æ
æ

# A Schilder type theorem for super-Brownian motion

Klaus Fleischmann, Jürgen Gärtner and Ingemar Kaj

Abstract. Let $X$ be a $d$-dimensional continuous super-Brownian motion with branching rate $\varepsilon$, which might be described symbolically by the "stochastic equation" $dX_t = \Delta^* X_t\,dt + \sqrt{2\varepsilon X_t}\,dW_t$ with $dW_t/dt$ a space-time white noise. A Schilder type theorem is established concerning large deviation probabilities of $X$ on path space as $\varepsilon \to 0$, with a representation of the rate functional via an $L^2$-functional on a generalized "Cameron-Martin space" of measure-valued paths.

## 0. Introduction

**0.1. Super-Brownian motion.** With a $d$-dimensional (*continuous*) *super-Brownian motion* with branching rate $\varrho \geq 0$, fixed diffusion constant $\kappa \geq 0$ and initial measure $\mu$ we mean the continuous random process $t \mapsto X_t$ on the time interval $I := [0,T]$, $T > 0$, with values in a set $\mathcal{M}$ of measures on $R^d$ (specified below) and with distribution $P_\mu = P_\mu^{\kappa,\varrho}$ defined as the solution of the following well-posed

**Martingale problem 0.1.1.**

(i) At initial time $t = 0$ : $\qquad P_\mu\big[X_0 = \mu\big] = 1\,.$

(ii) For each sufficiently regular map $t \mapsto f_t$ with $f_t$ in a set of smooth test functions (specified below), there is a continuous zero mean $P_\mu$-martingale $M_t = M_t(f)$, $t \in I$, such that $P_\mu$-a.s.

$$\langle X_t\,, f_t\rangle = \langle \mu, f_0\rangle + \int_0^t \langle X_s\,, \dot{f}_s + \kappa\Delta f_s\rangle\,ds + M_t\,, \qquad t \in I\,,$$

---

*AMS 1991 Subject classifications.* Primary: 60J80. Secondary: 60F10, 60G57.

*Key words and phrases.* Schilder's Theorem, super-Brownian motion, superprocess, large deviations, rate functional, Cameron-Martin space, cumulant equation, complete blow-up.

and with quadratic variation $\langle M\rangle$ given by

$$\langle M\rangle_t = 2\varrho \int_0^t \langle X_s\,, f_s^2\rangle\, ds\,, \qquad t\in I\,, \quad P_\mu\text{-a.s.} \qquad \diamondsuit$$

In the boundary case $\varrho = 0$ the solution of the martingale problem is the degenerate distribution $P_\mu^{\kappa,0} = \delta_\eta$ concentrated on the *heat flow* $\eta$. In the present framework this is the (deterministic) $\mathcal{M}$-valued continuous function $t\mapsto \eta_t\,,\ t\in I$, which solves the equation

$$\langle \eta_t\,, f_t\rangle = \langle \mu, f_0\rangle + \int_0^t \big\langle \eta_s\,, \dot f_s + \kappa\Delta f_s\big\rangle\, ds\,, \qquad t\in I\,. \tag{0.1.2}$$

Now we replace the branching rate $\varrho$ by $\varepsilon\varrho$ with $\varepsilon > 0$, and let $\varepsilon \to 0$. Then the formulas above suggest that the corresponding super-Brownian motions $X^{(\varepsilon)}$ with distribution $P_\mu^{\kappa,\varepsilon\varrho}$ converge in law towards the heat flow $\eta$ starting at $\mu$.

Our *purpose* is to find a nice description of the *rate functional* $S_\mu = S_\mu^{\kappa,\varrho}$ in a related *large deviation principle (LDP)*, which roughly says that

$$\varepsilon \log \mathcal{P}\big[X^{(\varepsilon)}\in A\big] \ \overset{\varepsilon\searrow 0}{\longrightarrow}\ -\inf_{\nu\in A} S_\mu(\nu) \tag{0.1.3}$$

for each $S_\mu$-continuity set $A$ of continuous measure-valued paths.

**0.2. Schilder's Theorem for Brownian motion.** Let us first recall *Schilder's Theorem for Brownian motion* $t\mapsto \xi_t$ on $I=[0,T]$ with values in $R^d$, starting at the origin and with generator $\kappa\Delta,\ \kappa>0$.

Let $H_0$ denote the set of all absolutely continuous functions $t\mapsto x_t\in R^d$ on $I$ starting at $x_0 = 0$ and with square integrable derivative $\dot x$ (*Cameron-Martin space*). Then the family of (scaled) Brownian motions $\xi^{(\varepsilon)} := \sqrt{\varepsilon}\,\xi$ satisfies a large deviation principle as in (0.1.3), but with rate functional

$$S_0(x) = S_0^\kappa(x) := \begin{cases} \frac{1}{4\kappa}\displaystyle\int_I |\dot x_t|^2\, dt\,, & x\in H_0 \\ \infty\,, & x\in \mathcal{C}(I,R^d)\setminus H_0\,. \end{cases} \tag{0.2.1}$$

**0.3. Schilder type theorem for super-Brownian motion.** Next we want to introduce the rate functional $S_\mu$ in the super-Brownian motion case in an informal way (precise definitions are given in § 1.3 below).

Let $H_\mu = H_\mu^\kappa$ denote the set of all paths $t\mapsto\nu_t\in\mathcal{M},\ t\in I=[0,T]$,

(i) with $\nu_0 = \mu$,

(ii) which are absolutely continuous (in time) with derivative denoted by $\dot\nu$ and with Laplacian $\Delta^*\nu$ both in a generalized sense,

(iii) such that the (generalized) Radon-Nikodym derivative $d(\dot\nu_t - \kappa\Delta^*\nu_t)/d\nu_t$ exists for almost all $t\in I$, and, moreover,

(iv) such that the following functional $\mathcal{F} = \mathcal{F}^\kappa$ of $\nu$ is finite:

$$(0.3.1)\qquad \mathcal{F}(\nu) := \int_I \big\| d(\dot{\nu}_t - \kappa\Delta^*\nu_t)/d\nu_t \big\|^2_{L^2(\nu_t)}\, dt = \int_I \Big\langle \nu_t\,, \big(d(\dot{\nu}_t - \kappa\Delta^*\nu_t)/d\nu_t\big)^2 \Big\rangle\, dt.$$

Here $L^2(m)$ denotes the usual Lebesgue space of functions square-integrable with respect to a measure $m$. In formal analogy to (0.2.1) the rate functional for the family $X^{(\varepsilon)}$ will be proportional to this functional $\mathcal{F}$. Indeed, setting (for $\kappa, \varrho > 0$)

$$(0.3.2)\qquad S_\mu(\nu) = S_\mu^{\kappa,\varrho}(\nu) := \begin{cases} \frac{1}{4\varrho}\mathcal{F}(\nu)\,, & \nu \in H_\mu\,, \\ \infty\,, & \text{otherwise}\,, \end{cases}$$

we can loosely formulate our *result* as follows:

**Theorem 0.3.3.** *(Schilder type theorem for super-Brownian motion). Let $\kappa, \varrho > 0$. The family of super-Brownian motions $X^{(\varepsilon)}$ with branching rate $\varepsilon\varrho$ and starting at $\mu \neq 0$ satisfies the following large deviation principle: For all $S_\mu$-continuity sets $A$ of continuous $\mathcal{M}$-valued paths*

$$\varepsilon \log \mathcal{P}\big[X^{(\varepsilon)} \in A\big] \xrightarrow{\varepsilon\searrow 0} -\inf_{\nu\in A} S_\mu(\nu)\,.$$

A large deviation principle for $X^{(\varepsilon)}$ but with attention restricted to a *fixed* time point $t > 0$ (instaed of looking at whole paths), was already derived in Fleischmann and Kaj (1994), hereafter referred to as [FK]. Of course, the representation (0.3.2) of the rate functional $S_\mu$ is only possible in the path-valued setting.

Our concept for the proof of Theorem 0.3.3 is as follows. First a weak LDP is established in a weak topology along the lines of [FK]. Opposed to our original preprint [9], an extension to a full LDP in a stronger topology is provided based on a nice exponential tightness result recently due to Schied (1994,1995). Following again [FK], the rate functional $S_\mu$ is identified as Legendre transform of the log-Laplace functional of $X$ via an infinite-dimensional version of Cramér's Theorem in the present situation where exponential moments are infinite as a rule. For the derivation of the final representation as written in (0.3.2) we then apply some methods from Dawson and Gärtner (1987).

So far we said that $X$ is a $d$-dimensional super-Brownian motion. But actually we distinguish between *several cases*:

(i) $X$ is defined on the whole $R^d$ where certain infinite (tempered) measures are admitted as states;

(ii) $X$ is defined only on the closure $\bar{G}$ of a bounded domain $G \subset R^d$ with either reflecting or killing boundary $\partial G$.

For simplicity, we call (i) the *non-compact* and (ii) the *compact* case.

We have to stress the fact that in the non-compact case we prove the formula (0.3.2) for the rate functional $S_\mu$ only up to a statement on a *"local" blow-up property* of some exponential moments of $X$ (or equivalently, of solutions of the cumulant equation) which we include as a hypothesis; see 1.2.4 below.

Of course this introductory presentation is a bit vague. The next section contains more concise formulations. There we present first in detail the more complicated non-compact case (§§ 1.1–1.4), whereas modifications which apply to the compact case are sketched in §§ 1.5–1.6. In Section 2 we state some results for the relevant nonlinear cumulant equation and engage in the identification of its solutions with the appropriate log-Laplace transition functionals of $X$, both topics again for the non-compact case. Here the behaviour at blow-up complicates the picture. We prove a complete blow-up statement (under $\kappa, \varrho > 0$) closely related to a corresponding property of the solutions of the cumulant equation. For this we use the local blow-up Hypothesis 1.2.4 just mentioned. The weak LDP, its mentioned strengthening, and the representation of the rate functional as a Legendre transform are given in Section 3. The subsequent section is devoted to the identification of the rate functional as indicated in (0.3.2). Only in the final Section 5 we come back to the more simple compact case sketching aspects different from the given development.

Our standard reference for the general theory of large deviations is Dembo and Zeitouni (1993). For the theory of superprocesses we refer to Dawson (1993). Concerning classical differential equation theory, see Friedman (1964).

## 1. Statement of results

**1.1. Preliminaries: Spaces (non-compact case).** Fix a dimension $d \geq 1$. Fix also a constant $a \geq 0$ and let $\varphi^a$ denote the *reference function*

$$\varphi^a(y) := (1 + |y|^2)^{-a/2}, \qquad y \in R^d, \tag{1.1.1}$$

(in practice we are going to take $a > d$).

Let $\Phi = \Phi(R^d)$ denote the set of continuous real-valued functions $\varphi$ on $R^d$ with the property that the ratio $\varphi(y)/\varphi^a(y)$ has a finite limit as $|y| \to \infty$. Equip this linear space $\Phi$ with the norm

$$\|\varphi\| := \sup_{y \in R^d} \frac{|\varphi(y)|}{\varphi^a(y)}, \qquad \varphi \in \Phi.$$

The separable Banach space $\{\Phi, \|\cdot\|\}$ serves as our basic space of test functions.

The set $\Phi_I := \mathcal{C}(I, \Phi)$ of continuous maps $t \mapsto \psi_t$ of the interval $I = [0,T]$, $T > 0$, into $\Phi$ with the supremum norm $\|\psi\|_I := \sup_{t \in I} \|\psi_t\|$ is also a Banach space. We denote by $\psi^a$ the "constant" element of $\Phi_I$ defined by $\psi^a_t := \varphi^a$, $t \in I$, (with the reference function $\varphi^a$ of (1.1.1)) and remark that $\|\psi^a\|_I = \|\varphi^a\| = 1$. Note also that $\|\cdot\|$ and $\|\cdot\|_I$ are not smaller than the corresponding supremum norms on $R^d$ and $I \times R^d$, respectively, since $0 < \varphi^a \leq 1$.

Let $\{\Phi^*, \|\cdot\|^*\}$ and $\{\Phi^*_I, \|\cdot\|^*_I\}$ refer to the dual Banach spaces of $\{\Phi, \|\cdot\|\}$ and $\{\Phi_I, \|\cdot\|_I\}$, respectively. In addition to the norm topologies in $\Phi^*$ and $\Phi^*_I$, we also consider theses spaces equipped with their weak*-topologies generated by $\Phi$ and $\Phi_I$, respectively.

We always let a measure on a topological space be defined on the relevant Borel $\sigma$-algebra, and let $\langle \mu, \varphi \rangle$ stand for the integral $\int \varphi(y)\, \mu(dy)$. The set $\mathcal{M}^a = \mathcal{M}^a(R^d)$ of all (locally finite non-negative) measures $\mu$ on $R^d$ with the property that $\langle \mu, \varphi^a \rangle < \infty$ can be considered as a subset of $\Phi^*$. In particular,

$$|\langle \mu, \varphi \rangle| \leq \|\varphi\| \langle \mu, \varphi^a \rangle < \infty , \qquad \mu \in \mathcal{M}^a, \quad \varphi \in \Phi ,$$

hence $\|\mu\|^* = \langle \mu, \varphi^a \rangle$. The induced weak*-topology in $\mathcal{M}^a$, the so-called *a-vague topology*, is just the coarsest topology such that for each $\varphi \in \Phi$ the mapping $\mu \mapsto \langle \mu, \varphi \rangle$ is continuous.

Note that $\mu \mapsto \varphi^a(y)\mu(dy)$ is a homeomorphism of $M^a$ onto the set $\mathcal{M}_f = \mathcal{M}_f(R^d)$ of all finite measures on $R^d$ equipped with the topology of weak convergence. Using the (Lévy-) Prohorov metric in $\mathcal{M}_f$, via this homeomorphism we define a metric $\rho_a$ on $\mathcal{M}^a$ (making the homeomorphism to an *isometry*). This way, $\mathcal{M}^a$ inherits the following properties: $\{\mathcal{M}^a, \rho_a\}$ is *Polish*, i.e. separable and complete, $\rho_a$ is *"subinvariant"*, that is $\rho^a(\mu, \nu) \geq \rho^a(\mu + \omega\, , \nu + \omega)$, $\mu, \nu, \omega \in \mathcal{M}^a$, and bounded by the image of the variational distance, that is $\rho_a(\mu, \nu) \leq \|\mu - \nu\|^*$, and, moreover, the open balls $B_\varepsilon(\mu)$ in $\mathcal{M}^a$ with center $\mu$ and $\rho_a$-radius $\varepsilon > 0$ are convex.

