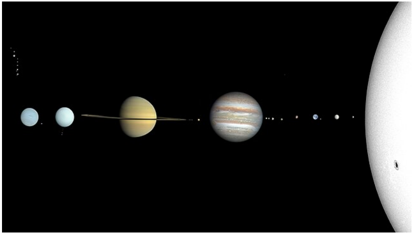
On the stability of the planar Sun-Jupiter-Saturn

Jordi-Lluís Figueras

Uppsala University



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Abstract

In this talk I will introduce the (planetary) N-body problem and briefly discuss its historical background: From Ptolemy, Copernicus, Kepler and Tycho Brahe to Newton; from Laplace, Euler and Lagrange, to Mittag-Leffler and Poincaré, ending up with Kolmogorov, Arnold and Moser. Then I will move to discuss the stability problem of our Solar system and, in particular, the Sun-Jupiter-Saturn. Is it stable? Do the planets orbit around the Sun in a (quasi)periodic fashion?

Disclaimer: I don't plan to give a thorough history description but just a few pointers that can help placing some ideas.

The question I will try to answer

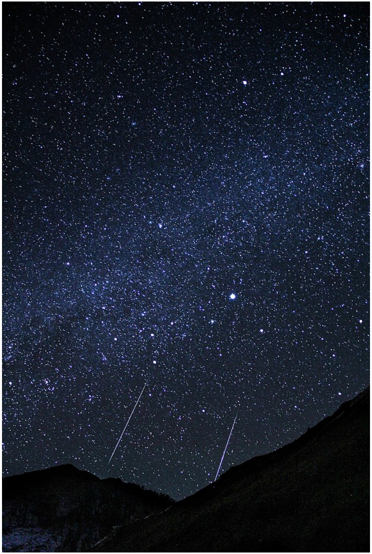
Let me be rude and insist in the question we want to tackle.

Do Jupiter and Saturn spin around the Sun with (quasi)periodic motion?

A History Snack

Antiquity

Since the very beginnings of history humankind has been fascinated by looking at the sky. All these bodies hanging there. Some very big (Sun/Moon), some very faint. Some moving, some fixed, some falling...



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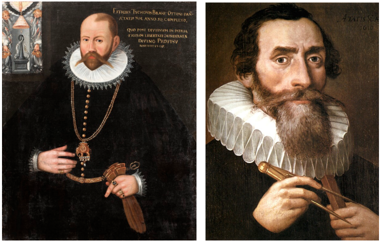
Proposing a model: Ptolemy

Ptolemy (around 100 AD), a roman astronomer, proposed in his work *The Planetary Hypotheses* laws governing the celestial motion: A geocentric model where the Sun and the Planets (the known ones) move around Earth in epicycles.

Source: wikimedia

Measuring the sky: Tycho Brahe
And modelling: Johannes Kepler

Tycho Brahe (1546-1601) was a very famous danish astronomer for doing very "precise" measurements of the cellestial bodies (without telescope![1]). This measurements by him where instrumental for Kepler (1571-1630) for proposing his three famous laws[2].



Source: wikimedia

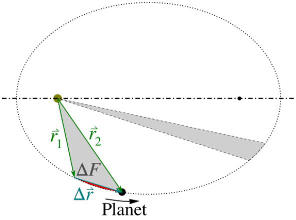
[1]: *First telescope appeared in 1608!*

[2]: *Copernicus (1473-1543) proposed a heliocentric model. Galileo Galilei (1564-1642) suported this theory.*

Kepler's laws of planetary motion

The first two laws appeared in 1609 and the third in 1619.

1. The orbit of a planet is an ellipse with the Sun at one of the two foci.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
3. The square of a planet's (orbital) period is proportional to the cube of the length of the semi-major axis of its orbit.



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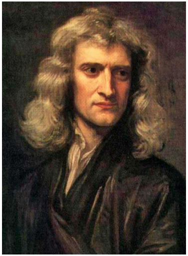
Universal Gravitational Law

Newton (1643-1727) made a huge advance for science and mathematics when he published his **Philosophiiæ Naturalis Principia Mathematica** in 1687.
Among other things, in this opera magna he settled the foundations of mechanics and the foundations of graviational mechanics:



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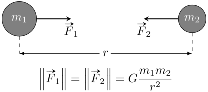
The force of attraction between two bodies is proportional to the product of the masses and inversely proportional to the square of the distance:

$$F \propto \frac{M_1 M_2}{r^2}.$$



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Newton can prove Kepler

With his gravitational law, Newton is successful in proving Kepler's three laws. By this we mean that he can prove the individual interactions between the very massive Sun and each of the planets[1].
A follow up question is if it is possible to see what happens when the planets start to influence each other.

