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The American put is log-concave in the log-price

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Abstract

We show that the American put option price is log-concave as a function of the log-price of the underlying asset. Thus the elasticity of the price decreases with increasing stock value. We also consider related contracts of American type, and we provide an example showing that not all American option prices are log-concave in the stock log-price.

Keywords: Log-concavity; Elasticity; American options

1. Introduction

In the absence of an explicit formula for the value of an American put option, there is a lot of interest in finding quantitative and qualitative properties of this price, in particular in the most fundamental case of the underlying asset being modeled by geometric Brownian motion. These questions are often, apart from their obvious relevance to applications, mathematically interesting and challenging. The literature in this field is extensive. Early references are [16] and [18] giving the first formulation of the price of the American style options as the solution of free boundary problems. The equivalence between the stochastic

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formulation and the free boundary problem for a put option is shown in [9]. Let us also mention the article [17] that gives an overview of early results in the area and contains many references.

To be more specific, consider a market consisting of a bank account with deterministic price process

$$B(u) = e^{ru} B(0)$$

and one risky asset with price process modeled, under a risk-neutral measure, by geometric Brownian motion,

$$dS(u) = rS\,du + \sigma S\,dW, \qquad S_0 = s.$$

Here the interest rate $r > 0$ and the volatility $\sigma > 0$ are assumed to be constants and W is a standard Brownian motion. The arbitrage free price V at time u of an American put option with maturity $T > u$ is given by $V(S(u), u)$ where

$$V(s,u) = \sup_{0 \leqslant \gamma \leqslant T-u} E e^{-r\gamma}\bigl(K - S(\gamma)\bigr)^+. \tag{1}$$

Here the supremum is taken over random times γ that are stopping times with respect to the filtration generated by the Brownian motion W, compare [11]. The function $s \mapsto (K - s)^+$ used here is called the contract function. The value of the American put option is thus the maximal discounted pay-off with the pay-off given by the contract function. Recall that a hedger who replicates a claim (for example, an American put option) at each instant u should have a portfolio consisting of $V_s(S(u), u)$ stocks, where V_s denotes the partial derivative with respect to the first variable. It is thus evident that convexity properties of the price function are of great interest: the American put option price is indeed convex in $S(u)$, which thus in particular means that the number of stocks in the hedging portfolio increases with increasing asset value. In fact, this convexity does not only hold in the case of geometric Brownian motion, but for virtually any time- and level-dependent volatility as long as the contract function is convex, compare [6–8]. To prove this one might approximate the American option by so-called Bermudan options that can only be exercised at a discrete set of times. On each subinterval between the times allowed for exercise, this option can be priced as a European option. The American option price can then be obtained as the limit (as the set of possible exercise times gets denser) of European option prices which are known to be convex if the contract function is convex.

Another such qualitative property of interest related to convexity is log-concavity. Recall that a non-negative function v defined on the set of positive real numbers is said to be log-concave in the log-variable if

$$v\bigl(s_1^{\lambda} s_2^{1-\lambda}\bigr) \geqslant v(s_1)^{\lambda} v(s_2)^{1-\lambda} \tag{2}$$

for all $0 < \lambda < 1$ and $s_1, s_2 > 0$. If v is strictly positive, then (2) is equivalent to the function $x \mapsto \ln v(e^x)$ being concave. An elementary computation shows that log-concavity of the price function in the log-variable, for each fixed time u, is equivalent to the so-called *elasticity* of the option price being decreasing. The elasticity is defined to be

$$\Omega(s,u) = \frac{sV_s(s,u)}{V(s,u)}. \tag{3}$$

Perhaps not as well-known as $\Delta = V_s$ or $\Gamma = V_{ss}$, the elasticity Ω, sometimes referred to as η, is often included as one of the "Greeks" for option prices. We recall that the hedger described above should have $V_s(S(u), u)$ stocks in his hedging portfolio and thus the amount $S_u V_s(S(u), u)$ invested in the stock. Therefore the elasticity Ω represents the fraction of the hedging portfolio that should be invested in the stock. Roughly speaking, this means that if the stock price increases one percent, the option price increases $\Omega(s, u)$ percent.

In Section 2 we show that at every fixed time u the elasticity of the American put option is decreasing as a function of the stock price s, where the stock price is modeled by geometric Brownian motion. This was previously known for European options with contract functions that are log-concave in the stock log-price, see [2,3]. However, passing from the result for European options to the corresponding result for American options is not immediate as in the case of convexity, as discussed above, since the prices of Bermudan options in general are *not* log-concave. Instead we use the fact that the property of log-concavity after a change of coordinates is equivalent to $f \diamond f \leqslant 0$, where f is the option price in the new coordinates and $\diamond$ denotes a certain bilinear form defined below. We show by explicit calculations that $f \diamond f \leqslant 0$ along the "parabolic" boundary of the continuation region (compare Lemmas 2.3, 2.5, 2.6), and we also provide a "maximum principle" (compare Proposition 2.2) to conclude that $f \diamond f \leqslant 0$ also in the interior of the continuation region.

