Thank You:

First of all I want to thank ~~the~~ all the organizers to allow me to speak here, in this very special event, where we celebrate Piotr's 60th birthday. I hope that all of you would join with me in saying that Piotr has been very influential in the field. Happy birthday, Piotr. You really deserve this party (and more!).

KAM THEOREM FOR ATTACKING SUN-JUPITER-SATURN (PLANAR)

Today I want to present a work that has been going on for quite a long time. People involved in the project are Alex Haro (UB) and Alejandro Luque ([illegible]

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Our objective is to prove that in the Newtonian planar Sun-Jupiter-Saturn system there exists invariant tori very close to the observations.

This problem dates back to Newton (1687), when he ~~can~~ was able to prove Kepler's solution for two-body systems and conjectured in his

3rd book that the anomalies observed (at that time) in Jupiter's orbit are based on Saturn's influence, but he could no give ~~more~~ a proof of it.

Long story short: from Newton plenty of mathematicians work on this an related problems until Kolmogorov (1954) gave birth to what we know nowadays the theory of KAM (~~...~~ Arnold and Moser). This theory tried to prove that given a Hamiltonian system close to be integrable, one can perform a canonical change of coordinates that ~~gives~~ gives the form

$$H = \underbrace{\omega}_{\text{diophantine}} \cdot I + \underbrace{g(I, \varphi)}_{\text{quadratic in } I}$$

From this new form one obtains that $I = 0$

is an invariant torus (maximal) for the system.

The main tool used in this type of proof (which we called the NF-approach) is in applying successives changes of coordinates (must be canonical) and get better and better Hamiltonians. Then prove that these changes of coordinates converge.

Arnold was able to use this to prove existence of qp solutions in planar planetary systems with ratio of semi-major axis close to zero.

Later Herman & Fejoz, Chierchia & Pinzari advanced in the problem. It was point out by Hénon that applying Arnold's results in specific systems lead to very small values of the masses is ridiculously small, of order 10^{-333} (compare with $10^{-3}, 10^{-4}$).

Moving on forward in time, several attempts to attack this problem using NF-techniques plus Computer-Assisted Proofs has happened: some by the Italian school (Locatelli, Chierchia, Celletti...) and also Fejoz-Castany. Very recently (2024) Locatelli-Caracciolo have been able to announce that the can apply KAM+NF+CAP in exo-planetary systems in the planar case after ~~doing~~ truncating the "astronomer's" expansions.

Our approach is still KAM but not NF. It is what is called "the parameterization method" (de la Llave, Gonzalez, Jorba, Villanueva 2005). The idea is that, instead of performing canonical changes of coordinates on the system, work the problem as a zero of a functional.

	NF	Param
Data:	Hamiltonian	Parameterization of the torus
Step	Change of coordinates	Newton step in the parameterization
Dimension	$2n$	n
Init data	Intervalar	Numeric (approx)
Around the torus	Gives a description of dynamics around the torus	No descr. needs to be done.

What is the problem?

Our Hamiltonian in Delaunay coordinates is of the form

$$\mathbb{R} \qquad \Pi$$

$$H=\sum_{i=1}^{2}{}' -\frac{m_i^3}{2L_i^2}+\mu H_{coupling}(\underbrace{L,l}_{2\text{-}D},\underbrace{G,g}_{1\text{-}D})$$

$\mu\cdot m_i$ are the masses of the planets.

(Sun's mass $=1$)

$\mu\cdot m_J=0.9546\cdot 10^{-3}$, $\mu\cdot m_S=0.2856\cdot 10^{-3}$

$\mu=10^{-3}$.

- μ plays the perturbation parameter role.
- At $\mu=0$ ~~there~~ system is degenerate and integrable. G is missing and there are no angles.
- We are interested for very specific frequency vector. $\omega^T=(\underbrace{-10^{-2},-10^{-2}}_{\substack{\text{fast}\\ L_1,\ L_2}},\underbrace{-10^{-5}}_{\substack{\text{slow}\\ G}})$

Alex Haro in his talk will explain how to get solutions for this system with this data.