Do they still lay on ellipses? Our goal in this talk aligns with this question.

[1] The Sun and a planet spin around their common center of mass as the focus of their elliptic trajectories. But! One can do a change of coordinates and place the Sun at the origin. Then the planet has it as a focus and its motion is an ellipse.

Newton and the 3 Body Problem

In fact, already in **Philosophiiæ Naturalis Principia Mathematica** Book 3, Proposition XIII, Theorem XIII, Newton discusses the need to enlarge the study of his previous 2 Body Problem (Kepler orbits) into the Jupiter and Saturn planets.

He says that these two planets perturb each other and this perturbation must be taken into account[2]. He also says that the perturbation of all other planets is too small and can be disregarded.

[2] without doing anything else. Recall that Book 3 is the *Wishful thinking book of Newton*

The birth of scientific/mathematics field.

Newton's statement and question was simple: We understand the 2 Body Problem (elliptic/parabolic/hyperbolic motion) and we desire to understand the 3 (or more) Body Problem.

This marked the starting point of a lot of mathematicians to get into the game. Many names we know from the past where involved in questions involving Celestial Mechanics: Euler (1707-1783), Lagrange (1736-1813), Laplace (1749-1827), Gauss (1777-1855), Jacobi (1804-1851), Weierstrass (1815-1897), Kovalevskaya (1850-1891), Poincaré (1854-1912), Mittag-Leffler (1846-1927), Lindstedt (1854-1939), Painlevé (1863-1933), Levi-Civita (1873-1941), Sundman (1873-1949), Siegel (1896-1981)... The list is endless and I can not make justice to all of them. Many mathematical fields have tools and questions and answers related to this field.

Sundman and series expansion.

One remarkable result was by Poincaré (who else!?) who proved first that the (restricted) 3 Body Problem was integrable (wrong!) and then he proved that it was not integrable and had chaos (true!)[3]

Sundman proved that one can obtain series expansions of all (noncolliding solutions) that converge for all time t.

[3] This is the famous Mittag-Leffler-Poincaré-King Oskar II affair (1885).

Chaos and the 3 Body Problem

So, with Poincaré results we obtain that the (restricted) 3 Body Problem is chaotic. Do we have, then, a solution to our problem?

No, we don't have a solution to our problem. First, because it is a simplified model but, most important: not because a system has chaos means that all its orbits are chaotic or that chaos has full measure! In fact, Poincaré chaos has zero measure. (This is a claim that sometimes needs to be repeated, otherwise there are assertions (weather models) that are blantly false).

KAM Theory

Jumping to KAM

Jumping ahead a lot of steps, we land on the creation in 1954 of **KAM theory** (Kolmogorov (1903-1987) Arnold (1937-2010) Moser (1928-1999)): A theory for proving the existence of quasiperiodic motion in Hamiltonian Systems.

What is KAM about?

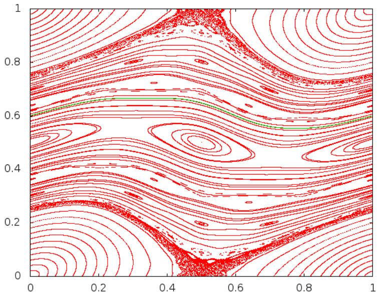
Original KAM theory starts with the observation that Integrable Hamiltonian systems $H(I, \varphi) = h(I)$, $I \in \mathbb{R}^n, \varphi \in \mathbb{T}^n$ have very simple dynamics:

$$\begin{cases} \dot{I} & = & 0 \\ \dot{\varphi} & = & -\nabla h(I) \end{cases}$$

This implies that the phase space $\mathbb{R}^n \times \mathbb{T}^n$ is foliated by tori with constant I_0 with inner dynamics $\varphi(t) = \varphi_0 - t\nabla h(I_0)$.

What is KAM about?

What is proved in the original KAM results is that, under a small perturbation of an integrable system, $H(I, \varphi) = h(I) + h_1(I, \varphi)$ with h_1 small, there exists a positive measure Cantor foliation of tori such that their are invariant and the inner dynamics is conjugated to $\varphi(t) = \varphi_0 + t\omega$ (these ωs are Diophantine vectors!).