In Section 2 we also provide an example which proves the existence of contract functions, in supremum norm arbitrarily close to the contract function of the put option, for which log-concavity in the stock log-price is not preserved for the corresponding American contract even in the case when the underlying asset is modeled by geometric Brownian motion. Thus there is no generalization of the results in [2,3] to general American contracts. The log-concavity of the American put option price is therefore rather delicate. In Section 3 we extend our result on the American put option to the case of American calls on a dividend paying stock.

The results about preservation of log-concavity for European options depend heavily on the stock price being modelled by a geometric Brownian motion. In fact, geometric Brownian motion is essentially the only model for which log-concavity of the contract function always is preserved, compare Theorem 1.2 in [13]. In spite of this, it is, however, of course still conceivable that log-concavity holds for the American put option for a more general class of models than geometric Brownian motion. We leave as an interesting problem to determine precisely which models that have this property. In an other direction, keeping the underlying asset modeled by geometric Brownian motion, we might ask which American options with log-concave contract functions have prices that are log-concave in the stock log-price. From the example in Section 2, we know that not all log-concave contracts have this property. Determining which contracts that do have this property is also an interesting open problem.

Instead of defining American put options as optimal stopping problems, we view them as solutions of free boundary problems, compare [9,12]. We show that the log-concavity in the stock log-price of the contract function is preserved by adapting techniques developed in [10] and by studying the behavior of the solution near the free boundary and near the singular point of the contract function.

2. The American put option

The main result in this article is the following theorem.

Theorem 2.1. *At each fixed time $u < T$, the elasticity $\Omega(s,u)$ of an American put option is decreasing as a function of the stock value s, or, equivalently, the American put price is log-concave in* $\ln s$.

This section mainly contains the proof of Theorem 2.1. Recall that there exists an optimal stopping time γ^* in (1) defined as

$$\gamma^* := \inf\bigl\{v \geqslant 0\colon \bigl(S(v), u+v\bigr) \notin \mathcal{D}\bigr\},$$

where the continuation region $\mathcal{D}$ is defined by

$$\mathcal{D} := \bigl\{(s,u) \in (0,\infty) \times (-\infty, T]\colon\ V(s,u) > (K-s)^+\bigr\}.$$

The continuation region can also be described as

$$\mathcal{D} = \bigl\{(s,u)\colon\ s > a(u)\bigr\}$$

for some time-dependent function $a(u) > 0$. This function defines the optimal stopping boundary

$$\Psi := \bigl\{(s,u)\colon\ s = a(u)\bigr\}.$$

It is well known that the function $a(u)$ is increasing and that

$$\lim_{u\to T} a(u) = K \quad \text{and} \quad \lim_{u\to-\infty} a(u) = \frac{2rK}{2r+\sigma^2}.$$

It is also known that this function is smooth, compare [4]. Moreover, the value V and the function a together solve the free boundary problem

$$\begin{cases} V_u + \frac{\sigma^2 s^2}{2} V_{ss} + rsV_s - rV = 0 & \text{if } s > a(u),\\ V = K - s & \text{if } s = a(u),\\ V_s = -1 & \text{if } s = a(u),\\ V(s,T) = (K-s)^+, \end{cases}$$

compare for example [9,12] or [17]. The equation $V_s(a(u),u) = -1$ is often referred to as the condition of smooth fit. Instead of working with $V(s,u)$, $a(u)$, s and u we work below with the dimensionless functions $f(x,t)$ and $b(t)$ and the variables x and t defined by

$$s = Ke^x, \qquad T - u = 2t/\sigma^2, \qquad a(u) = Ke^{b(t)} \quad \text{and} \quad V(s,u) = Kf(x,t). \tag{4}$$

It follows that $f(x,t)$ and $b(t)$ solve the free boundary problem

$$\begin{cases} f_t(x,t) = \mathcal{L}f(x,t) & \text{if } x > b(t),\\ f(x,t) = 1 - e^x & \text{if } x = b(t),\\ f_x(x,t) = -e^x & \text{if } x = b(t),\\ f(x,0) = (1-e^x)^+, \end{cases} \tag{5}$$

where $\mathcal{L}f = f_{xx} + (C-1)f_x - Cf$ and $C = 2r/\sigma^2$. The function $b(t)$ representing the free boundary in these coordinates is decreasing and satisfies

$$\lim_{t\to 0} b(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} b(t) = \ln\frac{C}{C+1}. \tag{6}$$

