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Applied Mathematics
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Sturm-Liouville problems

1. Describe the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{cases} y'' + \lambda y = 0, \qquad 0 < x < 1, \\ y'(0) = 0, \quad y(1) + y'(1) = 0. \end{cases}$$

Obtain an asymptotic approximation of large eigenvalues.

2. Determine the eigenvalues and eigenfunctions of the periodic Sturm-Liouville problem

$$\begin{cases} y'' + \lambda y = 0, \qquad 0 < x < L, \\ y(0) = y(L), \quad y'(0) = y'(L). \end{cases}$$

3. Transverse vibrations of a drumhead $D = \{(x, y) : x^2 + y^2 < a^2\}$ are governed by the two-dimensional wave equation. Writing the Laplacian in polar coordinates, the problem becomes:

$$\begin{cases} c^{-2}u_{tt} & = & u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}, \quad \text{for } 0 < r < a, \\ u & = & 0, \quad \text{when } r = a, \\ u, u_t & \text{are} & \text{given functions when } t = 0. \end{cases}$$

(a) Separating the variables as $u(r, \theta, t) = T(t)R(r)\Theta(\theta)$, show that

$$\begin{aligned} &T'' + \lambda c^2 T = 0, \\ &\Theta'' + \gamma\Theta = 0, \\ &R'' + \frac{1}{r}R' + \Big(\lambda - \frac{\gamma}{r^2}\Big)R = 0, \end{aligned}$$

for some constants λ, γ.
(b) Since Θ must satisfy periodic boundary conditions, use the result of Problem 2 to deduce that $\gamma = n^2$, $n = 0, 1, 2, ...$, so that

$$R'' + \frac{1}{r}R' + \Big(\lambda - \frac{n^2}{r^2}\Big)R = 0, \qquad \text{where } n = 0, 1, 2, ...$$

(c) Let $\rho = \sqrt{\lambda}r$ and put $w(\rho) = R(r)$. Show that w satisfies the *Bessel equation of order* n:

$$\rho^2 w'' + \rho w' + (\rho^2 - n^2)w = 0.$$

Bounded solutions $w = w_n$ are given by $w_n(\rho) = J_n(\rho)$, the Bessel functions of order n.
(d) The function J_n has an infinite number of positive zeros $\alpha_{n,m}$ such that

$$0 < \alpha_{n,1} < \alpha_{n,2} < \alpha_{n,3} < \cdots .$$

Show that the boundary condition at $r = a$ implies that the eigenvalues $\lambda = \lambda_{n,m}$ are given by

$$\lambda_{n,m} = \Big(\frac{\alpha_{n,m}}{a}\Big)^2.$$

It follows that the natural frequencies that the drum can produce are given by $c\alpha_{n,m}/a$.

Answers or hints:

1. The eigenvalues and eigenfunctions are $\lambda_n = \omega_n^2$ respectively $y_n(x) = \cos \omega_n x$, $n = 0, 1, 2, ...$, where ω_n are the positive solutions of the equation $\tan x = 1/x$. By drawing the graphs of these functions, one sees that $\omega_n \approx n\pi$, so that $\lambda_n \approx n^2\pi^2$, for large values of n.

2. The eigenvalues and eigenfunctions are:

$$\lambda_0 = 0, \quad y_0(x) = 1,$$
$$\lambda_n = \left(\frac{2n\pi}{L}\right)^2, \quad y_n(x) = A_n \cos \frac{2n\pi x}{L} + B_n \sin \frac{2n\pi x}{L}, \qquad \text{for } n = 1, 2, 3, ...$$