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Integral equations problems

1. Prove the Leibniz formula

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(x,y)\,dy = \int_{a(x)}^{b(x)} F_x(x,y)\,dy + F(x,b(x))b'(x) - F(x,a(x))a'(x).$$

2. Solve the Volterra equation

$$u(x) = 1 + \lambda \int_0^x u(y)\,dy$$

(a) by computing the Neumann series for u.
(b) using the Laplace transform.
(c) by first converting it into an equivalent initial value problem.

3. Reformulate the initial value problem

$$\begin{cases} u''(x) - \lambda u(x) = f(x), \quad x > 0, \\ u(0) = 1, u'(0) = 0, \end{cases}$$

as an equivalent Volterra integral equation. (Use the Leibniz formula to verify that the solution of the Volterra equation indeed satisfies the initial value problem.)

4. Solve the Fredholm equation

$$u(x) - \lambda \int_0^1 u(y)\,dy = 1$$

(a) using a Neumann series.
(b) by a direct approach.

5. Solve the Fredholm integral equation

$$u(x) = \lambda \int_0^1 (1-3xy)u(y)\,dy$$

for all values of λ.

6. Discuss the solvability of the Fredholm integral equation

$$u(x) - \lambda \int_0^1 (1-3xy)u(y)\,dy = f(x).$$

Answers or hints:

1. Let $G(x, \alpha, \beta) = \int_\alpha^\beta F(x, y)\, dy$. Compute $\frac{d}{dx} G(x, a(x), b(x))$ using the chain rule.

2. $u(x) = e^{\lambda x} = \sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!}$.

3. $u(x) - \lambda \int_0^x (x - y) u(y)\, dy = \int_0^x (x - y) f(y)\, dy$.

4. The integral equation is solvable iff $\lambda \neq 1$; then $u(x) = \frac{1}{1-\lambda}$. $\left[u(x) = \sum_{n=0}^{\infty} \lambda^n \text{ if } |\lambda| < 1.\right]$

5. We have the following cases:

- If $\lambda \neq \pm 2$, then $u(x) \equiv 0$.

- If $\lambda = 2$, then $u(x) = c(1 - x)$ where c is an arbitrary constant.

- If $\lambda = -2$, then $u(x) = c(1 - 3x)$ where c is an arbitrary constant.

(See **6**. below for further explanation.)

6. We have the following cases:

- If $\lambda \neq \pm 2$, then the integral equation is uniquely solvable for any f.

- If $\lambda = 2$, then the integral is solvable if and only if $\int_0^1 f(x)\, dx = \int_0^1 x f(x)\, dx$. If this condition is satisfied, then the integral equation has infinitely many solutions.

- If $\lambda = -2$, then the integral is solvable if and only if $\int_0^1 f(x)\, dx = 3 \int_0^1 x f(x)\, dx$. If this condition is satisfied, then the integral equation has infinitely many solutions.

Below follows a derivation of the above statements.

Note that the kernel is degenerate. Indeed, $k(x, y) = 1 - 3xy = \sum_{j=1}^{2} \alpha_j(x)\beta_j(y)$, where say $\alpha_1(x) = 1, \beta_1(y) = 1$, $\alpha_2(x) = x$ and $\beta_2(y) = -3y$.

The algebraic system associated with the Fredholm equation is

$$(I - \lambda A)\mathbf{c} = \mathbf{f}, \tag{1}$$

where

$$A = \begin{pmatrix} \int_0^1 \beta_1(x)\alpha_1(x)\, dx & \int_0^1 \beta_1(x)\alpha_2(x)\, dx \\ \int_0^1 \beta_2(x)\alpha_1(x)\, dx & \int_0^1 \beta_2(x)\alpha_2(x)\, dx \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ -3/2 & -1 \end{pmatrix},$$

$$\mathbf{f} = \begin{pmatrix} \int_0^1 f(x)\beta_1(x)\, dx \\ \int_0^1 f(x)\beta_2(x)\, dx \end{pmatrix} = \begin{pmatrix} \int_0^1 f(x)\, dx \\ -3\int_0^1 x f(x)\, dx \end{pmatrix}.$$

If this system is solvable, then solutions of the given integral equation are given by

$$u(x) = f(x) + \lambda(c_1\alpha_1(x) + c_2\alpha_2(x)) = f(x) + \lambda(c_1 + c_2 x). \tag{2}$$

Note that

$$\det(I - \lambda A) = \begin{vmatrix} 1 - \lambda & -\lambda/2 \\ 3\lambda/2 & 1 + \lambda \end{vmatrix} = 1 - \frac{\lambda^2}{4}.$$

So the system (1), hence the integral equation, is uniquely solvable if $\lambda \neq \pm 2$.

If $\lambda = 2$ the system (1) becomes

$$\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \int_0^1 f(x)\,dx \\ -3\int_0^1 xf(x)\,dx \end{pmatrix}.$$

This is solvable iff $\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx$, and the integral equation then has infinitely many solutions.

If $\lambda = -2$ the system (1) becomes

$$\begin{pmatrix} 3 & 1 \\ -3 & -1 \end{pmatrix} \mathbf{c} = \begin{pmatrix} \int_0^1 f(x)\,dx \\ -3\int_0^1 xf(x)\,dx \end{pmatrix}.$$

This is solvable iff $\int_0^1 f(x)\,dx = 3\int_0^1 xf(x)\,dx$, and the integral equation then has infinitely many solutions.

Of course, in case solutions do exist, they can be explicitly computed by solving the algebraic system (1) and using (2). For example, consider the homogenous case $f = 0$ of Problem 5. Then $\mathbf{f} = \mathbf{0}$. If $\lambda \neq \pm 2$ the homogeneous system (1) has only the trivial solution, and (2) gives $u(x) \equiv 0$. If $\lambda = 2$ putting $c_1 = a$ gives $c_2 = -a$. If we write $c = 2a$, then equation (2) gives $u(x) = c(1 - x)$. If $\lambda = -2$ putting $c_1 = a$ gives $c_2 = -3a$. Writing $c = -2a$, equation (2) gives $u(x) = c(1 - 3x)$.