Let $\mathcal{M}_I^a := \mathcal{C}(I, \mathcal{M}^a)$ denote the set of *continuous* maps $\nu : t \mapsto \nu_t$ from $I$ to $\mathcal{M}^a$. We regard $\mathcal{M}_I^a$ as a convex subset of $\Phi_I^*$ via the pairing

$$\langle \nu, \psi \rangle_I := \int_I \langle \nu_s\, , \psi_s \rangle \, ds\, , \qquad \nu \in \mathcal{M}_I^a, \quad \psi \in \Phi_I\, . \tag{1.1.2}$$

Since

$$|\langle \nu, \psi \rangle_I| \leq \|\psi\|_I \langle \nu, \psi^a \rangle_I\, , \qquad \nu \in \mathcal{M}_I^a, \quad \psi \in \Phi_I\, ,$$

we have

$$\|\nu\|_I^* = \langle \nu, \psi^a \rangle_I\, , \qquad \nu \in \mathcal{M}_I^a\, . \tag{1.1.3}$$

The subset topology inherited from the weak*-topology in $\Phi_I^*$ is the coarsest topology on $\mathcal{M}_I^a$ such that all functions $\nu \mapsto \langle \nu, \psi \rangle_I$, $\psi \in \Phi_I$, are continuous.

There is a metric $\rho_I$ on $\mathcal{M}_I^a$ which generates this topology, is subinvariant, satisfies

$$\rho_I(\mu, \nu) \leq \|\mu - \nu\|_I^*\, , \qquad \mu, \nu \in \mathcal{M}_I^a\, , \tag{1.1.4}$$

and is such that the open balls in $\mathcal{M}_I^a$ are convex. For the construction of such a metric, regard a weakly continuous finite-measure-valued path on $I$ as a finite measure on $I \times R^d$, and proceed as explained above in the case $\mathcal{M}^a$ (Prohorov metric on $\mathcal{M}_f(I \times R^d)$, isometry, etc.).

Note that $\mathcal{M}_I^a$ is *separable* (use the set of polygons on suitable equidistant partitions of $I$ with corner points in a dense countable subset of $\mathcal{M}^a$).

Additionally, we endow $\mathcal{M}_I^a = \mathcal{C}(I, \mathcal{M}^a)$ also with the *compact-open topology.* To distinguish symbolically between the two topologies we adopt the *convention* that the notation $\mathcal{M}_I^a$ always refers to the case of the first, weaker topology, whereas $\mathcal{C}(I, \mathcal{M}^a)$

refers to the path space equipped with the compact-open topology. But note that the Borel $\sigma$-fields of both spaces coincide (recall that $\{\mathcal{M}^a, \rho_a\}$ is separable).

We finish this subsection by introducing some spaces of smooth functions. Denote by $\mathcal{C}^\infty = \mathcal{C}^\infty(R^d)$ and $\mathcal{C}_I^\infty = \mathcal{C}_I^\infty(I \times R^d)$ the sets of real-valued functions $f$ on $R^d$ or $I \times R^d$, respectively, possessing continuous derivatives of all orders and put

$$\begin{aligned} \Phi^\infty = \Phi^\infty(R^d) &:= \bigl\{f \in \mathcal{C}^\infty : f, \Delta f \in \Phi\bigr\}, \\ \Phi_I^\infty = \Phi_I^\infty(I \times R^d) &:= \bigl\{f \in \mathcal{C}_I^\infty : f, \dot{f}, \Delta f \in \Phi_I\bigr\}, \end{aligned} \tag{1.1.5}$$

where again $\dot{f} = \frac{\partial}{\partial s} f$, and the Laplacian $\Delta$ acts on the $R^d$-variable of $f$.

**1.2. Super-Brownian motion in $R^d$ and the local blow-up hypothesis.** In line with standard settings we restrict from now on the parameter $a$ entering into the reference function $\varphi^a$ of (1.1.1) by assuming $a > d$. (In the context of a more general model, it was assumed in [FK] that in addition $a \le d + 2$; but this is not needed in the present super-Brownian case.) We then introduce formally the super-Brownian motion with state space $\mathcal{M}^a$ and paths realized in $\mathcal{M}_I^a$ (by no means the only option) as follows.

**Definition 1.2.1.** *(Super-Brownian motion in $R^d$).* For given diffusion and branching parameters $\kappa, \varrho \ge 0$, respectively, and for any $\mu \in \mathcal{M}^a$, we define the distribution $P_\mu = P_\mu^{\kappa,\varrho}$ on $\mathcal{C}(I, \mathcal{M}^a)$ of a continuous *super-Brownian motion $X$ in $R^d$* with $X_0 = \mu$ as the unique solution of the Martingale problem 0.1.1 with the functions $f$ taken from $\Phi_I^\infty$ defined in (1.1.5) (and $\mathcal{M} = \mathcal{M}^a$). We may easily extend the process $X$ from $I$ to all of $R_+$, keeping the notation $P_\mu = P_\mu^{\kappa,\varrho}$ for its laws. Occasionally we choose an initial time $s \in R$ instead of 0, in which case we write $P_{s,\mu} = P_{s,\mu}^{\kappa,\varrho}$ for the distributions. Finally, if $X$ starts off with a unit mass $\mu = \delta_y$, $y \in R^d$, we write $P_{s,y}$ for simplicity (and $E_{s,y}$ in the case of expectations). $\diamondsuit$

Next we want to introduce the *log-Laplace functional* $\Lambda = \Lambda^{\kappa,\varrho}$ of the pair $(X_T, X)$ related to the joint law of the process $X$ and its value $X_T$ at the "terminal" time $T$. For $(\varphi, \psi) \in \Phi \times \Phi_I$, set

$$\Lambda[\varphi,\psi](s,y) := \log E_{s,y}\Bigl[\exp\Bigl\{\langle X_T, \varphi\rangle + \int_s^T \langle X_r, \psi_r\rangle\, dr\Bigr\}\Bigr], \quad (s,y) \in I \times R^d. \tag{1.2.2}$$

Note that $\Lambda[\varphi,\psi]$ maps $I \times R^d$ into $(-\infty, +\infty]$. Define the following set of "uniform boundedness from above":

$$\mathcal{V} = \mathcal{V}^{\kappa,\varrho} := \Bigl\{(\varphi,\psi) \in \Phi \times \Phi_I : \sup_{s\in I,\, y\in R^d} \Lambda[\varphi,\psi](s,y) < +\infty\Bigr\}. \tag{1.2.3}$$

As mentioned above our result in the present non-compact case is partly based on a hypothetical statement. (We have no doubts that this statement is true.) It concerns the blow-up behaviour of $\Lambda[\varphi,\psi]$ for $(\varphi,\psi)$ in the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Indeed, a basic tool in the study of super-Brownian motion is that $\Lambda[\varphi,\psi]$ (at least for $\varphi,\psi\le 0$) represents the solution of a nonlinear parabolic equation, often called the cumulant equation. From this connection we can and will infer the following facts (taken from [FK], see Lemma 2.1.3 and Proposition 2.2.2 below): $\mathcal{V}$ defined in (1.2.3) is an open subset of $\Phi\times\Phi_I$, and if $(\varphi,\psi)$ belongs to $\partial\mathcal{V}$ then the supremum of $\Lambda\bigl[\theta(\varphi,\psi)^+-(\varphi,\psi)^-\bigr]$ on all of $I\times R^d$ blows up as $\theta\nearrow 1$. (Here the usual conventions $x^+ := x\vee 0$ and $x^- := (-x)^+$ are applied coordinate- and point-wise.) Intuitively it is clear that this blow-up should occur in a *bounded* region (recall that $\varphi$ and $\psi$ have a power decay as $|y|\to\infty$). In particular, the following hypothesis should be true:

**Hypothesis 1.2.4.** *(Local blow-up in $\psi$).* Fix $\psi\in\Phi_I$ such that $(0,\psi)$ belongs to the boundary $\partial\mathcal{V}$ of $\mathcal{V}$ (defined in (1.2.3)). Then there is a *compact* subset $K$ of $R^d$ (depending on $\psi$) such that

$$\sup_{s\in I,\,y\in K}\Lambda\bigl[0,\theta\psi^+-\psi^-\bigr](s,y)\ \longrightarrow\ +\infty\qquad\text{as}\quad \theta\nearrow 1. \qquad\diamondsuit$$

The study of local blow-up properties for solutions of related nonlinear equations is prevalent in the literature; see, for instance, Friedman (1986), Bebernes and Bricher (1992), Velazquez (1994), and references therein. (But we could not find a statement in the generality as formulated in the hypothesis.) Recall that this hypothesis is superflous in the compact phase space case we deal with in §§ 1.5–1.6 below.

**1.3. Distribution-valued functions and absolute continuity (non-compact case).** Consider the Schwartz space $\mathcal{D}$ of test functions $\varphi$ with compact support $\operatorname{supp}\varphi$ in $R^d$ and continuous derivatives of all orders. As usual, $\mathcal{D}$ is furnished with its inductive topology via the subspaces $\mathcal{D}_K=\{\varphi\in\mathcal{D}:\operatorname{supp}\varphi\subseteq K\}$, where the sets $K\subset R^d$ are compact. Let $\mathcal{D}^*$ denote the dual space of real *distributions* (generalized functions) on $R^d$. Since $\mathcal{M}^a$ can be considered as a subset of $\mathcal{D}^*$, we extend the notation $\varphi\mapsto\langle\vartheta,\varphi\rangle$ used for $\vartheta=\mu\in\mathcal{M}^a$ to any $\vartheta\in\mathcal{D}^*$.

A map $t\mapsto\vartheta_t\in\mathcal{D}^*$ defined on $I$ is said to be *absolutely continuous* if for each compact set $K\subset R^d$ there exists a neighborhood $U_K$ of the zero function 0 in $\mathcal{D}_K$ and an absolutely continuous real-valued function $k_K$ on $I$ such that

$$\bigl|\langle\vartheta_t,\varphi\rangle-\langle\vartheta_s,\varphi\rangle\bigr|\le\bigl|k_K(t)-k_K(s)\bigr|,\qquad s,t\in I,\ \ \varphi\in U_K\,.$$

For such an absolutely continuous map $t\mapsto\vartheta_t$ the derivatives $\dot\vartheta_t\in\mathcal{D}^*$ at $t$ exist in the distribution sense for Lebesgue almost all $t\in I$. Moreover, the *integration by parts formula*

$$\text{(1.3.1)}\qquad \langle\vartheta_t,f_t\rangle-\langle\vartheta_s,f_s\rangle=\int_s^t\langle\dot\vartheta_r,f_r\rangle\,dr+\int_s^t\langle\vartheta_r,\dot f_r\rangle\,dr\,,\qquad 0\le s<t\le T\,,$$

holds for each $f\in\mathcal{C}_I^{\infty,\mathrm{comp}}$, the set of all those functions $f\in\mathcal{C}_I^\infty$ having compact support. For these definitions and facts, see Dawson and Gärtner (1987), § 4.1, in particular their Lemma 4.3.

**Definition 1.3.2.** *(Radon-Nikodym derivative).* We say that the generalized function $\vartheta \in \mathcal{D}^*$ is *absolutely continuous* with respect to the measure $\mu \in \mathcal{M}^a$ if there exists a function $g \geq 0$ which is locally $\mu$-integrable and satisfies $\langle \vartheta, \varphi \rangle = \langle \mu, g\varphi \rangle$, $\varphi \in \mathcal{D}$. In this case we write $g = d\vartheta/d\mu$ and call $g$ the *Radon-Nikodym derivative* of $\vartheta$ with respect to the measure $\mu$. $\diamondsuit$

Note that this is a natural generalization of a notion in the measure case (if besides $\mu$ also $\vartheta$ belongs to $\mathcal{M}^a$).

**1.4. Main result (non-compact case).** For $\mu \in \mathcal{M}^a$ let $\Delta^*\mu$ denote the element in $\mathcal{D}^*$ defined by $\langle \Delta^*\mu, \varphi \rangle = \langle \mu, \Delta\varphi \rangle$, $\varphi \in \mathcal{D}$. By the Definition 1.3.2, given $t$, the Radon-Nikodym derivative $d(\dot{\nu}_t - \kappa\Delta^*\nu_t)/d\nu_t$ appearing in the functional $\mathcal{F}$ of (0.3.1) entering into the representation (0.3.2) of the rate functional should be an element $h_t$ in $L^2(\mu)$, which satisfies

$$(1.4.1) \qquad \bigl\langle \dot{\nu}_t - \kappa\Delta^*\nu_t \, , \varphi \bigr\rangle = \langle \nu_t \, , \, h_t\varphi \rangle \, , \qquad \varphi \in \mathcal{D} \, .$$

To put this and hence (0.3.2) on a firm base, for each $\mu \in \mathcal{M}^a$ we introduce a subset $H_\mu = H_\mu^\kappa$ of $\mathcal{M}_I^a$ which will play the same role as the *Cameron-Martin space* $H_0$ in Schilder's Theorem for Brownian motion.

**Definition 1.4.2.** *("Cameron-Martin space" $H_\mu$).* Fix $\mu \in \mathcal{M}^a$. Let $H_\mu = H_\mu^\kappa$ denote the set of all paths $\nu \in \mathcal{M}_I^a$ having the following properties:

(i) $\nu_0 = \mu$.

(ii) The $\mathcal{D}^*$-valued map $t \mapsto \nu_t$ defined on $I = [0, T]$ is absolutely continuous.

(iii) $\dot{\nu}_t - \kappa\Delta^*\nu_t \in \mathcal{D}^*$ is absolutely continuous with respect to $\nu_t$ , for almost all $t \in I$; the related Radon-Nikodym derivatives are denoted by $h_t$ (recall the Definition 1.3.2).