What is KAM about?

More concretely, starting with the Hamiltonian $H = H_0$, one proposes an iterative method of canonical[4] (symplectic) change of coordinates Φ_k onto the Hamiltonian obtaining

$$H_k = H_{k-1} \circ \Phi_{k-1}$$

such that

$$\lim_{k \to +\infty} H_k = H_\infty = \omega \cdot I + \mathcal{O}(I^2).$$

This Hamiltonian H_∞ has the invariant torus $I = 0$ with inner dynamics $\dot{\varphi} = \omega$.

[4] These are the changes of coordinates $(p, q) \to (P, Q)$ preserving the ODE structure $\dot{q} = \nabla_p H, \dot{p} = -\nabla_q H$.

What is KAM about?

The proof is not trivial in nature and relies on a very advanced fixed point scheme (something that, later on, will evolved to be called Nash-Moser schemes).

Non-invertible linearization

All boils down that one wants to apply a Newton-scheme method but the linerazation is not invertible into the same space but analyticity (or derivatives) is lost. This comes from solving the equation

$$\omega_1 \partial_1 f + \omega_2 \partial_2 f = g$$

with f, g periodic.

Why Diophantine numbers appear here?

Diophantine numbers[5] appear here because these are the most irrational numbers that exist. They are the furthest from being rational. And rational or close-to-rational are to be avoided. Why? Due to ressonances[6]! It is hidden in the notation and synthesis of the presentation, but a ressonance makes that stable motion is not possible.

[5] A vector ω is Diophantine if there exists γ, τ such that $|\langle \omega, k \rangle| \geq \frac{\gamma}{|k|^\tau}$ for all $k \in \mathbb{Z}^n$ nonzero.

[6] The same word **ressonance** as in a bridge falling down due to wind blowing to it or soldiers stepping on it

KAM and the 3 Body Problem

In principle, one has that the 3 Body Problem of Sun and two planets looks like the previous setting. The Hamiltonian is a sum of an integrable part (2 Kepler problems) plus a small interaction between the planets. And this is true! But, the problem suffers of a degeneracy that makes it even harder. At the integrable limit, there are some variables missing: the angular momenta. The system is too integrable (superintegrable).

Nevertheless, KAM schemes where able to be applied by Arnold and it was proven that, there exists quasiperiodic motions when the perturbation is small.

However, as it was pointed by Hénon, applying KAM for realistic problems (like ours) lead to very pessimistic results: For example, the masses of Jupiter and Saturn are forced to be of order 10^{-333} (and not of order 10^{-3}).

Smallness and KAM

In fact, at some point there was the pessimistic idea in the mathematics and physics community that KAM techniques only apply to too very small values of the perturbations and were not good enough for applications.

Some more historical context on the planetary problem

As said, a crucial advancement was performed by Arnold in *On the classical perturbation theory and the stability problem of planetary systems*, *Small denominators and problems of stability of motion in classical and celestial mechanics* where he proved the persistence of quasiperiodic motion for the planar three body problem for a ratio of the semi-major axis close to zero. The theory was later completed for the spatial N body problem in remarkable works by Herman and Féjoz *Démonstration du 'théorème d'Arnold' sur la stabilité du système planétaire (d'après Herman)*, and Chierchia and Pinzari *The planetary N -body problem: symplectic foliation, reductions and invariant tori*.

Some more historical context. CAP

Then, trying to attack the realistic problem (with realistic data), we encounter the works of Chierchia, Celletti and Locatelli, Giorgilli. We stress that the work *Invariant tori in the Sun-Jupiter-Saturn system* is remarkable by giving evidences that the normal form approach seems to work in this setting.

Other remarkable works on the 3 Body Problem or variants along these lines are by Celletti, Chierchia, Fejoz and Castan.

Some more historical context. Chaos

There is also very strong numerics done by Laskar showing that the Solar System with all its planets has a lot of chaos.

KAM and the Parameterization Method

All this changed when more suitable tools/perspectives were introduced. Among them we focus on the *Parameterization Method.*[7] This gives a new optic on how to do KAM: classic KAM is applying (symplectic) changes of coordinates that converge to a desired Hamiltonian system; the Parameterization Method is parameterizing the wanted quasiperiodic orbit (the torus where it lays) and perform successive corrections to this parameterization.