The value function f can be expressed in terms of the free boundary b and the fundamental solution

$$\Gamma(x,t) = \frac{1}{2\sqrt{\pi t}}\exp\left\{-\frac{(x+(C-1)t)^2}{4t} - Ct\right\}$$

to the equation $f_t = \mathcal{L}f$. Indeed, at points (x,t) in the continuation region we have

$$f(x,t) = \int_{-\infty}^{0}(1-e^y)\Gamma(x-y,t)\,dy + C\int_0^t\int_{-\infty}^{b(t-\tau)}\Gamma(x-y,\tau)\,dy\,d\tau. \tag{7}$$

Here the first integral is the price of the European put option (given in our new coordinates), whereas the second integral represents the extra value of the American option.

We begin our analysis by introducing a bi-linear form appearing naturally in our calculations. Let

$$f\diamond g := (fg_{xx} - 2f_xg_x + f_{xx}g)/2.$$

Since the value of the American put option is smooth in the continuation region it is straightforward to check that the elasticity $\Omega(s,u)$ defined in (3) is decreasing in s for every fixed time u if and only if $f\diamond f = ff_{xx} - f_x^2 \leqslant 0$ at all points (x,t).

Before we prove Theorem 2.1, we need a couple of results.

Proposition 2.2. *Let $g \in C^4(I)$ for some open set $I \subseteq \mathbb{R}$. Assume that $g\diamond g = 0$ at a point $x_0 \in I$, that $g \diamond g \leqslant 0$ in a neighborhood of x_0 and that $g(x_0) \neq 0$. Then*

$$g \diamond \mathcal{L}g \leqslant 0$$

at the point x_0.

Remark. Note that if $g = g(x,t)$ satisfies $g_t = \mathcal{L}g$ and the conditions of Proposition 2.2 are satisfied at some point (x_0,t_0), then

$$\partial_t(g\diamond g) = 2g\diamond g_t = 2g\diamond\mathcal{L}g \leqslant 0$$

at this point. Thus, the inequality of Proposition 2.2 shows that geometric Brownian motion is a robust model for conservation of log-concavity. More precisely, if the log-concavity is almost lost at some point, i.e., $g\diamond g = 0$, then the time-derivative of this expression satisfies the inequality necessary for preserving log-concavity.

Proof. By assumption, the function $g\diamond g = gg_{xx} - g_x^2$ has a local maximum 0 at x_0. Therefore

$$(g\diamond g)_x = 2g\diamond g_x = gg_{xxx} - g_xg_{xx} = 0 \tag{8}$$

and

$$(g \diamond g)_{xx} = gg_{xxxx} - g_{xx}^2 \leqslant 0$$

at x_0. Now, assume that $g_{xx}(x_0) \neq 0$. Then straightforward calculations yield that at the point x_0

$$\begin{aligned}
g \diamond (\mathcal{L}g) &= g \diamond \big(g_{xx} + (C-1)g_x - Cg\big) \\
&= g \diamond g_{xx} + (C-1)g \diamond g_x - Cg \diamond g \\
&= g \diamond g_{xx} \\
&= \big(gg_{xxxx} - 2g_x g_{xxx} + g_{xx}^2\big)/2 \\
&\leqslant g_{xx}^2 - g_x g_{xxx} \\
&= \frac{g_{xx}}{g}\left(gg_{xx} - \frac{gg_x g_{xxx}}{g_{xx}}\right) \\
&= \frac{g_{xx}}{g}\big(gg_{xx} - g_x^2\big) = 0,
\end{aligned}$$

where we have used that $g(x_0) \neq 0$. Finally, if $g_{xx}(x_0) = 0$, then it follows from (8) that $g_{xxx}(x_0) = 0$. Using this, we find that

$$g \diamond (\mathcal{L}g) \leqslant g_{xx}^2 - g_x g_{xxx} = 0,$$

which finishes the proof. $\square$

Let $\mathcal{C} := \{(x,t)\colon\ x > b(t),\ t > 0\}$ be the continuation region in the (x,t)-coordinates. In the proof of Theorem 2.1 we need to check that $f \diamond f \leqslant 0$ at the boundary

$$\partial\mathcal{C} = \big\{(x,0)\colon\ x > 0\big\} \cup \big\{(0,0)\big\} \cup \big\{(x,t)\colon\ x = b(t),\ t > 0\big\}$$

of the continuation region (strictly speaking $f \diamond f$ is not defined at the origin, so we instead look at $\limsup_{(x,t)\to(0,0)} f \diamond f$, compare Lemma 2.5 below). Dealing with the part of $\partial\mathcal{C}$ which consists of the optimal stopping boundary is easy.