(iv) $t \mapsto h_t = d(\dot{\nu}_t - \kappa\Delta^*\nu_t)/d\nu_t$ belongs to $L^2(\nu) := L^2\bigl(I \times R^d \, ; \, dr\,\nu_r(dy)\bigr)$ , that is

$$\mathcal{F}(\nu) = \mathcal{F}^\kappa(\nu) := \int_I \bigl\| d(\dot{\nu}_t - \kappa\Delta^*\nu_t)/d\nu_t \bigr\|_{L^2(\nu_t)}^2 \, dt = \int_I \langle \nu_t \, , \, h_t^2 \rangle \, dt = \langle \nu, h^2 \rangle_I < \infty \, .$$

We call $h$ the *Radon-Nikodym derivative* of $\dot{\nu} - \kappa\Delta^*\nu$ with respect to the measure-valued path $\nu$. $\diamondsuit$

Note that $\nu \in H_\mu$ implies (by definition) that (1.4.1) holds for *almost all* $t \in I$. Now we are in a position to restate Theorem 0.3.3 in the non-compact case. (For the compact case, see Theorem 1.6.1.)

**Theorem 1.4.3.** *(Schilder type Theorem for super-Brownian motion in $R^d$). Let $\kappa \geq 0$, $\varrho > 0$, $\mu \in \mathcal{M}^a$, $\mu \neq 0$. Then the laws $P_\mu^{\kappa,\varepsilon\varrho}$ on $\mathcal{C}(I, \mathcal{M}^a)$, equipped with the compact-open topology, satisfy a large deviation principle as $\varepsilon \to 0$ with a good convex rate functional $S_\mu = S_\mu^{\kappa,\varrho} : \mathcal{C}(I, \mathcal{M}^a) \mapsto [0, \infty]$. That is,* *(i)* $\liminf_{\varepsilon\to 0} \varepsilon \log P_\mu^{\kappa,\varepsilon\varrho}[X \in A] \ \ge\ -\inf_{\nu\in A} S_\mu(\nu)$, *for all open* $A \subseteq \mathcal{C}(I,\mathcal{M}^a)$,

*(ii)* $\limsup_{\varepsilon\to 0} \varepsilon \log P_\mu^{\kappa,\varepsilon\varrho}[X \in A] \ \le\ -\inf_{\nu\in A} S_\mu(\nu)$, *for all closed* $A \subseteq \mathcal{C}(I,\mathcal{M}^a)$,

*(iii)* *the level sets* $\{\nu \in \mathcal{C}(I,\mathcal{M}^a) : S_\mu(\nu) \le N\}$, $N > 0$, *are compact in* $\mathcal{C}(I,\mathcal{M}^a)$.

*If* $\kappa > 0$ *and under Hypothesis 1.2.4, the rate functional is given by*

$$S_\mu(\nu) = \begin{cases} \frac{1}{4\varrho}\mathcal{F}(\nu), & \nu \in H_\mu\,, \\ \infty\,, & \nu \in \mathcal{C}(I,\mathcal{M}^a) \setminus H_\mu\,. \end{cases} \tag{1.4.4}$$

**Remark 1.4.5.** (i) By projection, the theorem implies in particular Theorem 4.1.1 of [FK].
(ii) Our main objective is the proof of the representation formula (1.4.4), the validity of the LDP in the compact-open topology is due to Schied (1995).
(iii) It has meanwhile been verified that the representation (1.4.4) of the rate functional also holds in the case $\kappa = 0$ (branching without motion component); see Schied (1995), Theorem 5, which is a clarification and extension to paths of the fixed time point result Theorem 1.5.4 in [FK]. We prove (1.4.4) only in the case of a positive diffusion constant $\kappa$, since our method of proof is limited to that case. In fact, under $\kappa = 0$ a local blow-up does not imply a complete blow-up, not even in the compact case.
(iv) $S_\mu(\nu) = 0$ if and only if $\nu$ equals the heat flow $\eta$ on the time interval $I = [0,T]$ with degenerate law $P_\mu^{\kappa,0}$. ♢

**1.5. Preliminaries: Spaces (compact case).** The purpose of this subsection is to adapt the framework introduced in the §§ 1.1–1.4 to the compact case.

Let $G$ be a bounded domain in $R^d$ with smooth boundary $\partial G$. With this we mean a bounded connected (non-empty) open set $G \subset R^d$ where we impose that its boundary $\partial G$ belongs to the Hölder class $\mathcal{C}^{2+\gamma}$, for a fixed index $\gamma \in (0,1)$.

As the basic function space $\Phi = \Phi(\bar G)$, we take the Banach space $\mathcal{C}(\bar G)$ of all continuous functions $\varphi$ on the closure $\bar G = G \cup \partial G$ of $G$, equipped with the supremum norm $\|\cdot\|$. Formally this fits into the constructions of the non-compact case by setting $a = 0$ in the definition of the reference function $\varphi^a$ of (1.1.1) (and restricting to $\bar G$). Define $\Phi_I = \Phi_I(\bar G)$ analogously, i.e. as $\mathcal{C}(I,\Phi)$ with the supremum norm, and let $\Phi^*$ and $\Phi_I^*$ denote the dual Banach spaces, respectively.

The now relevant set $\mathcal{M}^0 = \mathcal{M}^0(\bar G)$ of all *finite measures* on $\bar G$ is again to be considered as a subset of $\Phi^*$ endowed with the induced weak*-topology. This is the topology of *weak convergence* generated by the Prohorov metric $\rho_0$. Note that $\mathcal{M}^0$ is a locally compact Polish space. Define $\mathcal{M}_I^0 = \mathcal{C}(I,\mathcal{M}^0)$ analogously, keeping our convention how to refer to the two needed topologies.

Next we prepare for a replacement of the space $\Phi^\infty$ of smooth functions. For our purpose, it is convenient to work with *Hölder continuous* functions. Recall the constant

$0 < \gamma < 1$, used for the definition of $G$. Consider the usual Hölder space $\mathcal{C}^{2+\gamma} = \mathcal{C}^{2+\gamma}(\bar{G})$ with norm $\|\cdot\|_{2+\gamma}$, defined with respect to Euclidean distance on $G$, see, for instance, Friedman (1964), § 3.2, (dropping there the time coordinate). Roughly speaking, $\mathcal{C}^{2+\gamma}$ consists of all continuous functions $\varphi$ on $\bar{G}$ with continuous first and second order partial derivatives, and moreover with $\gamma$-Hölder continuous second order derivatives.

Aimed to the two cases of boundary behavior of the desired super-Brownian motion $X$ in $\bar{G}$, we let $\frac{\partial}{\partial n}$ refer to the normal derivative at $\partial G$ and impose on $\varphi \in \mathcal{C}^{2+\gamma}$ always one of the following *boundary conditions*:

(i) $\quad \varphi = 0 \quad$ on $\partial G$ (Dirichlet boundary condition),

(ii) $\quad \frac{\partial}{\partial n}\varphi = 0 \quad$ on $\partial G$ (Neumann boundary condition).

Write $\left(\Phi^{2+\gamma}, \|\cdot\|_{2+\gamma}\right)$ for the Banach space of those functions $\varphi$ in $\mathcal{C}^{2+\gamma}$ which are such that, either, all functions satisfy condition (i), or, all functions satisfy condition (ii). Note that we use the same symbol for both separable Banach spaces.

In place of $\Phi_I^\infty$ we take $\left(\Phi_I^{2+\gamma}, \|\cdot\|_{2+\gamma}^I\right)$. This is the separable Banach space of all those functions $f$ of the usual Hölder space $\mathcal{C}_I^{2+\gamma} = \mathcal{C}^{2+\gamma}(I \times \bar{G})$ relative to the metric $\left(|t-t'| + |x-x'|^2\right)^{1/2}$ on $I \times G$ (see Friedman (1964), § 3.2), which additionally satisfy the Dirichlet boundary condition (i) at every time $t$. Alternatively, the Neumann boundary condition (ii) is imposed on all functions $f$ at all times. (Recall that in defining the Hölder space $\mathcal{C}_I^{2+\gamma}$ the $\frac{\gamma}{2}$-Hölder continuous first derivative with respect to the time variable $t \in I = [0, T]$ has to be taken into account.)

The space $(\Phi^{2+\gamma})^*$ is the topological dual of $\Phi^{2+\gamma}$, but equipped with the weak*-topology. Of course, it depends on the choice of the boundary condition (i) or (ii) in the definition of $\Phi^{2+\gamma}$.

We consider the *Laplacian* $\Delta$ as operator on $\Phi^{2+\gamma}$, that is, the domain of $\Delta$ essentially depends on the choosen boundary condition. Its dual operator $\Delta^*$, defined on $(\Phi^{2+\gamma})^*$ by

$$\langle \Delta^* \vartheta, \varphi \rangle = \langle \vartheta, \Delta\varphi \rangle, \qquad \vartheta \in (\Phi^{2+\gamma})^*, \ \varphi \in \Phi^{2+\gamma},$$

then also depends on the original choice of the boundary condition.

We regard the set $\mathcal{M}^0$ of finite measures on $\bar{G}$ also as a topological subset of $(\Phi^{2+\gamma})^*$. For $\vartheta \in (\Phi^{2+\gamma})^*$ and $\mu \in \mathcal{M}^0$, we define the *absolute continuity* of $\vartheta$ with respect to $\mu$ and the *Radon-Nikodym derivative* $d\vartheta/d\mu$ just as in Definition 1.3.2 (with $\mathcal{D}, \mathcal{D}^*$ replaced by $\Phi^{2+\gamma}, (\Phi^{2+\gamma})^*$).

Similarly as in § 1.3, a map $\vartheta : I \mapsto (\Phi^{2+\gamma})^*$ is said to be *absolutely continuous,* if there is an absolutely continuous real-valued function $k$ on $I$ such that

$$\left|\langle \vartheta_t, \varphi\rangle - \langle \vartheta_s, \varphi\rangle\right| \le \left|k(t) - k(s)\right|, \qquad s, t \in I, \ \varphi \in \Phi^{2+\gamma}, \ \|\varphi\|_{2+\gamma} \le 1.$$

An absolutely continuous map $\vartheta$ possesses a time derivative $\frac{d}{dt}\vartheta_t = \dot{\vartheta}_t \in (\Phi^{2+\gamma})^*$ (in the weak* sense), for almost all $t \in I$. (This is due to the fact that the polar of the unit ball in

$\Phi^{2+\gamma}$ is sequentially weak* compact.) As a consequence, for such maps $\vartheta$ the integration by parts formula (1.3.1) holds for each $f\in\Phi_I^{2+\gamma}$.

Now we can introduce the *"Cameron-Martin space"* $H_\mu = H_\mu^\kappa$, $\mu\in\mathcal{M}^0$, and for $\nu\in H_\mu$ the *functional* $\mathcal{F}(\nu)=\mathcal{F}^\kappa(\nu)$ in exactly the same way as in Definition 1.4.2, only replacing $\mathcal{D}^*$ by $(\Phi^{2+\gamma})^*$.

**1.6. Main result (compact case).** We introduce the *super-Brownian motion $X$ in $\bar G$* with either *killing* or *reflection* at the boundary $\partial G$ of $G$ just as in Definition 1.2.1, but by using $\Phi_I^{2+\gamma}$ as the class of test functions $f$ (depending on the choice of boundary condition in the definition of $\Phi_I^{2+\gamma}$). For the existence of a unique solution to martingale problems of such kind, see Dawson (1978), Theorem 3.1. (Note that in the case of killing the total mass process $t\mapsto X_t(\bar G)$ is no more a martingale; but the function constantly 1 on $\bar G$ does also not satisfy the killing boundary condition.)

Without any additional hypothesis, our result in the compact case now reads more carefully as follows. Recall we assume $\varrho>0$ and $\mu\in\mathcal{M}^a\setminus\{0\}$.

**Theorem 1.6.1.** *(Schilder type Theorem for super-Brownian motion in $\bar G$). The super-Brownian motion $X$ in $\bar G$ with either killing or reflection at the boundary $\partial G$ satisfies a large deviation principle as in Theorem 1.4.3 with a good convex rate functional $S_\mu$. Under $\kappa>0$, it takes the form*

$$S_\mu(\nu) := \begin{cases} \frac{1}{4\varrho}\displaystyle\int_I \big\| d\big(\dot\nu_t-\kappa\Delta^*\nu_t\big)/d\nu_t\big\|^2_{L^2(\nu_t)}\,dt, & \nu\in H_\mu\,,\\ \infty\,, & \nu\in\mathcal{C}(I,\mathcal{M}^a)\setminus H_\mu\,,\end{cases} \tag{1.6.2}$$

*where $\Delta^*$ is the dual to the Laplacian in $G$ with either killing or reflecting boundary condition.*

**Remark.** Also in the earlier non-compact case it is not really necessary to work with spaces of infinitely differentiable functions. On the other hand, countably many normed spaces are needed there, and so it is convenient to use the Schwartz distributions. $\diamondsuit$

**1.7. Two examples.** The *first example* concerns a LDP for a scaled version of the so-called *occupation time process* $\big\{Y_t := \int_0^t X_r\,dr\,:\,t\geq 0\big\}$ related to the super-Brownian motion $X$ in $R^d$. Suppose $d\geq 3$ (when $X$ has a steady state $X_\infty$ which satisfies $E[X_\infty(dx)] = dx$ with $dx$ denoting the Lebesgue measure). Apply for the moment our notation of function and path spaces in the non-compact case also to the infinite interval $I' := [0,\infty)$. Given $\psi\in\Phi_{I'}$ and $\nu\in\mathcal{M}^a_{I'}$ as well as a natural number $n$, define the rescaled function $\psi^n\in\Phi_{I'}$ by $\psi^n_s(x) := n^{-d}\psi_s(x/n)$ and the mass-time-space scaled path $\nu^n\in\mathcal{M}^a_I$ by $\langle\nu^n_s,\psi_s\rangle := \big\langle\nu_{n^2s},\psi^n_{n^2s}\big\rangle$, $s\in I$. We use the notation $\mu^n$ for analogously defined scaled measures. Set $\varepsilon := n^{-(d-2)}$. By the identity

$$\varepsilon\log P^{\kappa,\varepsilon\varrho}_\mu[X\in\cdot\,] = n^{-(d-2)}\log P^{\kappa;\varrho}_{\mu^n}[X^n\in\cdot\,]$$

(see for instance formula (4.6.4) in [FK]), the large deviation system $\{X^{(\varepsilon)}, \varepsilon\}$ with respect to $P_\mu^{\kappa,\varepsilon\varrho}$ as $\varepsilon \to 0$ along this particular $\varepsilon$-sequence is equivalent to the system $\{X^n, n^{-(d-2)}\}$ with respect to $P_{\mu^n}^{\kappa,\varrho}$ as $n \to \infty$.