This last methodology has shown being successful in problems far from integrable[8]: it was proven the existence of the golden curve in the Standard map for values 10^{-4}-close to breakdown (the proof involves the use of Computer-Assisted Proofs).

[7] de la Llave, González, Jorba, Villanueva (2001)
[8] F., Haro, Luque (2016)

Back to Sun-Jupiter-Saturn

What we have obtained is the application of all this accumulated knowledge and obtained an approximated solution to the planar Sun-Jupiter-Saturn that, when checking it against a (qualitative) theorem we have[9], we obtain[10] that (up to interval arithmetics) we have proven the existence of this solution. What we explain here is how we obtain this solution and the numerical check. (In the near future you will see the Computer validation).

[9] F., Haro, *A modified parameterization method for invariant lagrangian tori for partially integrable Hamiltonian systems* (2024)

[10] F., Haro *Sun-Jupiter-Saturn may exist: a verified computation of quasiperiodic solutions for the planar three body problem* (2024)

The road for proving quasiperiodic solutions to Sun-Jupiter-Saturn

The road that we follow is:

1. Write a qualitative theorem that, given the 3 Body Problem Hamiltonian system and an approximation to the solution (a parameterization to the torus), it produces some constants to be checked with this input.
2. Compute a very good numerical approximation of the parameterization to the torus.
3. Apply the qualitative theorem in step 1 for the torus in step 2. Check if the output of the theorem asserts the existence of the invariant torus.

The (planar) $(1 + n)$-Body Problem

The Equations: cartesian

The planar $(1+n)$-body problem (the Sun plus n planets) in Poincaré heliocentric cartesian coordinates has Hamiltonian [11] $H_{\mathrm{C}} : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$
\begin{aligned}
H_{\mathrm{C}}(x,y) &= \sum_{i=1}^{n} \left(\frac{\|y_i\|^2}{2m_i} - \frac{m_i}{\|x_i\|} \right) \\
&\quad + \mu \left(\sum_{i=1}^{n} \frac{\|y_i\|^2}{2} + \sum_{1 \le i < j \le n} \left(y_i \cdot y_j - \frac{m_i m_j}{\|x_i - x_j\|} \right) \right) \\
&= H^0_{\mathrm{C}}(x,y) + \mu \; H^1_{\mathrm{C}}(x,y).
\end{aligned}
$$

(The mass of Sun is fixed to 1 and the other masses m_i are then relative to the former. μ plays the role of capturing that the other masses are small compared to the Sun's. So, think of the largest planet having $m = 1$ and μ being its relative mass).

[11] The Sun is fixed at the origin because linear momentum is preserved (first integral)

The Equations: Delaunay

As it is customary in the planar $(1+n)$-body problem, we change to Delaunay coordinates: The Delaunay coordinates of the i-th body are $(\ell_i, g_i, L_i, G_i) \in \mathbb{T}^2 \times \mathbb{R}^2$, with $G_i < L_i$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, is mapped to the Cartesian coordinates $(x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}) \in \mathbb{R}^4$ through the following steps:

$$e_i = \sqrt{1 - \left(\frac{G_i}{L_i}\right)^2}, \; a_i = \frac{(L_i)^2}{m_i^2}, \; b_i = \frac{m_i^2}{L_i}, \; E_i = K(\ell_i, e_i),$$

$$\begin{pmatrix} q_{i,1} \\ q_{i,2} \end{pmatrix} = a_i \begin{pmatrix} \cos(E_i) - e_i \\ \frac{G_i}{L_i} \sin(E_i) \end{pmatrix}, \qquad \begin{pmatrix} x_{i,1} \\ x_{i,2} \end{pmatrix} = \begin{pmatrix} \cos(g_i) & -\sin(g_i) \\ \sin(g_i) & \cos(g_i) \end{pmatrix} \begin{pmatrix} q_{i,1} \\ q_{i,2} \end{pmatrix}$$

$$\begin{pmatrix} p_{i,1} \\ p_{i,2} \end{pmatrix} = \frac{b_i}{1 - e_i \cos(E_i)} \begin{pmatrix} -\sin(E_i) \\ \frac{G_i}{L_i} \cos(E_i) \end{pmatrix}, \qquad \begin{pmatrix} y_{i,1} \\ y_{i,2} \end{pmatrix} = \begin{pmatrix} \cos(g_i) & -\sin(g_i) \\ \sin(g_i) & \cos(g_i) \end{pmatrix} \begin{pmatrix} p_{i,1} \\ p_{i,2} \end{pmatrix}$$

where $E = K(\ell, e)$ denotes the solution of the Kepler equation $\ell = E - e\sin(E)$.

