, f - g_j\rangle| &= E|\langle Y_t^n, g - g^\eta\rangle| \\
&= E\Big|\int h(z)\mathrm{Im}(\langle Y_t^n, f_z\rangle)\, dadb - \sum_{m=1}^{k_\eta} a_m^\eta h(z_m)\mathrm{Im}(\langle Y_t^n, f_z\rangle)\Big| \\
&\le E\Big|\int_{K^c} h(z)\mathrm{Im}(\langle Y_t^n, f_z\rangle)\, dadb \\
&\quad + \sum_{m=1}^{k_\eta} E\Big|\int_{A_m} h(z)\mathrm{Im}(\langle Y_t^n, f_z\rangle) - h(z_m)\mathrm{Im}(\langle Y_t^n, f_{z_m}\rangle)\, dadb\Big| \\
&\le A_1 + A_2 + A_3,
\end{aligned}$$

where

$$A_1 = C_t \int_{K^c} |h(z)|(1+\frac{1}{b^q})\, dadb,$$

$$A_2 = \sum_{m=1}^{k_\eta} \int_{A_m} |h(z)|E|\mathrm{Im}(\langle Y_t^n, f_z - f_{z_m}\rangle)|\, dadb$$

and

$$A_3 = \int_{A_m} |h(z) - h(z_m)| E|\mathrm{Im}(\langle Y_t^n, f_{z_m}\rangle)| \, dadb.$$

By (17) A_1 is less then $C_T\epsilon$. Using (18) we estimate A_2 by

$$\begin{aligned} A_2 &\le \sum_{m=1}^{k_\eta} \int_{A_m} |h(z)||z - z_m| E|\mathrm{Im}(\langle Y_t^n, f_z f_{z_m}\rangle)| \, dadb \\ &\le \int_K |h(z)| \, dadb \, \eta \sup_{(z_1,z_2)\in K\times K} E|\langle Y_t^n, \mathrm{Im}(f_{z_1} f_{z_2})\rangle| \le C_T\epsilon. \end{aligned}$$

Finally (19) implies the estimate

$$\begin{aligned} A_3 &\le \sum_{m=1}^{k_\eta} \int_{A_m} C'\sqrt{2}|z - z_m| C_t (1 + \frac{1}{b_m^q}) \, dadb \\ &\le C_T C' \sqrt{2}(1 + \frac{1}{b_1^q}) \int_K dadb \, \eta \le C_T\epsilon \end{aligned}$$

and hence $E|\langle Y_t^n, f - g_j\rangle| \le 3C_T/j$. This estimate does not depend on n and hence the same estimate holds for Y_t and hence $\langle Y_t, f - g_i\rangle \to 0$ in distribution.

Property (iii) follows from Corollary 1, (iv) follows from Lemma 14 and finally (v) follows from Lemma 15 and 16. □

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