Lemma 2.3. *If f is the price of the American put option, then*

$$f \diamond f = C - (1+C)e^{b(t)} < 0$$

at points $(b(t),t)$, $t > 0$, of the optimal stopping boundary.

Proof. Differentiating the equality $f(b(t),t) = K - e^{b(t)}$ with respect to t and using the smooth fit condition one finds that $f_t(b(t),t) = 0$. Thus, from the equation $f_t = \mathcal{L}f$ it is seen that $f_{xx}(b(t)+,t) = C - e^{b(t)}$, so

$$f \diamond f = ff_{xx} - f_x^2 = \big(1 - e^{b(t)}\big)\big(C - e^{b(t)}\big) - \big(e^{b(t)}\big)^2 = C - (1+C)e^{b(t)}$$

at the optimal stopping boundary. Recall that the optimal stopping boundary is bounded from below by the stopping boundary of the perpetual option, i.e., for each fixed $t > 0$ we have $e^{b(t)} > \frac{C}{C+1}$, compare (6). Plugging this in above we find that $f \diamond f < 0$ at the optimal stopping boundary. $\square$

To study the log-concavity at the singular point in the origin is a little bit harder. What we need is Lemma 2.5, but first we consider the wedge

$$\mathcal{C}_1 := \big\{(x,t)\colon\ b(t) \leqslant x \leqslant (1-C)t,\ t>0\big\}$$

between the optimal stopping boundary and the line $x=(1-C)t$ for small times t. We also let $\mathcal{C}_2 := \mathcal{C} \setminus \mathcal{C}_1$.

Lemma 2.4. *In the region $\mathcal{C}_1$ we have $f_{xt} \geqslant 0$.*

Proof. We first claim that for each time $t>0$ there exists a value $x=\gamma(t)$ such that $f_{xt} \geqslant 0$ at points (x,t) with $b(t) \leqslant x \leqslant \gamma(t)$ and $f_{xt} \leqslant 0$ at points with $x \geqslant \gamma(t)$.

This statement follows from the approximation results of [5]. Indeed, in that paper a sequence of functions p^δ, $\delta>0$, is constructed so that $p^\delta \to f$ as $\delta \to 0$. Moreover, for each δ there is a continuous curve $x=\gamma^\delta(t)$ so that $p^\delta_{xt}<0$ (>0) if $x<\gamma^\delta(t)$ $(>\gamma^\delta(t))$. Since $p^\delta \to f$, it follows from standard interior estimates that $p^\delta_{xt} \to f_{xt}$ pointwise (use, for example, Theorem 4.9 in [15] and the fact that $(f_x)_t = \mathcal{L}(f_x)$). Since $f_t(b(t),t)=0$, it follows from the boundary version of the strong maximum principle that $f_{xt}>0$ at the free boundary. Moreover, since $f_t(x,t)$ tends to 0 as x grows to infinity for every fixed t, compare Lemma 2.6 below, there has to be points with $f_{xt}>0$ for every fixed t. Thus the regions where p^δ_{xt} are strictly positive and negative do not collapse as δ tends to 0. This proves the existence of $\gamma(t)$.

Now, recall that the function f_{xt} can be expressed in terms of the fundamental solution Γ as

$$f_{xt}(x,t) = \Gamma_x(x,t) + C\int_0^t \Gamma_x\big(x-b(t-\tau),\tau\big)\dot{b}(t-\tau)\,d\tau,$$

compare Lemma 3.1 in [4]. Inserting $x=(1-C)t$ the first term is 0 and using the known asymptotics $b(t) \sim -\sqrt{-2t\ln t}$ for small times t, compare [1,4] or [14], it is easily checked that the second term is strictly positive. This finishes the proof of the lemma. $\square$

Lemma 2.5. *The American option price satisfies*

$$\limsup_{(x,t)\to(0,0)} f \diamond f \leqslant 0.$$

Proof. First note that since the contract function of the put option is log-concave in the stock log-price, we only need to consider the above limit superior for points in the continuation region. We now claim that it suffices to check that

$$\limsup_{\mathcal{C}_2 \ni (x,t)\to(0,0)} f \diamond f \leqslant 0. \tag{9}$$

To see this, note that

$$\begin{aligned}(f \diamond f)_x &= f f_{xxx} - f_x f_{xx} \\ &= f f_{xt} - f_t f_x - (C-1) f \diamond f \\ &\geqslant f f_{xt} - (C-1) f \diamond f,\end{aligned}$$