For a fixed $\psi \in \Phi_{I'}$ and a Borel subset $A$ of $R$ we obtain (recall the notation (1.1.2))

$$
\begin{aligned}
P_\mu^{\kappa,\varepsilon\varrho}\big[\langle X, \psi\rangle_I \in A\big] &= P_{\mu^n}^{\kappa,\varrho}\Big[\int_0^T \langle X_{n^2 s}, \psi^n_{n^2 s}\rangle\, ds \in A\Big] \\
&= P_{\mu^n}^{\kappa,\varrho}\Big[n^{-2}\int_0^{n^2 T} \langle X_s, \psi^n_s\rangle\, ds \in A\Big].
\end{aligned}
$$

For simplicity, suppose $\mu = dx$, which is scaling invariant: $(dx)^n = dx$. Take $T = 1$ and set $n^2 =: t$. Then

$$
\begin{aligned}
t^{-(d-2)/2} \log P_{dx}^{\kappa,\varrho}\Big[t^{-1}\int_0^t \langle X_s, \psi_s^{\sqrt{t}}\rangle\, ds \in A\Big] &= \varepsilon \log P_{dx}^{\kappa,\varepsilon\varrho}\big[\langle X, \psi\rangle_I \in A\big] \\
&\longrightarrow -\inf\big\{S_\mu(\nu) : \nu \in \mathcal{M}_I^a,\ \langle \nu, \psi\rangle_I \in A\big\} \quad \text{as} \quad t \to \infty
\end{aligned}
$$

(if $A$ is a $S_\mu$-continuity set). Compare this with the large deviation principles derived in Iscoe and Lee (1993) ($d = 3, 4$) and Lee (1993) ($d \geq 5$) for the sequences

$$
a_t^{-1} \log P_{dx}^{\kappa,\varrho}\Big[t^{-1}\int_0^t \langle X_s, \varphi\rangle\, ds \in A\Big], \qquad a_t := \begin{cases} \sqrt{t}, & d = 3, \\ t/\log t, & d = 4, \\ t, & d \geq 5, \end{cases}
$$

where $\varphi \in \Phi$. Hence, opposed to our case, their results concern the unscaled ergodic limits $t^{-1} Y_t$ as $t \to \infty$.

We turn to the *second example*. It is devoted to the *total mass process* $t \mapsto |X_t|$ of the super-Brownian motion $X$ in $R^d$, or in $\bar{G}$ with reflecting boundary, where $|m| := m(R^d) = \langle m, 1\rangle$ denotes the total mass of a measure $m$. Assume that $\mu \neq 0$ is finite. Recall that the total mass process $|X|$ is a critical *Feller branching diffusion.* Since the total mass process is independent of the dimension we can formally set $d = 0$. Actually, throughout the paper we may admit this boundary case $d = 0$ with the obvious conventions as $R^0 = \{0\}$, $dy = \delta_0$, etc. In this case Hypothesis 1.2.4 is trivially true and the condition $\kappa > 0$ is irrelevant. Now $\Phi_I$ is the set of continuous functions $\psi$ on $I$ endowed with supremum norm. Moreover, $\mathcal{M}_I^a$ is the set of continuous nonnegative functions $\lambda$ on $I$ equipped with the weakest topology such that $\lambda \mapsto \int_I \psi_t\, \lambda_t\, dt$ is continuous for all $\psi \in \Phi_I$. Furthermore, the Cameron-Martin space $H_\mu$ consists of all nonnegative, absolutely continuous functions $\lambda$ on $I$ with $\lambda_0 = \mu > 0$ and such that $\frac{d}{dt}\sqrt{\lambda_t}$ belongs to $L^2(I\,; dt)$. In this special case, $\mathcal{F}$ in the rate functional $S_\mu$ of (1.4.4) takes the form

$$
\mathcal{F}(\lambda) = \int_I \big(\tfrac{d}{dt}\lambda_t\big)^2 \lambda_t^{-1} 1_{\{\lambda_t > 0\}}\, dt = 4\int_I \big(\tfrac{d}{dt}\sqrt{\lambda_t}\big)^2\, dt, \qquad \lambda \in H_\mu.
$$

By Jensen's inequality, for $\lambda \in H_\mu$ we have the estimate $S_\mu(\lambda) \geq \big(\sqrt{\lambda_T} - \sqrt{\mu}\big)^2/\varrho T$. But this lower bound is attained at the particular path in $H_\mu$ which is given by the parabola piece $t \mapsto \omega_t := \left[\sqrt{\mu} + \left(\sqrt{\lambda_T} - \sqrt{\mu}\right) t/T\right]^2$. Hence, if we are interested only in large deviations of $|X_T|$ (for $T$ fixed) then the rate functional simplifies to

$$S_\mu(\lambda_T) = \left(\sqrt{\lambda_T} - \sqrt{\mu}\right)^2/\varrho T, \qquad \lambda_T \geq 0.$$

(Of course, this also follows from the representation Theorem 1.5.4 in [FK] concerning the case $\kappa = 0$.) In particular

$$\varepsilon \log P_\mu^{\kappa,\varepsilon\varrho}\left[|X_T| \leq c\right] \overset{\varepsilon\searrow 0}{\longrightarrow} -\left(\sqrt{c} - \sqrt{\mu}\right)^2/\varrho T\,, \qquad 0 \leq c < \mu.$$

**1.8. Remark on Sanov's Theorem.** Here we discuss the question of deriving the Schilder type theorem by use of the contraction principle starting from a Sanov Theorem for super-Brownian motion in $R^d$.

Let $\mathcal{M} = \mathcal{M}(\mathcal{M}_I^a)$ denote the vector space of finite signed measures $Q$ on $\{\mathcal{M}_I^a, \rho_I\}$ (the metric $\rho_I$ was introduced before (1.1.4)) and $\mathcal{M}_1 = \mathcal{M}_1(\mathcal{M}_I^a)$ the subset of probability measures in $\mathcal{M}$. Furthermore, let $\mathcal{C}_b = \mathcal{C}_b(\mathcal{M}_I^a)$ denote the collection of bounded continuous functions $F$ on $\mathcal{M}_I^a$ equipped with the supremum norm $\|F\| = \sup\left\{|F(\nu)| : \nu \in \mathcal{M}_I^a\right\}$ and topologize $\mathcal{M}$ with the weak topology. Then $\mathcal{M}_1$ is a closed subspace of $\mathcal{M}$ and its relative topology is metrized by the Prohorov metric.

Let $X^1, X^2, \ldots$ denote a sequence of independent super-Brownian motions in $R^d$ identically distributed according to $P_\mu \in \mathcal{M}_1$. Then the associated *empirical distributions* converge weakly towards $P_\mu$, that is,

$$n^{-1}\textstyle\sum_{i=1}^n \delta_{X^i} \Longrightarrow P_\mu \qquad \text{as} \quad n \to \infty\,.$$

As in standard results as e.g. stated in Dembo and Zeitouni (1993), Theorems 6.2.10 and 4.5.14, the sequence of empirical distributions should satisfy a *full large deviation principle* with rate functional

$$\widetilde{S}_\mu(Q) = \widetilde{S}_\mu^{\kappa,\varrho}(Q) := \sup\left\{\langle Q, F\rangle - \log\langle P_\mu\,, e^F\rangle : F \in \mathcal{C}_b\right\}, \qquad Q \in \mathcal{M}_1\,.$$

Moreover, by taking averages in the empirical distributions (evaluating at the identity map $H(\nu) = \nu$ on $\mathcal{M}^a$), one expects the *law of large numbers*

$$n^{-1}\textstyle\sum_{i=1}^n X^i \Longrightarrow E_\mu[X] \qquad \text{as} \quad n \to \infty$$

for the *empirical means*. But it is clear from the Martingale problem 0.1.1 that the expression on the left side in the preceding relation has distribution $P_\mu^{\kappa,\varrho/n}$ and that the deterministic measure on the right side is the heat flow $\eta$. Formally we are then back at studying the limit of $X^{(\varepsilon)}$ along $\varepsilon = 1/n$. However, the projection map

$$\pi : \mathcal{M}_1 \ni Q \longmapsto E_Q[X] = \left\{t \mapsto \textstyle\int \nu_t\, Q(d\nu)\right\} \in \mathcal{M}_I^a$$

(projection to expectations) is *not continuous*, for the set-up chosen. This prevents the conclusion of a Schilder type LDP with $S_\mu(\nu) = \inf\{\widetilde{S}_\mu(Q) : E_Q[X] = \nu\}$.

A related situation is when one expects that the family of marginal distributions of $X$ appears as limit of the *empirical process*, i.e.

$$\left\{t \mapsto n^{-1}\textstyle\sum_{i=1}^n \delta_{X^i(t)}\right\} \Longrightarrow \left\{t \mapsto P_\mu \circ X_t^{-1}\right\} \qquad \text{as} \quad n \to \infty\,.$$

Again the appropriate projection mapping may not be continuous. For a discussion of such problems, see Dawson and Gärtner (1987) and Feng (1995).

**1.9. Further remark.** In contrast to the fixed time LDP in [FK], in this work we restrict to the super-Brownian motion rather than working with the class of continuous *super-$\alpha$-stable motions in $R^d$*, $0 < \alpha \le 2$. Originally, this was motivated by the fact that in the proof of Lemma 4.1.2 below we use that infinite differentiability of $\varphi$ and $\psi$ in the partial differential equation (2.1.4) (cumulant equation) implies that the corresponding classical solution $u_{\varphi,\psi}$ is also infinitely differentiable. However, (under the additional restriction $d < a \le d + \alpha$ on $a$) equation (2.1.4) makes sense also with $\Delta$ replaced by the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$ (see, for instance, Yosida (1978), formula (9.11.5)), if it is understood as an ordinary differential equation in a Banach space. Then the mentioned $\mathcal{C}^\infty$-property carries over, and the representation formula (1.4.4) of the rate functional (in the non-compact case and under a local blow-up hypothesis) is valid for $\alpha < 2$ when replacing $\Delta^*$ by $\Delta^*_\alpha$ and working from the beginning with the $\alpha$-setting of [FK] instead of restricting to $\alpha = 2$.

## 2. Cumulant equation and exponential moments

Recall that in the Sections 2–4 we are only concerned with the non-compact case.

**2.1. Cumulant equation.** For $\kappa > 0$ let $\mathcal{T}_t^\kappa$, $t \ge 0$, denote the Brownian evolution semigroup on $\Phi = \Phi(R^d)$ defined for $t > 0$ by

$$\text{(2.1.1)} \qquad \mathcal{T}_t^\kappa \varphi(x) = (4\pi\kappa t)^{-d/2} \int e^{-|x-y|^2/4\kappa t} \varphi(y)\, dy\,, \qquad x \in R^d\,, \quad \varphi \in \Phi\,,$$

and for $t = 0$ by $\mathcal{T}_0^\kappa \varphi = \varphi$. Extend to $\kappa = 0$ by setting $T_t^0 \varphi = \varphi$, $t \ge 0$.

For given $\kappa, \varrho \ge 0$, consider the nonlinear *integral equation*

$$\text{(2.1.2)} \quad u_s = \mathcal{T}_{T-s}^\kappa \varphi + \int_s^T \mathcal{T}_{r-s}^\kappa \psi_r\, dr + \varrho \int_s^T \mathcal{T}_{r-s}^\kappa (u_r^2)\, dr\,, \quad s \in I\,, \ (\varphi, \psi) \in \Phi \times \Phi_I\,,$$

(in backward time formulation and terminal time condition $u_T = \varphi$). It is well-known that such equations can be solved at least for $\varphi, \psi \le 0$, and that their solutions have probabilistic interpretations in terms of certain Laplace functionals of the super-Brownian motion in $R^d$. Here we restate some results on the existence of solutions of (2.1.2) and their properties, taken from [KF].

**Lemma 2.1.3.** *(Cumulant equation).*

*(i)* (uniqueness). *For each* $(\kappa, \varrho, \varphi, \psi) \in R_+^2 \times \Phi \times \Phi_I$ *there is at most one solution* $u_{\varphi,\psi} = u_{\varphi,\psi}^{\kappa,\varrho} \in \Phi_I$ *of the integral equation* (2.1.2).

*(ii)* (existence). *The set* $\mathcal{U}$ *of all those* $(\kappa, \varrho, \varphi, \psi) \in R_+^2 \times \Phi \times \Phi_I$ *such that there exists a solution* $u_{\varphi,\psi}^{\kappa,\varrho}$ *of* (2.1.2) *in* $\Phi_I$ *is open and includes* $\{(\varphi, \psi) \leq 0\}$ *and* $\{\varrho = 0\}$. *In particular,* $u_{0,0}^{\kappa,\varrho} \equiv 0$.

*(iii)* (continuity). *The solutions* $u_{\varphi,\psi}^{\kappa,\varrho}$ *are continuous in* $(\kappa, \varrho, \varphi, \psi) \in \mathcal{U}$.