$\longmapsto$

What we want to concentrate is in which environment we are moving on: The system around this data ($\mu = 10^3$) satisfies:

(a) The frequency vector has slow components.

(b) Nearby tori to our torus have frequency vectors very close to our original one: torsion is very small.

We need a KAM theorem that mitigates (a) & (b) and also that is explicit:

The theorem must admit CAP procedure: compute a finite number of upper bounds and get a constant to be checked less than 1.

$\longmapsto$

Our KAM theorem solution to this problem appears in F & Haro Physica D 2024 and is based on the parametrization method + ~~sharp~~ sharp use of Diophantine constants (as in Villanueva 2017) + explicit control on the torsion (good control on "nearby tori") + explicit constants.

THE PARAMETERIZATION METHOD

We have a Hamiltonian

$$H : \mathbb{R}^n \times \mathbb{T}^n \longrightarrow \mathbb{R}$$

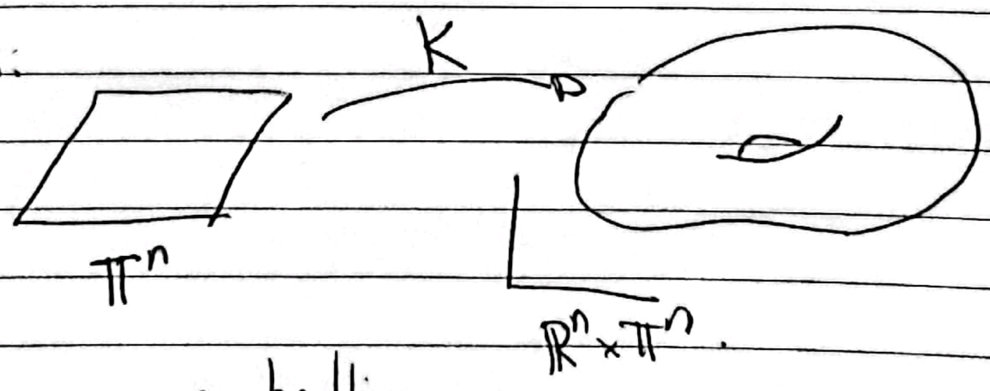
inducing dynamics

$$\dot{z} = -J(DH)^T \quad \text{with } J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$\boxed{X_H := -J(DH)^T}$$

We want to ~~find~~ prove the existence of an invariant torus (in our case primary) with inner dynamics conjugated to rigid flow with vector w.

Torus:



embedding.

- Invariance condition $\Leftrightarrow$ the v.f. is tangent to the torus $\Leftrightarrow$

$$X_H(K(\theta)) \in DK(\theta)$$

- Inner dynamics conj to rigid ω $\Leftrightarrow$ the inner dynamics are $\dot{\theta} = v(\theta)$ with $DK(\theta)\, v(\theta) = X_H(K(\theta))$. To be conj to rigid ω is equivalent so $\theta = s(\alpha)$ with

$$\dot{\alpha} = \omega = Ds(\alpha)^{-1} v(s(\alpha))$$ [Reparametrization on the torus].

Joining all this leads to the invariance equation

$$\mathcal{L}_\omega K(\theta) + X_H(K(\theta)) = 0.$$

with $\mathcal{L}_\omega K(\theta) := DK(\theta) \cdot \omega$

So, given an approximate solution to the invariance equation,

$$\mathcal{L}_\omega K(\theta) + X_H(\theta) = \underset{\text{small}}{E(\theta)},$$

how do we prove the existence of a nearby "true" invariant torus? The idea is by doing Newton steps and refine at each step the torus (getting smaller E's).