where we have used the equations $f_t = \mathcal{L}f$, $(f_x)_t = \mathcal{L}(f_x)$, $f_t \geqslant 0$ and $f_x \leqslant 0$. Now assume that there exist a sequence of points $(x_n, t_n) \in \mathcal{C}_1$ converging to the origin such that $(f \diamond f)(x_n, t_n) > \varepsilon$ for some $\varepsilon > 0$. Since $(f \diamond f)_x \geqslant -(C-1) f \diamond f$ in $\mathcal{C}_1$, we see that

$$(f \diamond f)\big((1-C)t_n, t_n\big) \geqslant \varepsilon e^{-(C-1)((1-C)t_n - x_n)}$$

for all n. As the sequence of points converges to the origin, the distance $(1-C)t_n - x_n$ shrinks to 0, so it follows that the limit superior in (9) is at least ε. Thus, if the limit of $f \diamond f$ is positive along a sequence of points in the wedge $\mathcal{C}_1$, then it is also positive along the line $x = (1-C)t$, so it suffices to show (9), i.e., to check the limit superior for sequences of points in $\mathcal{C}_2$.

To do this we decompose the American put price f as $f = f^E + p$ where

$$f^E(x,t) = \int_{-\infty}^{0} (1 - e^y)\Gamma(x-y,t)\,dy$$

and

$$p(x,t) = C \int_0^t \int_{-\infty}^{b(t-\tau)} \Gamma(x-y,\tau)\,dy\,d\tau,$$

compare Eq. (7). Then

$$\begin{aligned} f \diamond f &= f^E \diamond f^E + 2 f^E \diamond p + p \diamond p \\ &= f^E \diamond f^E + f^E p_{xx} - 2 f_x^E p_x + f_{xx}^E p + p p_{xx} - p_x^2. \end{aligned}$$

Recall that $f^E \diamond f^E \leqslant 0$, see [2,3]. Moreover, $f_x^E \leqslant 0$ and

$$p_x(x,t) = -C \int_0^t \Gamma\big(x - b(t-\tau), \tau\big)\,d\tau \leqslant 0,$$

so

$$f \diamond f \leqslant f^E p_{xx} + f_{xx}^E p + p p_{xx}. \tag{10}$$

It is straightforward to check that

$$p(x,t) \leqslant Ct \tag{11}$$

and

$$\begin{aligned} \big|f_{xx}^E(x,t)\big| &= \left|\Gamma(x,t) - \int_{-\infty}^{0} e^y \Gamma(x-y,t)\,dy\right| \\ &\leqslant \big|\Gamma(x,t)\big| + \left|\int_{-\infty}^{0} e^y \Gamma(x-y,t)\,dy\right| \\ &\leqslant \frac{1}{2\sqrt{\pi t}} + 1 = \mathcal{O}(1/\sqrt{t}\,) \end{aligned}$$

for small t, so the second term on the right-hand side of (10) certainly approaches 0 for t small. Moreover, for $x \geqslant (1-C)t$ we have

$$\begin{aligned} f^E(x,t) &= \int_{-\infty}^{0} (1-e^y)\Gamma(x-y,t)\,dy \\ &\leqslant \int_{-\infty}^{0} (1-e^y)\Gamma\big((1-C)t-y,t\big)\,dy \\ &\leqslant \int_{-\infty}^{0} (1-e^y)\frac{1}{2\sqrt{\pi t}}\exp\left\{-\frac{y^2}{4t}\right\}dy \\ &\leqslant \frac{1}{2\sqrt{\pi t}}\int_{0}^{\infty} y\exp\left\{-\frac{y^2}{4t}\right\}dy = \sqrt{t/\pi}. \end{aligned}$$

Thus, considering (11), it suffices to show that $p_{xx} = o(1/\sqrt{t}\,)$ for $(x,t) \in \mathcal{C}_2$ close to the origin. Now let

$$\begin{aligned} p_{xx}(x,t) &= -C\int_0^t \Gamma_x\big(x-b(t-\tau),\tau\big)\,d\tau \\ &= -C\int_0^{t/2} \Gamma_x\big(x-b(t-\tau),\tau\big)\,d\tau - C\int_{t/2}^{t} \Gamma_x\big(x-b(t-\tau),\tau\big)\,d\tau \\ &=: I_1 + I_2. \end{aligned}$$