*(iv)* (blow-up). *The set* $\mathcal{U}$ *is different from* $R_+^2 \times \Phi \times \Phi_I$ *and if* $(\kappa_n, \varrho_n, \varphi_n, \psi_n) \in \mathcal{U}$ *converges to a boundary point of* $\mathcal{U}$ *as* $n \to \infty$, *then*

$$\sup_{s\in I,\, y\in R^d} u_{\varphi_n,\psi_n}^{\kappa_n,\varrho_n}(s,y) \to +\infty.$$

*(v)* (regularity). *For fixed* $(\kappa, \varrho) \in R_+^2$, *if* $(\varphi, \psi)$ *in the (open) section*

$$\mathcal{U}[\kappa, \varrho] := \big\{(\varphi', \psi') \in \Phi \times \Phi_I : (\kappa, \varrho, \varphi', \psi') \in \mathcal{U}\big\}$$

*of* $\mathcal{U}$ *belongs to* $\Phi^\infty \times \Phi_I^\infty$ *(defined in (1.1.5)), then* $u_{\varphi,\psi}$ *is also an element of* $\Phi_I^\infty$ *and solves the partial differential equation*

$$-\frac{\partial}{\partial s} u_s = \kappa \Delta u_s + \varrho\, u_s^2 + \psi_s\,, \qquad u_s|_{s=T-} = \varphi\,. \tag{2.1.4}$$

**Proof:** The statements (i) to (iv) follow from the corresponding parts (i) to (iv) of Theorem 2.4.3 in [FK], (which concerns the cumulant equation for the more general super-stable processes in $R^d$). Regularity, as in (v), is known from classical differential equation theory. □

**2.2. Log-Laplace functional.** Recall the definition (1.2.2) of the log-Laplace functional $\Lambda$ and the related set $\mathcal{V}$ of uniform boundedness from above, introduced in (1.2.3). For the sake of completeness we work in this section also with the following stronger version of the local blow-up hypothesis:

**Hypothesis 2.2.1.** *(Local blow-up).* Fix $(\varphi, \psi)$ in the boundary $\partial\mathcal{V}$ of $\mathcal{V}$. Then there is a *compact* subset $K$ of $R^d$ (depending on $\varphi, \psi$) such that

$$\sup_{s\in I,\, y\in K} \Lambda\big[\theta(\varphi,\psi)^+ - (\varphi,\psi)^-\big](s,y) \;\longrightarrow\; +\infty \qquad \text{as} \quad \theta \nearrow 1. \qquad \diamondsuit$$

**Proposition 2.2.2.** *(Log-Laplace functional). Fix* $\kappa, \varrho \geq 0$.

*(i)* (identification). *The set* $\mathcal{V} = \mathcal{V}^{\kappa,\varrho}$ *coincides with* $\mathcal{U}[\kappa, \varrho]$ *defined in Lemma 2.1.3*(v), *and* $u_{\varphi,\psi} = \Lambda[\varphi, \psi]$ *on this set.*

*(ii)* ("star shape"). *If* $(\varphi,\psi)\in\partial\mathcal{V}$ *and* $\theta<1$, *then* $(\varphi_\theta\,,\psi_\theta):=\theta(\varphi,\psi)^+-(\varphi,\psi)^-$ *belongs to the open set* $\mathcal{V}$.

*(iii)* (complete blow-up.) *Let* $\kappa>0$. *Under Hypothesis 2.2.1, if* $(\varphi,\psi)$ *does not belong to the closure of* $\mathcal{V}$ *in* $\Phi\times\Phi_I$ *then* $\Lambda[\varphi,\psi](0,y)=+\infty$ *for all* $y\in R^d$, *whereas under Hypothesis 1.2.4 this statement is true in the special case* $\varphi=0$.

Clearly, such complete blow-up statements are closely related to corresponding notions expressed in terms of the cumulant equation; see e.g. Baras and Cohen (1987), Lacey and Tzanetis (1988).

**Proof:** Part (i) of the proposition is a consequence of Theorem 3.3.1 in [FK] (see also Corollary 3.3.4 there), specialized to the present situation.

As a preliminary for the proof of (ii), note that for the moment we can allude to the case $d=0$ as in the second example in § 1.5 and apply (i) to the total mass process $|X|$. First, by time-homogeneity and monotonicity, for $\theta\geq 0$ we have

$$(2.2.3)\quad E_{s,y}\Big[\exp\Big\{\theta\int_s^T|X_r|\,dr\Big\}\Big]=E_{0,0}\Big[\exp\Big\{\theta\int_0^{T-s}|X_r|\,dr\Big\}\Big]\leq E_{0,0}\big[\exp\big\{\theta\,|Y_T|\big\}\big]\,.$$

Here again $Y$ denotes the occupation time process related to $X$. But in dimension 0 the partial differential equation (2.1.4) simplifies to

$$\frac{d}{dt}u_\theta(t)=\varrho\,u_\theta^2(t)+\theta\,,\qquad t>0\,,\qquad u_\theta(0+)=0\,,$$

where $\theta\in R$ (for convenience we passed to a forward time setting). For $\varrho>0$ its solution is

$$u_\theta(t)=\begin{cases}-\sqrt{-\theta/\varrho}\;\tanh\big[t\sqrt{-\theta\varrho}\,\big], & \text{if}\quad\theta\leq 0,\quad t\geq 0,\\ \sqrt{\theta/\varrho}\;\tan\big[t\sqrt{\theta\varrho}\,\big], & \text{if}\quad\theta\geq 0,\quad 0\leq t\sqrt{\theta\varrho}<\pi/2\,.\end{cases}$$

Hence, with this $u_\theta(t)$,

$$(2.2.4)\qquad E_{0,0}\big[\exp\big\{\theta\,|Y_T|\big\}\big]=\begin{cases}\exp\big\{u_\theta(t)\big\}<\infty\,, & \text{if}\quad t^2\theta\varrho<\pi^2/4\,,\\ \infty\,, & \text{otherwise}\,.\end{cases}$$

Recall at this place that

$$(2.2.5)\qquad E_{0,0}\big[\exp\big\{\theta\,|X_t|\big\}\big]=\begin{cases}\exp\big\{\theta/(1-t\theta\varrho)\big\}<\infty\,, & \text{if}\quad t\theta\varrho<1\,,\\ \infty\,, & \text{otherwise,}\end{cases}$$

(cf. e.g. step $1^o$ of proof of Theorem 5.2.1 in [FK]).

Going back to any dimension $d$, we are now ready to prove (ii). For notational ease, as an extension of (1.1.2) and by an abuse of notation, *from now on we write*

$$(2.2.6)\quad \big\langle X,(\varphi,\psi)\big\rangle_{[s,t]}:=\langle X_t\,,\varphi\rangle+\int_s^t\langle X_r\,,\psi_r\rangle\,dr\,,\qquad(\varphi,\psi)\in\Phi\times\Phi^I,\quad 0\leq s<t\leq T\,.$$

We have to show that for $(\varphi,\psi)\in\partial\mathcal{V}$ and $0<\theta<1$,

$$\sup_{s\in I,\,y\in R^d} E_{s,y}\Big[\exp\big\langle X,(\varphi_\theta\,,\psi_\theta)\big\rangle_{[s,T]}\Big]<\infty\,. \tag{2.2.7}$$

Fix $\varepsilon\in(0,1-\theta)$ and set $p:=1/\varepsilon$ and $q:=1/(1-\varepsilon)$. Note that $p^{-1}+q^{-1}=1$ and $0<\theta q<1$. Select $(\varphi_n\,,\psi_n)\in\mathcal{V}$ such that $(\varphi_n\,,\psi_n)\to(\varphi,\psi)$ as $n\to\infty$, and put $(\varphi_{n,\theta}\,,\psi_{n,\theta}):=\theta(\varphi_n\,,\psi_n)^+-(\varphi_n\,,\psi_n)^-$. Decompose $(\varphi_\theta\,,\psi_\theta)=\big[(\varphi_\theta\,,\psi_\theta)-(\varphi_{n,\theta}\,,\psi_{n,\theta})\big]+(\varphi_{n,\theta}\,,\psi_{n,\theta})$ in (2.2.7), and apply Hölder's inequality to get

$$E_{s,y}\Big[\exp\big\langle X,(\varphi_\theta\,,\psi_\theta)\big\rangle_{[s,T]}\Big]\le A^{1/p}\,B^{1/q}\,,$$

where

$$A:=E_{s,y}\Big[\exp\big\langle X,p\,(\varphi_\theta-\varphi_{n,\theta}\,,\psi_\theta-\psi_{n,\theta})\big\rangle_{[s,T]}\Big],\quad B:=E_{s,y}\Big[\exp\big\langle X,q\,(\varphi_{n,\theta}\,,\psi_{n,\theta})\big\rangle_{[s,T]}\Big].$$

First of all, $p\,(\varphi_\theta-\varphi_{n,\theta})+p\,(\psi_\theta-\psi_{n,\theta})\le 2\varepsilon^{-1}\big(\|\varphi-\varphi_n\|+\|\psi-\psi_n\|_I\big)=:\varepsilon_n\to 0$ as $n\to\infty$. Then by (2.2.3) the term $A$ is bounded from above by $E_{0,0}\big[\exp\big\{\varepsilon_n\big(|X_T|+|Y_T|\big\}\big]$, which is finite for a sufficiently small $\varepsilon_n$ by (2.2.4) and (2.2.5) (apply the Cauchy-Schwarz inequality). On the other hand, $q\,(\varphi_{n,\theta}\,,\psi_{n,\theta})\le(\varphi_n\,,\psi_n)$, hence the quantity $B$ is finite uniformly in $s$ and $y$ for each fixed $n$. Thus (ii) is verified.

The proof of (iii) will be postponed to § 2.4 below. □

**2.3. Estimates involving exponential moments.** The purpose of this subsection is to prepare for the proof of the complete blow-up statements (iii) of Proposition 2.2.2. Recall the notation (2.2.6). First of all, "enlarging" the test function, the exponential moments "grow" *uniformly* on a sufficiently small "time-space ball":

**Lemma 2.3.1.** *Fix $(\varphi,\psi)\in\Phi\times\Phi_I$ and $\theta>1$. Then there exist positive constants $\varepsilon, C$ such that*

$$E_{s,y}\Big[\exp\big\langle X,(\varphi,\psi)\big\rangle_{[s,t]}\Big]\;\le\;C\Big(E_{s+h,y+z}\Big[\exp\big\langle X,\theta(\varphi,\psi)\big\rangle_{[s+h,t+h]}\Big]\Big)^{1/\theta}$$

*for all $y,z\in R^d$ and $s,t,s+h,t+h\in I$ with $s<t$, whenever $|h|,|z|\le\varepsilon$.*

**Proof:** Choose first $q$ such that $\theta^{-1}+q^{-1}=1$ and then $\delta>0$ for which (recall (2.2.4) and (2.2.5))

$$C:=\Big(\sup_{0\le s<t\le T}E_{s,0}\big[\exp\big\{\delta q\big(|X_t|+|Y_t|\big)\big\}\big]\Big)^{1/q}<\infty\,.$$

Moreover, by uniform continuity, select $\varepsilon>0$ such that $\varphi\le\varphi(\cdot-z)+\delta$ and $\psi\le\psi_{(\cdot-h)}(\cdot-z)+\delta$ provided that $|h|,|z|\le\varepsilon$. Then the statement follows from Hölder's inequality. □

Next we give an estimate concerning a small change at the upper end of the interval of time integration.

**Lemma 2.3.2.** *Fix $(\varphi,\psi)\in\Phi\times\Phi_I$ and $\theta>1$. Then there exist positive constants $h_0$ and $C$ such that*

$$E_{s,y}\Big[\exp\big\langle X,(\varphi,\psi)\big\rangle_{[s,t]}\Big] \;\le\; C\Big(E_{s,y}\Big[\exp\big\langle X,\theta(\varphi,\psi)\big\rangle_{[s,t+h]}\Big]\Big)^{1/\theta},$$

*for all $y\in R^d$ and $0\le s\le t, t+h\le T$ where $|h|\le h_0$.*

**Proof:** Again set $\theta^{-1}+q^{-1}=1$. By (2.2.5) we may choose $\varepsilon>0$ with

$$C^q := \sup_{r\in I} E_{0,0}\big[\exp\{\varepsilon|X_r|\}\big] < \infty\,, \tag{2.3.3}$$

and then by (2.2.4) a time point $h_0>0$ such that

$$\log\sup_{|h|\le h_0} E_{0,0}\Big[\exp\Big\{q\,\|\varphi\|\big(1+|X_{|h|}|\big)+q\,\|\psi\|_I\,|Y_{h_0}|\Big\}\Big] \le \varepsilon\,. \tag{2.3.4}$$

Let $0\le|h|<h_0$. Clearly,

$$\begin{aligned}\langle X_t,\varphi\rangle \;&\le\; \langle X_{t+h},\varphi\rangle+\|\varphi\|\big(|X_t|+|X_{t+h}|\big),\\ \langle X,\psi\rangle_{[s,t]} \;&\le\; \langle X,\psi\rangle_{[s,t+h]}+\|\psi\|_I\int_{t\wedge(t+h)}^{t\vee(t+h)}|X_r|\,dr\,.\end{aligned}$$

Hence,

$$\big\langle X,(\varphi,\psi)\big\rangle_{[s,t]}\le\big\langle X,(\varphi,\psi)\big\rangle_{[s,t+h]}+\|\varphi\|\big(|X_t|+|X_{t+h}|\big)+\|\psi\|_I\int_{t\wedge(t+h)}^{t\vee(t+h)}|X_r|\,dr\,.$$

Together with this estimate we apply Hölder's inequality. To see that the claim comes out, let $X'$ be an independent copy of $X$ and $Y'$ the related occupation time process. Condition on time $t\wedge(t+h)$ and use

$$\begin{aligned}E_{t\wedge(t+h)\,,\,X_{t\wedge(t+h)}}&\Big[\exp\Big\{q\,\|\varphi\|\big(|X'_t|+|X'_{t+h}|\big)+q\,\|\psi\|_I\int_{t\wedge(t+h)}^{t\vee(t+h)}|X'_r|\,dr\Big\}\Big]\\ &\le E_{0\,,\,X_{t\wedge(t+h)}}\big[\exp\big\{q\,\|\varphi\|\big(|X'_0|+|X'_{|h|}|\big)+q\,\|\psi\|_I\,|Y'_{h_0}|\big\}\big]\,.\end{aligned}$$

By the branching property we may continue with

$$\le\exp\int X_{t\wedge(t+h)}(dy)\,\log E_{0,0}\big[\exp\big\{q\,\|\varphi\|\big(1+|X'_{|h|}|\big)+q\,\|\psi\|_I\,|Y'_{h_0}|\big\}\big]$$

and by (2.3.4) in a second step with $\le\exp\big\{|X_{t\wedge(t+h)}|\,\varepsilon\big\}$. Then by time-homogeneity and the choice of $\varepsilon$ in (2.3.3), the proof is complete. □

An infinite exponential moment becomes *uniformly* infinite if $X$ starts earlier and the test function is simultaneously magnified by a factor $\theta>1$:

**Lemma 2.3.5.** *If $\kappa > 0$ and $E_{s,y}\big[\exp\langle X,(\varphi,\psi)\rangle_{[s,t]}\big] = \infty$ for some $\varphi \in \Phi$, $\psi \in \Phi_I$, $0 < s < t \le T$ and $y \in R^d$, then*

$$E_{0,z}\Big[\exp\big\langle X,\theta(\varphi,\psi)\big\rangle_{[0,t]}\Big] = \infty\,, \qquad z \in R^d\,, \quad \theta > 1\,. \tag{2.3.6}$$

**Proof:** Choose $\varepsilon, C > 0$ as in Lemma 2.3.1. By the Markov property the expectation expression in (2.3.6) has the lower bound

$$E_{0,z}\Big[\exp\langle X,\theta\psi\rangle_{[0,s]}\, E_{s,X_s}\big[\exp\big\langle X',\theta(\varphi,\psi)\big\rangle_{[s,t]}\big]\,;\, X_s(B_\varepsilon(y)) > 0\Big]$$

where $B_\varepsilon(y)$ denotes the open ball in $R^d$ with radius $\varepsilon$ and center $y$. But

$$E_{s,X_s}\Big[\exp\big\langle X',\theta(\varphi,\psi)\big\rangle_{[s,t]}\Big] \ge \exp\int_{|x-y|<\varepsilon} X_s(dx)\,\log E_{s,x}\Big[\exp\big\langle X'_t,\theta(\varphi,\psi)\big\rangle_{[s,t]}\Big]$$

by the branching property. Then, by Lemma 2.3.1 (with $h = 0$), the assumption implies that the logarithmic term is infinite for $x \in B_\varepsilon(y)$. Therefore the integral is infinite if $X_s(B_\varepsilon(y)) > 0$. But using the Brownian semigroup $\mathcal{T}^\kappa$ introduced in (2.1.1), we get

$$E_{0,z}\big[X_s(B_\varepsilon(y))\big] = \mathcal{T}^\kappa_s(1_{B_\varepsilon(0)})\,(y-z) > 0\,,$$

since $s, \varepsilon, \kappa > 0$ by assumption. Hence the event $\big\{X_s(B_\varepsilon(y)) > 0\big\}$ has positive $P_{0,z}$-probability, and the claim follows. $\square$

**2.4. Proof of the complete blow-up statements.** We now prove Proposition 2.2.2 (iii), restricting our attention to the more general Hypothesis 2.2.1 case. To overcome unpleasant boundary effects, we choose $I' := [0,T']$ with $T' > T$, and count each $\psi \in \Phi_I$ also as an element of the corresponding function space $\Phi_{I'}$, by setting $\psi_s \equiv \psi_T$ on $[T,T']$. In the following we will apply the results of § 2.3 with $I$ replaced by $I'$.