NEWTON STEP:

Given $K(\theta)$ find $\Delta(\theta)$ s.t. $K(\theta)+\Delta(\theta)$ has a smaller error of invariance $E(\theta)$.

$$\mathcal{L}_\omega K(\theta) + \mathcal{L}_\omega \Delta(\theta) + X_H(K(\theta)) + DX_H(K(\theta))\Delta(\theta)$$
$$+ \text{h.o.t.}(K(\theta), \Delta(\theta)) = 0.$$

We discarded h.o.t. and obtain

(*) $\mathcal{L}_\omega \Delta(\theta) + DX_H(K(\theta))\,\Delta(\theta) = -E(\theta)$.

Questions to be answered.

(a) how do we solve (*)? How do we estimate the size of $\Delta(\theta)$?

(b) How small is the new error?

$= \|\text{h.o.t.}(K(\theta), \Delta(\theta))\|$

quadratic in Δ.

The key point here is by noticing that:

(i) $L(\theta) := DK(\theta)$ satisfies

tangent bundle to the torus

$$\mathcal{L}_\omega L(\theta) + DK(\theta) L(\theta) = \text{small}(E(\theta))$$

(ii) $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ helps me to build a normal bundle.

$$N(\theta) = J\,L(\theta)\,(L(\theta)^T L(\theta))^{-1}.$$

$N(\theta)$ is almost symplectic conjugated to $L(\theta)$:

$$L(\theta)^T J L(\theta) = o(E)$$
$$L(\theta)^T J N(\theta) = -Id + o(E)$$
$$N(\theta)^T J N(\theta) = o(E)$$
$$N(\theta)^T J L(\theta) = Id + o(E)$$

All these $o(E)$ need to have explicit constants.

More importantly, and this is the key ingredient in all this!,

$$\boxed{\begin{aligned} &\mathcal{L}_\omega L + DX\, L = o(E) \\ &\mathcal{L}_\omega N + DX\, N = L\underbrace{T}_{\text{explicit matrix.}} + o(E) \end{aligned}}$$

Reducibility condition

T has explicit expression w.r.t. X_H, L, N.

$$T = N^T J(\mathcal{L}_\omega N + DXN)$$

With this insight one gets that

$$\Delta(\theta) = L(\theta)\xi^L(\theta) + N(\theta)\xi^N(\theta)$$

satisfies

$$\begin{cases} \mathcal{L}_\omega \xi^L(\theta) + T(\theta)\xi^N(\theta) = \eta^L(\theta) & (*)_1 \\ \mathcal{L}_\omega \xi^N(\theta) + 0 = \eta^N(\theta) & (*)_2 \end{cases}$$

with

$$\eta^L(\theta) = -N^T J E$$

$$\eta^N(\theta) = L^T J E$$

$(*)_2$ has a solution, small divisors,

$$\xi^N_k = \frac{\eta^N_k}{i\langle k, \omega\rangle}$$

ξ^N_0 free.

ξ^N_0 is determined in $(*)_1$ by imposing zero average ($\Rightarrow$ we need torsion $\Rightarrow$ $\langle T\rangle$ invertible!!!!)

Actually, what we do in our paper is something a little bit different. Instead of having the new torus as $K(\theta)+\Delta(\theta)$ we do

$$K(\theta+\xi^L(\theta)) + N(\theta+\xi^L(\theta))\,\xi^N(\theta+\xi^L(\theta))$$

which is equivalent up to second order. This help us to deal with better ~~the~~ fact that we have slow sub frequencies.

Summary:

At each Newton step what we know is

$\|E\|_g$, $\|K\|_g$, $\|L\|_g$, $\|N\|_g$, ~~$\|T\|_g$~~ $\|\langle T\rangle^{-1}\|$ and more auxiliary terms, and after that we have

$\|E_{new}\|_{s-3\delta}$, $\|K\|_{new, s-3\delta}$, $\|L\|_{new, s-3\delta}$, $\|N\|_{new, s-3\delta}$, [illegible] $\langle t\rangle^{-1}$

+ auxiliary terms.

This is called the iterative lemma.

So, just starting with this data we can.

ⓐ Get estimates of the next one

or

ⓑ See if it converges.

By combining ⓐ & ⓑ we can prove existence. (See Alex talk).