Formula sheet for the course Applied Mathematics

Fourier series

Functions with period 2π

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt$$

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n})$$

Parseval's formula:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Functions with period T

Put $\Omega = 2\pi/T$

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t),$$

where

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\Omega t}\, dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\Omega t\, dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\Omega t\, dt.$$

Parseval's formula:

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Some trigonometric formulas

$$2 \sin a \sin b = \cos(a - b) - \cos(a + b)$$

$$2 \sin a \cos b = \sin(a - b) + \sin(a + b)$$

$$2 \cos a \cos b = \cos(a - b) + \cos(a + b)$$

$$2 \sin^2 t = 1 - \cos 2t, \qquad 2 \cos^2 t = 1 + \cos 2t$$

The Laplace transform

$f(t)$	$\tilde{f}(s) = F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
General rules	
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$e^{at} f(t)$	$F(s-a)$
$f(at), \quad a > 0$	$\frac{1}{a} F(\frac{s}{a})$
$f(t-a)H(t-a), \quad a > 0$	$e^{-as} F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0)$ $\quad -s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$
$f * g(t) = \int_0^t f(u)g(t-u)\,du$	$F(s)G(s)$
Special functions	
$\delta(t)$	1
$H(t)$	$\frac{1}{s}$
$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$t \cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$\frac{1}{\sqrt{\pi t}}\, e^{-\frac{a^2}{4t}}, \quad a \geq 0$	$\frac{1}{\sqrt{s}}\, e^{-a\sqrt{s}}$
$\frac{a}{\sqrt{4\pi}\, t^{3/2}}\, e^{-\frac{a^2}{4t}}, \quad a > 0$	$e^{-a\sqrt{s}}$
$1 - \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right), \quad a > 0$	$\frac{1}{s}\, e^{-a\sqrt{s}}$

The Fourier transform

$f(t)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$
General rules	
$\alpha f(t) + \beta g(t)$	$\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$
$e^{i\alpha t} f(t)$	$\hat{f}(\omega - \alpha)$
$f(t - t_0)$	$e^{-it_0\omega} \hat{f}(\omega)$
$f(-t)$	$\hat{f}(-\omega)$
$f(at) \quad (a \neq 0)$	$\frac{1}{\lvert a \rvert} \hat{f}(\frac{\omega}{a})$
$tf(t)$	$i \frac{d\hat{f}}{d\omega}$
$f'(t)$	$i\omega \hat{f}(\omega)$
$\hat{f}(t)$	$2\pi f(-\omega)$
$f * g(t) = \int_{-\infty}^{\infty} f(u) g(t-u)\, du$	$\hat{f}(\omega)\hat{g}(\omega)$
Special functions	
$\chi_{[-a,a]}$	$\frac{2 \sin a\omega}{\omega}$
$e^{-\lvert t \rvert}$	$\frac{2}{1+\omega^2}$
$\frac{1}{1+t^2}$	$\pi e^{-\lvert \omega \rvert}$
$e^{-t^2/2}$	$\sqrt{2\pi} e^{-\omega^2/2}$

Plancherel's formulas:

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\, d\omega$$

$$\int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega)\, \overline{\hat{g}(\omega)}\, d\omega$$