The Equations: Delaunay

The Hamiltonian is then written in Delaunay coordinates (ℓ, g, L, G) as a function $H_{\mathrm{D}} : \mathbb{T}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$
\begin{aligned}
H_{\mathrm{D}}(\ell, g, L, G) &= \sum_{i=1}^{n} \frac{-m_i^3}{2L_i^2} + \mu \; H_{\mathrm{C}}^1 \circ D(\ell, g, L, G) \\
&= H_{\mathrm{D}}^0(L) + \mu \; H_{\mathrm{D}}^1(\ell, g, L, G),
\end{aligned}
$$

where D denotes the *Delaunay map* from Delaunay coordinates (ℓ, g, L, G) to Cartesian coordinates (x, y) described above.

(Notice how clear it is in this expression the famous degeneracy in the $(1+n)$-body problem: the integrable part H_{D}^0 only depends on the actions L and not on the G actions!)

First integral: total angular momentum

The $(1+n)$-body problem has a first integral: the total angular momentum $\hat{G}_n = \sum_{1\le k\le n} G_k$. This helps us to reduce even further the number of coordinates of the system by 2 (the first integral and its associated angular variable). We obtain then the reduced Hamiltonian $H_{\hat{G}_n} : \mathbb{T}^{2n-1} \times \mathbb{R}^{2n-1} \to \mathbb{R}$ given by

$$H_{\hat{G}_n}(\ell, \hat{g}, L, \hat{G}) = H^0_{\mathrm{D}}(L) + \mu H^1_{\hat{G}_n}(\ell, \hat{g}, L, \hat{G}),$$

with $\hat{g} = (\hat{g}_1, \ldots, \hat{g}_{n-1})$ and $\hat{G} = (\hat{G}_1, \ldots, \hat{G}_{n-1})$. [12]

(Evaluating $H_{\hat{G}_n}$ is not "*difficult*": From Cartesian to Delaunay requires only to solve on easy equation, composing then is easy, and reducing $\hat{G}_n$ is explicit. Also, getting partial derivatives of it is also easy: Use automatic differentiation or do it by hand if the order is small.)

[12] $\hat{G}_s := \sum_{1\le k\le s} G_k$, and g_s is the symplectic conjugate to $\hat{G}_s$

The (planar) Sun-Jupiter-Saturn

Sun-Jupiter-Saturn system

In our case, we set $n = 2$ (the number of planets). Moreover, the relative masses of Jupiter and Saturn we use are $0.9546 \cdot 10^{-3}$ and $0.2856 \cdot 10^{-3}$, so $m_1 = 0.9546$, $m_2 = 0.2856$, and $\mu = 10^{-3}$.
By using their orbital elements (semiaxes a_i and excentricities e_i) we get that two of the frequencies are

$$\omega^I = (8.3954928870254630120 4{\cdot}10^{-2}, 3.3824011705930435825 9{\cdot}10^{-2}).$$

The third frequency[13], the relative precession[14], is

$$\omega^{\hat{g}_1} = -1.85007988077595000000 \cdot 10^{-5}.$$

These frequencies are almost Diophantine.[15]

[13] Recall that we are working on a 3 degrees of freedom problem

[14] obtained via *frequency analysis*

[15] Theorem: If $\omega = (\omega^I, \omega^{\hat{g}_1})$, for any $k \neq 0$ it is satisfied $|k \cdot \tilde{\omega}| \geq \frac{\gamma}{|k|_1^\tau}$ for some $\tilde{\omega}$ satisfying $|\tilde{\omega} - \omega| < 10^{-80}$, $\gamma = 1.69 \cdot 10^{-6}$ and $\tau = 2.4$.

What we have done in this project is two things:

1. Obtain a good approximation of a parameterization $K : \mathbb{T}^3 \to \mathbb{T}^3 \times \mathbb{R}^3$ that is (numerically) invariant. It satisfies the equation
$$\mathfrak{L}_\omega K(\theta) + X_{H_{\hat{G}_n}}(K(\theta)) = 0,$$
where $\mathfrak{L}_\omega K(\theta) = -(DK(\theta))^\top \cdot \omega$ and $X_{H_{\hat{G}_n}} = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} (DH_{\hat{G}_n})^\top$.
2. Check that the given approximation fulfills all the hypotheses of our quantitative KAM theorem[16].

