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Applied Mathematics
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Partial differential equations problems

1. Solve the heat problem

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad t > 0, \\ u(x,0) = x(\pi - x), \quad 0 < x < \pi. \end{cases}$$

2. Solve the following problem for a vibrating string:

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, \quad t > 0, \\ u(x,0) = 3\sin 2x, \ u_t(x,0) = 5\sin 3x, \quad 0 < x < \pi. \end{cases}$$

3. Solve the heat problem

$$\begin{cases} u_t - u_{xx} = 0, \qquad 0 < x < \pi, \quad t > 0, \\ u(0,t) = u_x(\pi,t) = 0, \qquad t > 0, \\ u(x,0) = \sin \frac{1}{2}x + 3\sin \frac{5}{2}x, \qquad 0 < x < \pi. \end{cases}$$

4. Consider the following telegraph problem:

$$\begin{cases} u_{tt} + u_t - c^2 u_{xx} = 0, \quad a < x < b, \quad t > 0, \\ u(a,t) = 0, \ u_x(b,t) = 0, \quad t \geq 0, \\ u(x,0) = f(x), \ u_t(x,0) = g(x), \quad a \leq x \leq b. \end{cases}$$

Use the energy method to prove that the problem has at most one solution.

5. Consider the following Cauchy problem for the wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = f(x) = \begin{cases} 2, & |x| \leq 1, \\ 0, & |x| > 1, \end{cases} \\ u_t(x,0) = g(x) = 0, \quad x \in \mathbb{R}. \end{cases}$$

Draw the graphs of the solution $u(x,t)$ at times $t_j = j/2$, where $j = 0, 1, 2, 3$.

6. Consider the equation

$$xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0.$$

Find the domain where the equation is elliptic, and the domain where the equation is hyperbolic.

Answers or hints:

1. $u(x,t) = \frac{8}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} e^{-(2k+1)^2 t} \sin(2k+1)x$.

2. $u(x,t) = 3\cos 2t \sin 2x + \frac{5}{3}\sin 3t \sin 3x$.

3. $u(x,t) = e^{-\frac{1}{4}t}\sin\frac{1}{2}x + 3e^{-\frac{25}{4}t}\sin\frac{5}{2}x$.

4. Use the same energy function $E(t)$ as for the wave equation.

5. By d'Alemberts formula $u(x,t) = \frac{1}{2}[f(x+t) + f(x-t)]$.

6. The equation is elliptic when $xy < 0$ and hyperbolic when $xy > 0$. (It is parabolic when $xy = 0$, but this is not a domain.)