We deal with I_1 and I_2 separately. First, note that if $\tau \leqslant t$ and t is small, then the asymptotic behavior of b implies

$$x-(1-C)\tau-b(t-\tau) \geqslant (1-C)(t-\tau)-b(t-\tau) \geqslant 0$$

for points in $\mathcal{C}_2$. Thus

$$\begin{aligned} \frac{4\sqrt{\pi}}{C}I_1 &= \int_0^{t/2}\bigg(\frac{x-(1-C)\tau-b(t-\tau)}{\tau^{3/2}} \\ &\qquad \times\exp\left\{-\frac{(x-(1-C)\tau-b(t-\tau))^2}{4\tau}-C\tau\right\}\bigg)\,d\tau \\ &\leqslant \int_0^{t/2}\frac{x+|1-C|t-b(t)}{\tau^{3/2}}\exp\left\{-\frac{(x-|1-C|t-b(t/2))^2}{4\tau}\right\}d\tau \\ &\leqslant 2\sqrt{\pi}\,\frac{x+|1-C|t-b(t)}{x-|1-C|t-b(t/2)} \end{aligned}$$

$$= 2\sqrt{\pi}\left(1 + \frac{2|1-C|t + b(t/2) - b(t)}{x - |1-C|t - b(t/2)}\right),$$

where we in the last inequality have used that

$$\int_0^t \frac{k}{\tau^{3/2}} \exp\left\{\frac{-k^2}{4\tau}\right\} d\tau = 2\int_{k/\sqrt{t}}^{\infty} e^{-y^2/4}\, dy \leqslant 2\sqrt{\pi}$$

for some constant K_1 (use the coordinate change $y = k/\sqrt{\tau}$). Thus it is straightforward, using the known asymptotics $-b(t) \sim \sqrt{-2t\ln t}$ for the optimal stopping boundary, to show that I_1 is uniformly bounded at points $(x,t) \in \mathcal{C}_2$ close to the origin. Next,

$$\begin{aligned}
\frac{4\sqrt{\pi}}{C} I_2 &= \int_{t/2}^{t} \bigg(\frac{x - (1-C)\tau - b(t-\tau)}{\tau^{3/2}} \\
&\qquad \times \exp\left\{-\frac{(x-(1-C)\tau - b(t-\tau))^2}{4\tau} - C\tau\right\}\bigg)\, d\tau \\
&\leqslant \int_{t/2}^{t} \frac{x + |1-C|t - b(t/2)}{\tau^{3/2}}\, d\tau \\
&= \big(x + |1-C|t - b(t/2)\big) 2(\sqrt{2}-1)\frac{1}{\sqrt{t}}
\end{aligned}$$

so $I_2 = o(1/\sqrt{t}\,)$ for small t. Consequently, $p_{xx} = I_1 + I_2 = o(1/\sqrt{t}\,)$ for small t, which finishes the proof. $\square$

Next we deal with log-concavity of f at infinity. Using the formulas for f, f_x and f_{xx} in terms of b and Γ, compare Lemma 3.1 in [4], the next lemma is easily proved. We omit the details.

Lemma 2.6. *As x_0 tends to infinity, the suprema of f, f_x and f_{xx} in the region $\{(x,t)\colon x \geqslant x_0\}$ all tend exponentially to zero.*

It is clear that $f \diamond f = 0$ for all points in $\{(x,0)\colon x > 0\}$. From Lemmas 2.3 and 2.5 it thus follows that $f \diamond f \leqslant 0$ at the "parabolic boundary." Therefore, if the function $(f \diamond f)(x,t)$ satisfied some appropriate parabolic equation, then Theorem 2.1 would follow from the maximum principle. However, it is not clear to us how to find such an equation.

Instead we introduce for $\varepsilon > 0$ the function

$$f^{\varepsilon}(x,t) := f(x,t) - \varepsilon(x+M)$$

for some constant M large enough so that $Cx + CM + 1 - C \geqslant 0$ for all points (x,t) in the continuation region. We have

$$\begin{aligned}
f^{\varepsilon} \diamond f^{\varepsilon} &= f \diamond f - 2\varepsilon(x+M) \diamond f + \varepsilon^2 (x+M) \diamond (x+M) \\
&= f \diamond f - \varepsilon(x+M) f_{xx} + 2\varepsilon f_x - \varepsilon^2. \qquad (12)
\end{aligned}$$

We concentrate our study of f^ε to the region $\mathcal{U}$ defined as

$$\mathcal{U} := \{(x,t)\colon\ f_{xx}(x,t) > 0\} \cap \mathcal{C}.$$

Note that if $C \geqslant 1$, then it follows from $f_t \geqslant 0$ and $f_x \leqslant 0$ that

$$f_{xx} = f_t - (C-1)f_x + Cf \geqslant 0$$

so the set $\mathcal{U} = \mathcal{C}$. On the other hand, if $C < 1$, then $\mathcal{C} \setminus \mathcal{U}$ is non-empty. This can be seen from the equality $f_{xx} = C - e^x$ which holds for points (x,t) at the optimal stopping boundary. Also note that $f \diamond f = f f_{xx} - f_x^2 \leqslant 0$ if $f_{xx} \leqslant 0$, so it remains to check that $f \diamond f \leqslant 0$ at points in $\mathcal{U}$.