[16] F., Haro, *A modified parameterization method for invariant lagrangian tori for partially integrable Hamiltonian systems* (2023)

Obtaining the approximation

Obtaining the good parameterization

The main obstacle we have is that for $\mu = 0$ there exists a lot of tori with our prescribed frequency: Any $\hat{G}_n$ works. Moreover, all our numerical algorithms fail if we try to start from there.
Our strategy: First solve the problem of finding an invariant torus $K : \mathbb{T}^3 \to \mathbb{T}^3 \times \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ for the problem with Hamiltonian

$$H = H_{\hat{G}_n} + \lambda G_n$$

and with the constraint

$$\langle \Pi_{\hat{G}_n} \circ K \rangle - \hat{G}_0 = 0.$$

Obtaining the good parameterization

This is equivalent to solving the system

$$\begin{cases} \mathfrak{L}_\omega K(\theta) + X_{H_{\hat{G}_n}}(K(\theta)) + X_{\hat{G}_n}(K(\theta))\lambda & = 0 \\ \langle \Pi_{\hat{G}_n} \circ K \rangle - \hat{G}_0 & = 0 \end{cases},$$

for the pair (K, λ) doing homotopy from $\mu = 0$ to $\mu = 10^{-3}$. If we pick correctly the value $\hat{G}_0$ we get that for $\mu = 10^{-3}$ the value of λ will be equal to zero[17].
Second: Refine on our Hamiltonian itself: We do it and obtain that the error of invariance is 10^{-54}.

[17] And this happens!

Obtaining the good parameterization: Refinment

The refinment is made on applying several Newton steps on the equation

$$\mathfrak{L}_\omega K(\theta) + X_{H_{\hat{G}_n}}(K(\theta)) = E(\theta),$$

where $E(\theta)$ is the invariance error. This is done by doing a change of coordinates,

$$P(\theta) = \begin{pmatrix} DK(\theta) & N(\theta) \end{pmatrix}$$

with $N(\theta)$ is the symplectic conjugated to $DK(\theta)$.

Under this change of coordinates the system is transformed into the form

$$\begin{cases} \mathfrak{L}_\omega \xi^L(\theta) & + T(\theta)\xi^N(\theta) & = & \eta^L(\theta) \\ \mathfrak{L}_\omega \xi^N(\theta) & & = & \eta^N(\theta) \end{cases}$$

Checking the KAM theorem

Finally, checking if it works

We then need to check on our theorem that all the conditions hold. The theorem is monstruous and I don't plan to write it here, but I can summarize the kind of inputs that needs: It needs control on the size of parameterization, its derivatives, the size of the Hamiltonian and its derivatives, the Diophantine constants, size of a linear frame built from the parameterization, and a transversality condition[18] (from all this previous data). Finally, it needs to know the error of invariance.

With all this information we obtain that for the theorem to guarantee the existence of the torus it needs to have error of invariance smaller than 10^{-47}, but we got 10^{-54} *!!!*.

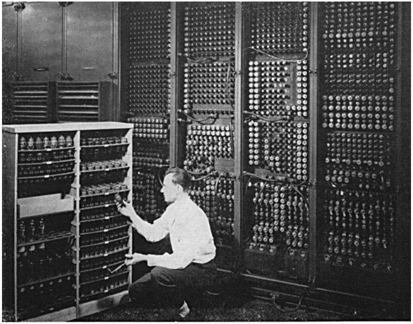
---

[18] Called torsion

Obstacles while computing

All this sounds nice and so, but we suffered quite a lot of obstacles. These can be classified as: theoretical and computational. The first ones are normal, problems are difficult to know how to solve. The second ones are more severe. We were required to use Uppmax, Uppsala's supercomputer for several reasons: Our computations need around half a TB of RAM and 4 TB of memory storage. A step of computations is around 10 days in a node of 16 cores. All this and add that the supercomputer gets frozen once every month, and it takes around $3 \sim 7$ days to get your job starting on it. Summary: it took us some good long years just to compute this parameterization.

Thank you very much!



Replacing a bad tube meant checking among ENIAC's 19,000 possibilities.

Jordi-Lluís Figueras **figueras@math.uu.se**