Let $\kappa > 0$ and suppose that $(\varphi,\psi)$ does not belong to the closure of the convex open set $\mathcal{V}$. Then there exists a $\theta_c \in (0,1)$ such that $\theta_c(\varphi,\psi)^+ - (\varphi,\psi)^- \in \partial\mathcal{V}$. By Hypothesis 2.2.1, we may choose $\theta_n \nearrow \theta_c$ and in $I \times R^d$ a point $(s,y)$ as well as a sequence $(s_n\,,y_n) \to (s,y)$ as $n \to \infty$ such that

$$\Lambda\big[\theta_n(\varphi,\psi)^+ - (\varphi,\psi)^-\big](s_n\,,y_n) \;\longrightarrow\; +\infty \qquad \text{as} \quad n \to \infty\,. \tag{2.4.1}$$

From (2.2.3) and (2.2.4) we know that $s < T$.

Choose constants $\theta_1\,,\theta_2\,,\theta_3$ satisfying $\theta_c < \theta_1 < \theta_2 < \theta_3 < 1$. From (2.4.1) and $\theta_n < \theta_c$ we conclude that

$$E_{s_n,y_n}\Big[\exp\Big\langle X,\big(\theta_c\varphi^+ - \varphi^-,\theta_c\psi^+ - \psi^-\big)\Big\rangle_{[s_n,T]}\Big] \longrightarrow \infty \qquad \text{as} \quad n \to \infty\,.$$

Apply Lemma 2.3.1 with $(\varphi,\psi)$ replaced by $\theta_c(\varphi,\psi)^+ - (\varphi,\psi)^-$, with $\theta = \theta_1/\theta_c$, with $(s_n\,,y_n)$ instead of $(s,y)$, and with $z = y - y_n$ and $h = \delta + s - s_n$. By monotonicity at a final step, there exists an $\varepsilon > 0$ such that for all $\delta \in (0,\varepsilon)$

$$E_{s+\delta,y}\Big[\exp\Big\langle X,\big(\theta_1\varphi^+ - \varphi^-,\theta_1\psi^+ - \psi^-\big)\Big\rangle_{[s+\delta,\,T+\delta+s-s_n]}\Big] \longrightarrow \infty \quad \text{as} \quad n \to \infty\,.$$

Then Lemma 2.3.2 with $h = s_n - s$ yields

$$E_{s+\delta,y}\Big[\exp\Big\langle X, \big(\theta_2\varphi^+ - \varphi^-, \theta_2\psi^+ - \psi^-\big)\Big\rangle_{[s+\delta,\, T+\delta]}\Big] = \infty$$

for all sufficiently small $\delta > 0$. (Here and in the remaining steps again some obvious monotonicities are applied.) Now we exploit Lemma 2.3.5 to obtain

$$E_{0,z}\Big[\exp\Big\langle X, \big(\theta_3\varphi^+ - \varphi^-, \theta_3\psi^+ - \psi^-\big)\Big\rangle_{[0,T+\delta]}\Big] = \infty\,, \qquad z \in R^d,$$

for sufficiently small $\delta > 0$. Finally, use Lemma 2.3.2 (with $h = -\delta$) to get

$$E_{0,z}\big[\exp\big\langle X, (\varphi, \psi)\big\rangle_I\big] = \infty\,, \qquad z \in R^d,$$

that is, $\Lambda[\varphi, \psi](0, z) \equiv +\infty$. This verifies part (iii) of Proposition 2.2.2, hence completes the proof of that proposition. □

## 3. Existence and first representation of the rate functional

Parts of the existence proof and the identification of the rate functional as the Legendre transform of a log-Laplace functional are developed for the fixed time case in Sections 4 and 5 of [FK] in such a way that with some modifications they carry over to the path-valued setting. Therefore, at this place we only sketch the corresponding arguments. In this section we take $\kappa \geq 0$ and $\varrho > 0$.

**3.1. Large deviation principle.** First we mention the following simple *scaling property* of super-Brownian motion:

(3.1.1) If $X$ has the law $P_\mu^{\kappa,\varrho}$ then $cX$, $c > 0$, is distributed according to $P_{c\mu}^{\kappa,c\varrho}$.

This implies that rather than studying the limit $\varepsilon \log P_\mu^{\kappa,\varepsilon\varrho}[X \in \cdot\,]$ as $\varepsilon \to 0$, we may *equivalently* consider $r^{-1} \log P_{r\mu}\big[r^{-1}X \in \cdot\,\big]$ as $r \to \infty$.

**Lemma 3.1.2.** *(Supermultiplicativity). Fix $\mu \in \mathcal{M}^a$ and a convex Borel subset $A$ of $\mathcal{M}_I^a$. The function*

$$f(r) := P_{r\mu}\big[r^{-1}X \in A\big]\,, \qquad r > 0\,,$$

*is supermultiplicative: $f(r+s) \geq f(r)\, f(s)$, $r, s > 0$.*

**Proof:** Exploit the assumed convexity of $A$ and the branching property; see [FK], proof of Lemma 4.2.1. □

**Lemma 3.1.3.** *If, in addition, $A$ is open and if $f(r) > 0$ for some $r > 0$, then $f$ is bounded away from 0 on some nonempty open interval.*

**Proof:** Follow the proof of Lemma 4.2.3 in [FK] by using the separability of $\mathcal{M}_I^a$, the inequality (1.1.4), subinvariance of the metric $\rho_I$, convexity of balls in $\mathcal{M}_I^a$, the supermultiplicativity Lemma 3.1.2, the identity (1.1.3) and an exponential inequality in connection with the fact that $\Lambda[0, \theta\psi^a] < \infty$ for $\theta > 0$ sufficiently small, which holds via the identification (i) in Proposition 2.2.2. □

For the fixed $\mu \in \mathcal{M}^a$ and convex open $A \subseteq \mathcal{M}_I^a$ in the preceding lemmas, by passing to the *subadditive function* $g(r) := -\log f(r),\ r > 0$, as in [FK], § 4.3, we conclude for the existence of $\lim_{r\to\infty} r^{-1} g(r)$ in $[0, +\infty]$, which we denote by $S_\mu(A)$. Applying this to open balls $A = B_\varepsilon(\nu)$ in $\mathcal{M}_I^a$, we obtain in the monotone limit as $\varepsilon \to 0$ a lower semicontinuous *convex* functional denoted by $S_\mu(\nu),\ \nu \in \mathcal{M}_I^a$, serving as *rate functional* in the desired *weak LDP* (taking into account the equivalent setting as indicated after (3.1.1)).

To get the LDP in the strong version as formulated in Theorem 1.4.3 (i)–(iii), we first apply an *exponential tightness* result due to Schied, which is carried out in § 10.1 of [17] and based on [16]: To each $N < \infty$ there is a compact set $K_N \subseteq \mathcal{C}(I, \mathcal{M}^a)$ such that

$$\limsup_{r\to\infty} r^{-1} \log P_{r\mu}\big[r^{-1}X \notin K_N\big] < -N.$$

But the embedding of $\mathcal{C}(I, \mathcal{M}^a)$ onto $\mathcal{M}_I^a$ is continuous, hence the $K_N$ are also compact in $\mathcal{M}_I^a$, and we get exponential tightness in $\mathcal{M}_I^a$. Consequently we have the (full) LDP in $\mathcal{M}_I^a$ with the good rate functional $S_\mu$ (see e.g. Dembo and Zeitouni (1993), Lemma 1.2.18). Using again the continuous embedding and the exponential tightness in $\mathcal{C}(I, \mathcal{M}^a)$, by the inverse contraction principle (e.g. [7], Theorem 4.2.4) we get the desired *LDP in $\mathcal{C}(I, \mathcal{M}^a)$ with good convex rate functional* $S_\mu$. We are left to prove the representation of $S_\mu$ as stated in (1.4.4).

**3.2. Representation of $S_\mu$ as Legendre transform.** The purpose of this subsection is to express as an intermediate result the rate functional $S_\mu$ in terms of exponential moments of $X$.

Recalling (1.2.2) and (1.1.2), for $\mu \in \mathcal{M}^a$ and $\psi \in \Phi_I$, set

$$\Lambda_\mu(\psi) := \int \Lambda[0, \psi](0, y)\, \mu(dy) = \log E_\mu\big[\exp\langle X, \psi\rangle_I\big]. \tag{3.2.1}$$

Note that the latter identity holds by the branching property of $X$, and that $\Lambda_\mu(\psi) > -\infty$ by Jensen's inequality since $E_\mu[X] = \eta \in \mathcal{M}_I^a$.

**Lemma 3.2.2.** *(Representation as Legendre transform). For $\mu \in \mathcal{M}^a$,*

$$S_\mu(\nu) = \Lambda_\mu^*(\nu) := \sup\Big\{\langle \nu, \psi\rangle_I - \Lambda_\mu(\psi) :\ \psi \in \Phi_I\Big\}, \qquad \nu \in \mathcal{M}_I^a.$$

**Proof:** By the contraction principle, from Theorem 1.4.3 (i) and (ii) we conclude (appealing again to the reformulation using the scaling (3.1.1)) that for fixed $\mu \in \mathcal{M}^a$, the sequence

$$\big\{P_{n\mu}[n^{-1}X \in \cdot\,] : n = 1, 2, \ldots\big\}$$

of laws on $\mathcal{M}_I^a$ satisfies an LDP with rate functional $S_\mu$. If in this LDP we can replace $S_\mu$ by $\Lambda_\mu^*$, the claim will be true by uniqueness of the rate functional.

Fix for the moment $\psi \in \Phi_I$. From (3.2.1),

$$\Lambda_\mu(\psi) \equiv n^{-1} \log E_{n\mu}\big[\exp\langle X, \psi\rangle_I\big].$$

Then the (weak) LDP implies, for the continuous functional $\langle\cdot, \psi\rangle$ on $\mathcal{M}_I^a$,

$$\Lambda_\mu(\psi) \ge \langle \nu, \psi\rangle_I - S_\mu(\nu), \qquad \nu \in \mathcal{M}_I^a,$$

(see e.g. [7], Lemma 4.3.4). Hence, $-S_\mu \le -\Lambda_\mu^*$ giving the desired large deviation upper bound. It remains to demonstrate that for each non-empty open $G \subseteq \mathcal{M}_I^a$

$$\liminf_{n\to\infty} n^{-1} \log P_{n\mu}\big[n^{-1}X \in G\big] \ \ge\ -\Lambda_\mu^*(\nu_0)\,, \qquad \nu_0 \in G\,. \tag{3.2.3}$$

But for $\nu_0 \in G$ we can find a finite sequence $\psi_1, ...\psi_n \in \Phi_I$ and an $\varepsilon > 0$ such that

$$U := \Big\{\nu \in \mathcal{M}_I^a \ :\ \big|\langle\nu, \psi_i\rangle - \langle\nu_0\,, \psi_i\rangle\big| < \varepsilon\,,\ i = 1, ..., m\Big\} \subseteq G.$$

On the other hand, note that by the definition of $\Lambda_\mu^*$ in Lemma 3.2.2,

$$\Lambda_\mu^*(\nu_0) \ \ge\ \sup\Big\{\big\langle\nu_0\,, \textstyle\sum_i \theta_i\psi_i\big\rangle_I - \Lambda_\mu\big[\sum_i \theta_i\psi_i\big] \ :\ \theta = [\theta_1, ..., \theta_m] \in R^m\Big\},$$

where we abbreviate the r.h.s. by $\lambda_\mu^*\big(\big[\langle\nu_0\,, \psi_1\rangle_I\,, ..., \langle\nu_0\,, \psi_m\rangle_I\big]\big)$. Therefore we get (3.2.3) if we show that already

$$\liminf_{n\to\infty} n^{-1} \log P_\mu\Big[n^{-1}\textstyle\sum_{j=1}^n X^j \in U\Big] \ \ge\ -\lambda_\mu^*\Big(\big[\langle\nu_0\,, \psi_1\rangle_I\,, ..., \langle\nu_0\,, \psi_m\rangle_I\big]\Big),$$

where $X^1, X^2, ...$ are independent copies of $X$ distributed according to $P_\mu$. (In fact, by uniqueness in the Martingale problem 0.1.1, the process $X$ with law $P_{n\mu}$ coincides in distribution with $X^1 + \ldots + X^n$.) But the latter estimate follows from the lower large deviation bound of the classical Cramér Theorem in $R^m$, since the exponential moments of $\sum_i \theta_i\langle X^1, \psi_i\rangle_I$ are finite for all $\theta$ in a neigbourhood of 0. (See e.g. [7], Corollary 6.1.6.) This finishes the proof. $\square$

**Remark.** Note that we just established an infinite dimensional version of *Cramér's Theorem* for some i.i.d. elements $X^1, X^2, ...$ in the "half-space" $\mathcal{M}_I^a$, having finite exponential moments of $\varepsilon\langle X^1, \psi^a\rangle_I$ only if $\varepsilon > 0$ is sufficiently small. (Compare with Theorem 4.5.14 of [7].) $\diamondsuit$

**3.3. Some consequences.** To get an immediate consequence of Lemma 3.2.2, apply Jensen's inequality to (3.2.1) to obtain $S_\mu(\eta) \le 0$. Hence $S_\mu(\eta) = 0$, proving one of the statements in Remark 1.4.5 (iv). That $\eta$ is the single $\nu$ where $S_\mu$ disappears will be proved in the end of § 4.3.

