

## Transform theory problems

1. Solve the initial value problem

$$x''(t) - x'(t) - 2x(t) = 2, \quad x(0) = x'(0) = 1.$$

2. Solve the integral equation

$$y(t) = t + \int_0^t \sin(t - \tau)y(\tau) d\tau, \quad t \geq 0.$$

3. Solve the problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x > 0, \quad t > 0, \\ u(0, t) = a \sin \omega t, & t > 0, \\ u(x, 0) = 0, \quad u_t(x, 0) = 0, & x > 0. \end{cases}$$

4. Determine the fouriertransformen  $\hat{f}(\omega)$  of the function  $f(t) = te^{-|t|}$ .

5. Find a function  $u \in L^1(\mathbb{R})$  such that

$$u(t) + \int_{-\infty}^{\infty} e^{-|t-s|} u(s) ds = e^{-|t|}.$$

6. Solve the problem

$$\begin{cases} u_t - u_x - u_{xx} = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = H(x+1) - H(x-1), & x \in \mathbb{R}. \end{cases}$$

(Here  $H(x)$  denotes the Heaviside function, which is equal to 1 if  $x$  is positive and 0 otherwise.)

**Answers or hints:**

1.  $x(t) = e^{2t} + e^{-t} - 1.$

2.  $y(t) = t + \frac{t^3}{6}.$

3.  $u(x, t) = a \sin\left(\omega\left(t - \frac{x}{c}\right)\right)H\left(t - \frac{x}{c}\right),$  where  $H$  denotes the Heaviside function.

4.  $\hat{f}(\omega) = -\frac{4i\omega}{(1+\omega^2)^2}.$

5.  $u(t) = \frac{1}{\sqrt{3}}e^{-\sqrt{3}|t|}.$

6.  $u(x, t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+t+1}{2\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x+t-1}{2\sqrt{t}}\right)\right],$  where  $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$