Proposition 2.7. *The function f^ε satisfies $f^\varepsilon \diamond f^\varepsilon < 0$ at all points at the boundary of $\mathcal{U}$ at which f_{xx} is well-defined. Moreover, at the origin*

$$\limsup_{\mathcal{U} \ni (x,t) \to (0,0)} f^\varepsilon \diamond f^\varepsilon < 0.$$

Proof. Boundary points can be of some different types. It suffices to check

(a) points in $\{(x,0)\colon\ x > 0\}$,
(b) points at the free boundary with $f_{xx} \geqslant 0$,
(c) points in $\mathcal{C}$ with $f_{xx} = 0$, and
(d) the origin.

Points of type (a) are easy to handle. Indeed, since $f = f_x = f_{xx} = 0$ at these points it follows from (12) that $f^\varepsilon \diamond f^\varepsilon = -\varepsilon^2$. Similarly, points of type (b) and (d) are taken care of by (12) and Lemmas 2.3 and 2.5, respectively. Finally, points of type (c) are handled using that $f \diamond f \leqslant -f_x^2 \leqslant 0$ if $f_{xx} = 0$. $\square$

Proof of Theorem 2.1. In the stopping region the option price equals the contract function, so the price is clearly log-concave. Moreover, at points in $\mathcal{C} \setminus \mathcal{U}$ we have $f \diamond f \leqslant 0$ as explained above, so it remains to show that $f \diamond f = f f_{xx} - f_x^2 \leqslant 0$ everywhere in the region $\mathcal{U}$. In order to do this we first show that $f^\varepsilon \diamond f^\varepsilon \leqslant 0$ in $\mathcal{U}$. For some $T_0 > 0$, define the set

$$\Lambda^\varepsilon := \{(x,t) \in \mathcal{U}\colon\ t \leqslant T_0 \text{ and } (f^\varepsilon \diamond f^\varepsilon)(x,t) > 0\},$$

and assume that $\Lambda^\varepsilon \neq \emptyset$. From (12) and Lemma 2.6 it follows that we can find a constant N such that $f^\varepsilon \diamond f^\varepsilon < 0$ for all points (x,t) with $x \geqslant N$. Thus Λ^ε is bounded, so the closure $\bar{\Lambda}^\varepsilon$ is compact. Let

$$t_0 = \inf\{t\colon\ (x,t) \in \bar{\Lambda}^\varepsilon \text{ for some } x\}.$$

Due to compactness there exists x_0 such that $(x_0,t_0) \in \bar{\Lambda}^\varepsilon$. By continuity, we have that $(f^\varepsilon \diamond f^\varepsilon)(x_0,t_0) = 0$, so it follows from Proposition 2.7 that this point is in the interior of $\mathcal{U}$. Thus, by the definition of t_0, the function

$$x \mapsto (f^\varepsilon \diamond f^\varepsilon)(x,t_0) \quad \text{for } x \in \mathcal{U}$$

attains a local maximum at x_0. Hence we have

$$f^\varepsilon \diamond f^\varepsilon = (f^\varepsilon \diamond f^\varepsilon)_x = 0 \quad \text{and} \quad (f^\varepsilon \diamond f^\varepsilon)_{xx} \leqslant 0$$

at the point (x_0, t_0). Note that $f^\varepsilon_x = f_x - \varepsilon < 0$ since $f_x \leqslant 0$. Therefore $0 = f^\varepsilon \diamond f^\varepsilon = f^\varepsilon f^\varepsilon_{xx} - (f^\varepsilon_x)^2$ at (x_0, t_0) implies that $f^\varepsilon(x_0, t_0) \neq 0$. Consequently Proposition 2.2 yields

$$f^\varepsilon \diamond \mathcal{L} f^\varepsilon \leqslant 0$$

at (x_0, t_0). At this point we also have, again by the definition of t_0,

$$(f^\varepsilon \diamond f^\varepsilon)_t \geqslant 0,$$

so

$$\begin{aligned} 0 &\leqslant (f^\varepsilon \diamond f^\varepsilon)_t - 2 f^\varepsilon \diamond (\mathcal{L} f^\varepsilon) \\ &= 2 f^\varepsilon \diamond f^\varepsilon_t - 2 f^\varepsilon \diamond \mathcal{L} f + 2\varepsilon f^\varepsilon \diamond \mathcal{L}(x + M) \\ &= 2\varepsilon f^\varepsilon \diamond \mathcal{L}(x + M) \\ &= 2\varepsilon C f_x - \varepsilon(CM + Cx + 1 - C) f_{xx} - 2\varepsilon^2 C \\ &< 2\varepsilon C f_x - \varepsilon(CM + Cx + 1 - C) f_{xx} \leqslant 0. \end{aligned}$$