Next we formulate a simple *time scaling property* of super-Brownian motion $X$, which follows immediately from its definition via the Martingale problem 0.1.1.

**Lemma 3.3.1.** *(Scaling). For $\varepsilon > 0$ fixed, the time scaled process $X'$ defined by $X'_s = X_{\varepsilon s}$, $0 \le s \le T/\varepsilon$, is a super-Brownian motion on $[0, T/\varepsilon]$ with diffusion constant $\varepsilon\kappa$ and branching rate $\varepsilon\varrho$.*

A further consequence of Lemma 3.2.2 is now the following fact.

**Lemma 3.3.2.** *$S_\mu(\nu) = \infty$ for $\nu \in \mathcal{M}_I^a$ with $\nu_0 \ne \mu \in \mathcal{M}^a$.*

**Proof:** Fix $\varepsilon \in (0,1)$. Consider a continuous function $g$ defined on $R_+$ with support in $I$. Define $\psi^{(\varepsilon)}$ by $\psi_s^{(\varepsilon)}(y) := \varepsilon^{-1} g(s/\varepsilon)\varphi^a(y)$, $s \in I$, $y \in R^d$. Clearly $\psi^{(\varepsilon)}$ belongs to $\Phi_I$ (and is approximately $\delta_0$-shaped in the $s$-coordinate). For $\nu \in \mathcal{M}_I^a$,

$$\langle \nu, \psi^{(\varepsilon)} \rangle_I = \varepsilon^{-1} \int_0^T g(s/\varepsilon) \, \langle \nu_s, \varphi^a \rangle \, ds = \int_0^T g(s) \, \langle \nu_{\varepsilon s}, \varphi^a \rangle \, ds$$

converges to $\langle \nu_0, \varphi^a \rangle \int_I g(s)\, ds$ as $\varepsilon \to 0$, by continuity and bounded convergence.

By the scaling Lemma 3.3.1, $X'_s = X_{\varepsilon s}$, $s \in I$, defines a super-Brownian motion $X'$ on $I$ with parameters $\varepsilon\kappa$ and $\varepsilon\varrho$. Hence, as above,

$$E_\mu^{\kappa,\varrho}\big[\exp\langle X, \psi^{(\varepsilon)} \rangle_I\big] = E_\mu^{\kappa,\varrho}\Big[\exp\Big\{ \int_0^T g(s) \, \langle X_{\varepsilon s}, \varphi^a \rangle \, ds \Big\}\Big] = E_\mu^{\varepsilon\kappa,\varepsilon\varrho}\big[\exp\langle X', \psi^{(1)} \rangle_I\big].$$

By Lemma 2.1.3 (ii), $\{\varrho = \kappa = 0\}$ belongs to the open set $\mathcal{U}$ of solutions of the cumulant equation. Thus, for sufficiently small $\varepsilon$ the quadruple $(\varepsilon\kappa, \varepsilon\varrho, 0, \psi^{(1)})$ belongs to $\mathcal{U}$ and so by the identification statement (i) of Proposition 2.2.2,

$$\Lambda_\mu[0, \psi^{(\varepsilon)}] = \log E_\mu^{\varepsilon\kappa,\varepsilon\varrho}\big[\exp\langle X', \psi^{(1)} \rangle_I\big] = \big\langle \mu, u_{0,\psi^{(1)}}^{\varepsilon\kappa,\varepsilon\varrho}(0) \big\rangle .$$

But the continuity property (iii) of Lemma 2.1.3 implies that this expression converges to

$$\big\langle \mu, u_{0,\psi^{(1)}}^{0,0}(0) \big\rangle = \langle \mu, \varphi^a \rangle \int_I g(s)\, ds$$

as $\varepsilon \to 0$. In summary, starting with the identity in Lemma 3.2.2, we have for all $\nu \in \mathcal{M}_I^a$

$$S_\mu(\nu) \;\ge\; \langle \nu, \psi^{(\varepsilon)} \rangle_I - \Lambda_\mu[0, \psi^{(\varepsilon)}] \;\xrightarrow{\varepsilon \searrow 0}\; \big\langle \nu_0 - \mu\,, \varphi^a \big\rangle \int_I g(s)\, ds\,.$$

Now it suffices to vary $g$ appropriately in order to finish the proof. □

## 4. Identification of the rate functional $S_\mu$

We assume in this section again that $\varrho > 0$, and consider a fixed non-vanishing $\mu \in \mathcal{M}^a$. If $\nu \in \mathcal{M}_I^a$ is such that $\nu_0 \ne \mu$, then it does not belong to our Cameron-Martin space $H_\mu$ defined in 1.4.2. Hence, it satisfies (1.4.4) by Lemma 3.3.2. For the rest of this section we therefore restrict all considerations to paths $\nu \in \mathcal{M}_I^a$ with $\nu_0 = \mu \ne 0$, which we call *admissible* for convenience.

**4.1. A further representation of $S_\mu$.** Let $\Phi_{I,0}^\infty$ denote the subspace of all functions $f$ in $\Phi_I^\infty$ (introduced in (1.1.5)) which satisfy $f_T = 0$. For an admissible path $\nu$ and for $f \in \Phi_{I,0}^\infty$ define

$$(4.1.1)\quad J_{s,t}^\nu(f) := \langle \nu_t, f_t \rangle - \langle \nu_s, f_s \rangle - \int_s^t \Big\langle \nu_r, \dot f_r + \kappa \Delta f_r + \varrho f_r^2 \Big\rangle \, dr, \qquad 0 \le s < t \le T.$$

As an intermediate step we are going to prove

**Lemma 4.1.2.** *For each admissible path* $\nu \in \mathcal{M}_I^a$,

$$S_\mu(\nu) \geq \sup\left\{J_{0,T}^\nu(f) : f \in \Phi_{I,0}^\infty\right\},$$

*whereas under* $\kappa > 0$ *and Hypothesis 1.2.4 also the opposite inequality holds.*

**Proof:** Let $\nu \in \mathcal{M}_I^a$ be admissible. By Lemma 3.2.2, we start with

$$S_\mu(\nu) = \sup\Big\{\langle \nu, \psi\rangle_I - \log E_\mu\big[\exp\langle X, \psi\rangle_I\big] : \psi \in \Phi_I\Big\}.$$

For fixed $f \in \Phi_{I,0}^\infty$, set $\psi_f := -\dot{f} - \kappa\Delta f - \varrho f^2 \in \Phi_I$. In the notation of the Martingale problem 0.1.1 we then have $P_\mu$-a.s.

$$\langle X, \psi_f\rangle_I = -\big\langle X, \dot{f} + \kappa\Delta f\big\rangle_I - \varrho\langle X, f^2\rangle_I = \langle \mu, f_0\rangle + M_T - \tfrac{1}{2}\langle M\rangle_T.$$

Since $M_t$, $t \in I$, is a zero mean $P_\mu$-martingale, the process $t \mapsto L_t := \exp\left\{M_t - \frac{1}{2}\langle M\rangle_t\right\}$, $t \in I$, is a positive $P_\mu$-local martingale, hence a $P_\mu$*-supermartingale* with $L_0 = 1$. But then

$$\log E_\mu\big[\exp\langle X, \psi_f\rangle_I\big] = \langle \mu, f_0\rangle + \log E_\mu[L_T] \leq \langle \mu, f_0\rangle.$$

So for $f \in \Phi_{I,0}^\infty$

$$S_\mu(\nu) \geq \langle \nu, \psi_f\rangle_I - \langle \mu, f_0\rangle = -\langle \mu, f_0\rangle - \big\langle \nu, \dot{f} + \kappa\Delta f + \varrho f^2\big\rangle_I = J_{0,T}^\nu(f),$$

hence, we get the first of the claimed inequalities.

For an arbitrary $\psi \in \Phi_I^\infty$ such that $(0,\psi) \in \mathcal{U}[\kappa,\varrho]$ we have $u_{0,\psi} \in \Phi_{I,0}^\infty$ and

$$\dot{u}_{0,\psi} + \kappa\Delta u_{0,\psi} + \varrho u_{0,\psi}^2 = -\psi,$$

according to the regularity statement (v) of Lemma 2.1.3. Hence (recalling the notations introduced in (4.1.1) and (3.2.1)),

$$J_{0,T}^\nu(u_{0,\psi}) = \langle \nu, \psi\rangle_I - \big\langle \mu, u_{0,\psi}(0)\big\rangle = \langle \nu, \psi\rangle_I - \Lambda_\mu(\psi), \tag{4.1.3}$$

where we also used the identification part (i) of Proposition 2.2.2. Thus, for such $\psi$,

$$\sup\left\{J_{0,T}^\nu(f) : f \in \Phi_{I,0}^\infty\right\} \geq \langle \nu, \psi\rangle_I - \Lambda_\mu(\psi). \tag{4.1.4}$$

Since $\Phi_I^\infty$ is a dense subset of $\Phi_I$, by continuity (recall (4.1.3) and Lemma 2.1.3(iii)) this inequality even holds for all $(0,\psi) \in \mathcal{U}[\kappa,\varrho] = \mathcal{V}$. Moreover, by Proposition 2.2.2(ii) we may pass monotonously to any $(0,\psi) \in \partial\mathcal{V}$, and (4.1.4) is still valid. Finally, under $\kappa > 0$ and Hypothesis 1.2.4, the *complete blow-up property* (iii) in Proposition 2.2.2 implies $\Lambda_\mu(\psi) = +\infty$ if $(0,\psi) \notin \mathcal{V} \cup \partial\mathcal{V}$ (since $\mu \neq 0$). Consequently, (4.1.4) is true for any $\psi \in \Phi_I$, and we get

$$\sup_{f \in \Phi_{I,0}^\infty} J_{0,T}^\nu(f) \geq \sup_{\psi \in \Phi_I}\left\{\langle \nu, \psi\rangle_I - \Lambda_\mu(\psi)\right\} = \Lambda_\mu^*(\nu) = S_\mu(\nu),$$

finishing the proof. □

**4.2. Properties of the latter representation of $S_\mu$.** To study properties of the supremum expression appearing in Lemma 4.1.2, we use techniques similar to those in the proof of Lemma 4.8 in Dawson and Gärtner (1987). Recall Definition 1.4.2 of our Cameron-Martin space $H_\mu$.

**Lemma 4.2.1.** *If $\nu$ belongs to $H_\mu$, $\mu \neq 0$, then*

$$\sup\left\{J^\nu_{0,T}(f) : f \in \Phi^\infty_{I,0}\right\} = \tfrac{1}{4\varrho}\,\mathcal{F}(\nu)$$

*(which is finite by the definition of $H_\mu$).*

**Proof:** Set $\mathcal{C}^{\infty,\mathrm{comp}}_{I,0} := \Phi^\infty_{I,0} \cap \mathcal{C}^{\infty,\mathrm{comp}}_I$ (the latter defined after (1.3.1)). By (4.1.1) and the integration by parts formula (1.3.1),

$$J^\nu_{0,T}(f) = \int_I \left[\left\langle \dot\nu_r - \kappa\Delta^*\nu_r\,,\, f_r\right\rangle - \left\langle \nu_r\,,\, \varrho f_r^2\right\rangle\right] dr$$

for $f \in \mathcal{C}^{\infty,\mathrm{comp}}_{I,0}$. Assume $\nu$ belongs to $H_\mu$. By Definition 1.4.2, the Radon-Nikodym derivative $g$ of $\dot\nu - \kappa\Delta^*\nu$ with respect to $\nu$ exists, and we can proceed as in (1.4.1) to get

$$J^\nu_{0,T}(f) = \int_I \left\langle \nu_r\,,\, g_r f_r - \varrho f_r^2\right\rangle dr = \left\langle \nu\,,\, gf - \varrho f^2\right\rangle_I \tag{4.2.2}$$

where $\langle\nu, g^2\rangle_I < \infty$. Since each $f \in \Phi^\infty_{I,0}$ is a limit in $L^2(\nu)$ of a sequence of $\mathcal{C}^{\infty,\mathrm{comp}}_{I,0}$-functions, the identity (4.2.2) even holds for all $f$ in $\Phi^\infty_{I,0}$. Hence, rewriting (4.2.2),

$$J^\nu_{0,T}(f) = \tfrac{1}{4\varrho}\langle\nu, g^2\rangle_I - \tfrac{1}{4\varrho}\left\langle \nu\,,(g - 2\varrho f)^2\right\rangle_I, \qquad f \in \Phi^\infty_{I,0}\,.$$

Because $g \in L^2(\nu)$ can in $L^2(\nu)$ be approximated by functions in $\Phi^\infty_{I,0}$ we have

$$\inf\left\{\left\langle \nu\,,(g - 2\varrho f)^2\right\rangle_I : f \in \Phi^\infty_{I,0}\right\} = 0\,,$$

and then

$$\sup\left\{J^\nu_{0,T}(f) : f \in \Phi^\infty_{I,0}\right\} = \tfrac{1}{4\varrho}\langle\nu, g^2\rangle_I = \tfrac{1}{4\varrho}\,\mathcal{F}(\nu)\,. \qquad \square$$