To arrive at this contradiction we have used $f_x \leqslant 0$ and $f_{xx} \geqslant 0$ in $\mathcal{U}$. From the contradiction it follows that $\Lambda^\varepsilon = \emptyset$. Since T_0 is arbitrary, $f^\varepsilon \diamond f^\varepsilon \leqslant 0$ in the region $\mathcal{U}$. Letting $\varepsilon \to 0$, we find that $f \diamond f \leqslant 0$ in the region $\mathcal{U}$. This finishes the proof. □

We end this section with an example showing that log-concavity in the stock log-price is not preserved in general for American options in the standard Black–Scholes model. Thus there is no direct generalization of the results by Borell in [2,3].

Example. Consider the American option with contract function given in the transformed coordinates by

$$h(x) = (1 - e^x)^+ + \varepsilon$$

for some constant $\varepsilon \in (0, 1)$. Then it is straightforward to check that the optimal stopping boundary consists of two curves, one of which has an x-coordinate strictly larger than 0 and the other one has an x-coordinate strictly smaller than 0. At the boundary where $x > 0$ one can show that the smooth fit condition $f_x = 0$ holds (for example, methods similar to the one used to prove Lemma 7.8 in [12] can be used). Moreover, since American option prices increase in the time to maturity we have $f_t \geqslant 0$ (actually, if the boundary is C^1, then the smooth fit condition implies that $f_t = 0$ at the boundary). Thus, at the part of the boundary where $x > 0$ we have $f_{xx} = f_t - (C - 1) f_x + C f \geqslant C f$, so $f \diamond f \geqslant C f^2 = C\varepsilon^2 > 0$.

3. American calls on a dividend paying stock

In this section we consider call options written on a stock which pays a continuous dividend yield $\delta > 0$. The stock price is thus modeled as

$$dS(u) = (r - \delta) S\, du + \sigma S\, dW$$

under the risk-neutral probability measure. Using the same change of coordinates (4) as in Section 2, one finds that the price f of the American call option then satisfies

$$\begin{cases} f_t(x,t) = \hat{\mathcal{L}} f(x,t) & \text{if } x > b(t), \\ f(x,t) = e^x - 1 & \text{if } x = b(t), \\ f_x(x,t) = e^x & \text{if } x = b(t), \\ f(x,0) = (e^x - 1)^+, \end{cases}$$

where

$$\hat{\mathcal{L}} f = f_{xx} + (C - D - 1) f_x - Cf \tag{13}$$

and $D = \frac{2\delta}{\sigma^2}$ (the functions f and b are of course not the same here as in the previous section).

We then have the following result for American call options.

Theorem 3.1. *For any fixed time $u < T$, the price of an American call option written on a dividend paying stock is log-concave as a function of the stock log-price.*

Proof. Recall that b is monotone increasing with

$$\lim_{t \to 0} e^{b(t)} = \frac{C}{D} \quad \text{and} \quad \lim_{t \to \infty} e^{b(t)} = \frac{\gamma}{\gamma - 1}, \tag{14}$$

where γ is the positive solution to the equation

$$\gamma^2 + (C - D - 1)\gamma - C = 0.$$

It is straightforward to check that

$$f \diamond f = De^{2x} - (1 + C + D)e^x + C$$

at the free boundary, and that (14) implies that $f \diamond f \leqslant 0$ at the free boundary. Noting that f is $C^{2,1}$ up to the point $(\ln \frac{C}{D}, 0)$, no extra analysis (corresponding to Lemma 2.5) needs to be performed in the vicinity of this point. Using

$$f(x,t) = \int_0^\infty (e^y - 1)\Gamma(x - y, t)\, dy + \int_0^t \int_{b(t-\tau)}^\infty (De^y - C)\Gamma(x - y, \tau)\, dy\, d\tau,$$

where

$$\Gamma(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{ -\frac{(x + (C - D - 1)t)^2}{4t} - Ct \right\},$$

it is straightforward to check that

$$\limsup_{(x,t) \to (0,0)} f \diamond f \leqslant 0.$$

It follows that $f \diamond f \leqslant 0$ at all boundary points of the continuation region. The proof is then completed in the same way as in the proof of Theorem 2.1 but with the function f^ε defined by

$$f^\varepsilon(x,t) := f(x,t) + \varepsilon(x - M)$$

for M large so that $C - D - 1 - Cx + CM \geqslant 0$ in the continuation region. $\square$

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