**Lemma 4.2.3.** *Each admissible $\nu \in \mathcal{M}^a_I$ with the property that*

$$\sup\left\{J^\nu_{0,T}(f) : f \in \Phi^\infty_{I,0}\right\} < \infty$$

*belongs to $H_\mu$.*

**Proof:** Approximating the indicator function of the interval $[s,t] \subseteq I$ by a decreasing sequence of smooth functions $\theta_n$ from $I$ to $[0,1]$, we conclude that for any $\nu$ with properties as in the lemma, $J^\nu_{0,T}(\theta_n f) \to J^\nu_{s,t}(f)$ as $n \to \infty$. (Note in particular, that $\langle\nu_t\,, f_t\rangle - \langle\nu_s\,, f_s\rangle$ will be "created" by a term involving $\langle\nu_r\,, \dot\theta_n(r) f_r\rangle$.) Hence

$$\sup_{f\in\Phi^\infty_{I,0}} J^\nu_{s,t}(f) \le \sup_{f\in\Phi^\infty_{I,0}\,,\, n\ge 1} J^\nu_{0,T}(\theta_n f) \le \sup_{f\in\Phi^\infty_{I,0}} J^\nu_{0,T}(f) < \infty\,, \qquad 0 \le s < t \le T\,.$$

Next we introduce the functions

$$\ell^\nu_{s,t}(f) := \langle\nu_t\,, f_t\rangle - \langle\nu_s\,, f_s\rangle - \int_s^t \left\langle \nu_r\,, \dot f_r + \kappa\Delta f_r\right\rangle dr\,, \qquad \varphi \in \Phi^\infty_{I,0}\,,$$

so that

$$J^{\nu}_{s,t}(f) = \ell^{\nu}_{s,t}(f) - \varrho \int_s^t \langle \nu_r\,, f_r^2 \rangle\, dr\,.$$

Moreover,

$$c\,\ell^{\nu}_{s,t}(f) - c^2 \varrho \int_s^t \langle \nu_r\,, f_r^2 \rangle\, dr = J^{\nu}_{s,t}(cf) \le \sup_{f' \in \Phi^{\infty}_{I,0}} J^{\nu}_{s,t}(f') < \infty\,, \tag{4.2.4}$$

for all $0 \le s < t \le T$, $c \in R$, and $f \in \Phi^{\infty}_{I,0}$. If $\int_s^t \langle \nu_r\,, f_r^2 \rangle\, dr > 0$, by maximizing the l.h.s. over all $c \in R$ we obtain

$$\ell^{\nu}_{s,t}(f)^2 \;\le\; 4\varrho \sup_{f' \in \Phi^{\infty}_{I,0}} J^{\nu}_{0,T}(f') \int_s^t \big\langle \nu_r\,, f_r^2 \big\rangle\, dr\,. \tag{4.2.5}$$

On the other hand, if $\int_s^t \langle \nu_r\,, f_r^2 \rangle\, dr = 0$, then (4.2.4) can hold for all $c \in R$ only if $\ell^{\nu}_{s,t}(f) = 0$. Hence, (4.2.5) is true in both cases. Therefore, we may consider $\ell^{\nu}_{s,t}$ as a linear bounded functional on the closure $\overline{\Phi^{\infty}_{I,0}}$ of $\Phi^{\infty}_{I,0}$ in $L^2_{s,t}(\nu) := L^2\big([s,t] \times R^d\,;\, dr\,\nu_r(dy)\big)$.

By the *Riesz representation Theorem* there exists an element $h^{s,t}$ in the Hilbert space $\overline{\Phi^{\infty}_{I,0}}$ such that

$$\ell^{\nu}_{s,t}(f) = \int_s^t \big\langle \nu_r\,, h_r^{s,t} f_r \big\rangle\, dr\,, \qquad f \in \Phi^{\infty}_{I,0}\,.$$

Next we use the additivity

$$\ell^{\nu}_{s,t}(f) + \ell^{\nu}_{t,t'}(f) = \ell^{\nu}_{s,t'}(f)\,, \qquad 0 \le s < t < t' \le T, \tag{4.2.6}$$

with $s = 0$ and $t' = T$ to get

$$\ell^{\nu}_{0,t}(f) = \int_0^t \big\langle \nu_r\,, h_r^{0,T} f_r \big\rangle\, dr + \int_t^T \Big\langle \nu_r\,, (h_r^{0,T} - h_r^{t,T}) f_r \Big\rangle\, dr\,, \qquad f \in \Phi^{\infty}_{I,0}\,.$$

But here the last integral must vanish since the other two terms are independent of the "remainder" $\big\{f_r : r \in [t,T]\big\}$. Hence,

$$\ell^{\nu}_{0,t}(f) = \int_0^t \big\langle \nu_r\,, h_r^{0,T} f_r \big\rangle\, dr\,, \qquad 0 < t < T,$$

and combining with the additivity (4.2.6),

$$\ell^{\nu}_{s,t}(f) = \int_s^t \big\langle \nu_r\,, h_r^{0,T} f_r \big\rangle\, dr\,, \qquad 0 < s < t < T.$$

In other words, abbreviating $h := h^{0,T}$,

$$\langle \nu_t\,, f_t \rangle - \langle \nu_s\,, f_s \rangle = \int_s^t \big\langle \nu_r\,, \dot f_r + \kappa \Delta f_r + h_r f_r \big\rangle\, dr\,, \qquad f \in \Phi^{\infty}_{I,0}\,, \quad 0 < s < t < T.$$

In particular, for elements $f$ in $\Phi^{\infty}_{I,0}$ satisfying $f_s \equiv \varphi \in \mathcal{D}$ on $[0, T-\varepsilon]$ for $0 < \varepsilon < T$,

$$\langle \nu_t\,, \varphi \rangle - \langle \nu_s\,, \varphi \rangle = \int_s^t \big\langle \nu_r\,, \kappa \Delta \varphi + h_r \varphi \big\rangle\, dr\,, \qquad 0 < s < t \le T - \varepsilon\,. \tag{4.2.7}$$

This shows that $\nu : I \mapsto \mathcal{D}^*$ is absolutely continuous. Furthermore, divide the previous relation by $t-s$ and let $t \to s \in (0,T)$ to obtain for Lebesgue almost all $s \in I$

$$\big\langle \dot\nu_s - \kappa\Delta^*\nu_s \,, \varphi\big\rangle = \langle \nu_s \,, h_s\varphi\rangle, \qquad \varphi \in \mathcal{D}\,.$$

Therefore by the Definition 1.3.2 the $L^2(\nu)$ element $h$ we have constructed yields that $\dot\nu_s - \kappa\Delta^*\nu_s$ is absolutely continuous with respect to $\nu_s$ with Radon-Nikodym derivative $h_s = d(\dot\nu_s - \kappa\Delta^*\nu_s)/d\nu_s$ for almost every $s$. Consequently $\nu$ is an element of the Cameron-Martin space $H_\mu$. □

**4.3. Completion of proof of Theorem 1.4.3.** Suppose $\kappa > 0$ and Hypothesis 1.2.4. By combining Lemma 4.1.2 and Lemma 4.2.1 we obtain that $S_\mu(\nu) = \mathcal{F}(\nu)/4\varrho < \infty$ for all $\nu$ in $H_\mu$, $\mu \neq 0$. On the other hand, the Lemmas 4.1.2 and 4.2.3 together show that each admissible $\nu$ in $\mathcal{M}_I^a$ with $S_\mu(\nu) < \infty$ belongs to $H_\mu$. Thus we have completed the proof of the representation formula (1.4.4) hence the proof of Theorem 1.4.3. □

As announced, we finally want to show that $S_\mu(\nu) = 0$ implies $\nu = \eta$. Since in the derivation of (1.4.4) the assumption $\kappa > 0$ was used only for the second part of Lemma 4.1.2, in the case $\kappa = 0$ the r.h.s. of (1.4.4) still gives a lower bound for $S_\mu(\nu)$. Hence, for general $\kappa \geq 0$, the assumption $S_\mu(\nu) = 0$ yields $\nu \in H_\mu$ and $\mathcal{F}(\nu) = 0$. Consequently, $\nu$ solves the heat flow equation $\dot\nu - \kappa\Delta^*\nu = 0$ in an $L^2(\nu)$ sense. More precisely, under $\mathcal{F}(\nu) = 0$, the $L^2(\nu)$ element $h$ in the Definition 1.4.2 (iv) vanishes, and the identity (4.2.7) implies

$$\langle \nu_t \,, \varphi\rangle - \langle \nu_0 \,, \varphi\rangle = \int_0^t \big\langle \nu_r \,, \kappa\Delta\varphi\big\rangle\, dr\,, \qquad t \in I, \quad \varphi \in \mathcal{D}\,.$$

But this is a version of the heat flow equation (recall (0.1.2)) which has the unique solution $\nu = \eta$, finishing the proof of the statement in Remark 1.4.5 (iv).

## 5. Modification of proof in the compact case

**5.1. Cumulant equation and exponential moments.** Let $\mathcal{T}_t^\kappa$, $t \geq 0$, denote the *Feller semigroup* on $\Phi = \Phi(\bar G) = \mathcal{C}(\bar G)$, related to the multiple $\kappa\Delta$ of the Laplacian $\Delta$ in $G$ with either Dirichlet boundary condition (*killing*) or Neumann boundary condition (*reflection*) at $\partial G$. In other words, $\mathcal{T}^\kappa$ is the evolution semigroup of a Brownian motion in $\bar G$ with either killing or reflection at $\partial G$.

With this modification of $\mathcal{T}^\kappa$, the integral equation (2.1.2) makes sense, and the cumulant equation Lemma 2.1.3 remains valid. Here in the regularity property (v) we replaced $\Phi^\infty \times \Phi_I^\infty$ by $\Phi^{2+\gamma} \times \Phi_I^{2+\gamma}$, and we have to add either of the boundary conditions

(i) $\quad \varphi = 0 \quad$ on $\; I \times \partial G \quad$ (Dirichlet boundary condition),

(ii) $\quad \frac{\partial}{\partial n}\varphi = 0 \quad$ on $\; I \times \partial G \quad$ (Neumann boundary condition),

to the partial differential equation (2.1.4). (Actually, concerning $\psi$ it suffices to assume only the smoothness $\mathcal{C}_I^\gamma$ to get the solution $u_{\varphi,\psi}$ in $\Phi_I^{2+\gamma}$.)

Similarly, the log-Laplace functional Proposition 2.2.2 is true in the present compact case (clearly, without the additional local blow-up Hypothesis 1.2.4).

**5.2. Existence and identification of the rate functional.** With these modifications, the existence and representation proof of the rate functional $S_\mu$ in Section 3 turns over to the present compact case. (Note in particular, that the scaling statements (3.1.1) and Lemma 3.3.1 are valid also in the compact case.)

Finally, also all the arguments for the identification of $S_\mu(\nu)$ for $\nu \in \mathcal{M}_I^0$ with $\nu_0 = \mu \neq 0$ of Section 4 carry over to the compact case. Here we only have to replace $\Phi_I^\infty$ by $\Phi_I^{2+\gamma}$.

Altogether, Theorem 1.6.1 is verified. $\square$

REFERENCES


1. P. Baras and L. Cohen, *Complete blow-up after $T_{\max}$ for the solution of a semilinear heat equation*, J. Functional Analysis **71** (1987), 142-174.

2. J. Bebernes and S. Bricher, *Final time blowup profiles for semilinear parabolic equations via center manifold theory*, SIAM J. Math. Anal. **23** (1992), 852-869.

3. D.A. Dawson, *Stochastic evolution equations and related measure processes*, J. Multivariate Anal. **5** (1975), 1-52.

4. D.A. Dawson, *Geostochastic calculus*, Canadian J. Statistics **6** (1978), 143-168.

5. D.A. Dawson, *Measure-valued Markov processes*, Ecole d'Eté de probabilités de Saint Flour XXI - 1991 (ed. P.L. Hennequin), Lecture Notes Math. **1541** (1993), 1-260.

6. D.A. Dawson and J. Gärtner, *Large deviations from the McKean-Vlasov limit for weakly interacting diffusions*, Stochastics **20** (1987), 247-308.

7. A. Dembo and O. Zeitouni, *Large deviations techniques and applications*, (Jones and Bartlett, Boston, 1993).

8. S. Feng, *Large deviations for empirical process of mean field interaction particle system with unbounded jump*, *Ann. Probab.* (1995, to appear).

9. K. Fleischmann, J. Gärtner and I. Kaj, *A Schilder type theorem for super-Brownian motion*, (Preprint Nr. 14, Dept. Math. Uppsala Univ., 1993).

10*. K. Fleischmann and I. Kaj, *Large deviation probabilities for some rescaled superprocesses*, Ann. Inst. Henri Poincaré **30** (1994), 607-645.

11. A. Friedman, *Partial differential equations of parabolic type*, (Prentice-Hall, London, 1964).

12. A. Friedman, *Blow-up of solutions of nonlinear parabolic equations*, In: Nonlinear diffusion equations and their equilibrium states, Vol. I, Proceedings, Aug. 25 – Sep. 12, 1986. Editor: W.-M. Ni, pp. 301-318. (Springer-Verlag, New York, 1988).

13. I. Iscoe and T.-Y. Lee, *Large deviations for occupation times of measure-valued branching Brownian motions*, Stochastics, Stochastic Reports **45** (1993), 177-209.

14. A.A. Lacey and D. Tzanetis, *Complete blow-up for a semilinear diffusion equation with a sufficiently large initial condition*, IMA J. Appl. Math. **41** (1988), 207-215.

15. T.-Y. Lee, *Some limit theorems for super-Brownian motion and semilinear differential equations*, Ann. Probab. **21** (1993), 979-995.

16. A. Schied, *Criteria for exponential tightness in path spaces*, (Preprint, University of Bonn, 1994).

**17.** A. Schied, *Sample path large deviations for super-Brownian motion*, (Preprint, Humboldt University, Berlin, 1995).

**18.** J.J.L. Velazquez, *Blow up for semilinear parabolic equations*, Recent advances in PDE (eds. M.A. Herrero, E. Zuazua), RAM (Wiley, Chichester, 1994) 131-145.

**19.** K. Yosida, *Functional analysis*, 5-th edition. (Springer-Verlag, Berlin, 1978).

---

* quoted as [FK] within the paper

*Weierstrass Institute for Applied Analysis*
*and Stochastics*
*Mohrenstrasse 39*
*D-10117 Berlin, Germany*
*fleischmann@iaas-berlin.d400.de*

*Department of Mathematics*
*Technical University*
*Strasse des 17. Juni 136*
*D-10623 Berlin, Germany*
*jg@math.tu-berlin.de*

*Department of Mathematics*
*Box 480*
*S-751 06 Uppsala, Sweden*
*ingemar.kaj@math.uu.se*