UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
Jörgen Östensson

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Dynamical systems problems

1. Find the general solution and sketch the phase portrait for each of the following systems. Characterize the systems as to type (node etc.) and stability.

(a)
$$\begin{cases} x' = -3x + 4y \\ y' = -2x + 3y \end{cases}$$

(b)
$$\begin{cases} x' = 7x + 6y \\ y' = 2x + 6y \end{cases}$$

(c)
$$\begin{cases} x' = -x + y \\ y' = -x - y \end{cases}$$

2. Determine the values of $b \in \mathbb{R}$ for which the system

$$\mathbf{x}' = \begin{pmatrix} 3 & b \\ 1 & 1 \end{pmatrix} \mathbf{x}$$

undergoes a bifurcation.

3. Determine the critical points of the system

$$\begin{cases} x' &= x - y \\ y' &= x^2 + y^2 - 2 \end{cases}$$

and investigate their nature and stability properties.

4. Consider the system

$$\begin{cases} x' &= 4x + 4y - x(x^2 + y^2), \\ y' &= -4x + 4y - y(x^2 + y^2). \end{cases}$$

(a) Show that there is a closed orbit in the region $1 \le r \le 3$, where $r^2 = x^2 + y^2$.
(b) Find the general solution. (Hint: Use polar coordinates.)

Answers or hints:

1. (a) The matrix has eigenvalues 1 and -1 with eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}$ respectively $\begin{pmatrix}2\\1\end{pmatrix}$.
So $(0,0)$ is an (unstable) saddle point.
(b) The matrix has eigenvalues 10 and 3 with eigenvectors $\begin{pmatrix}2\\1\end{pmatrix}$ respectively $\begin{pmatrix}3\\-2\end{pmatrix}$.
So $(0,0)$ is an unstable node.
(c) The matrix has eigenvalues $-1+i$ and $-1-i$ with eigenvectors $\begin{pmatrix}1\\i\end{pmatrix}$ respectively $\begin{pmatrix}1\\-i\end{pmatrix}$.
So $(0,0)$ is an asymptotically stable spiral point.

2. The eigenvalues are $2\pm\sqrt{1+b}$. Thus, $(0,0)$ is an unstable spiral point if $b<-1$. If $-1<b<3$ both eigenvalues are positive, so in this case $(0,0)$ is an unstable node. If $b>3$ one eigenvalue is positive and one is negative, so in this case $(0,0)$ is saddle point. Hence, the system bifurcates at $b=-1$ and $b=3$.

3. The critical points are at $(1,1)$ and $(-1,-1)$. At $(1,1)$ the linearized system has coefficient matrix:

$$A=\begin{pmatrix}1 & -1\\ 2 & 2\end{pmatrix}.$$

The eigenvalues are $\frac{3\pm\sqrt{7}i}{2}$. Thus, $(1,1)$ is an unstable spiral point.
At $(-1,-1)$ the linearized system has coefficient matrix:

$$A=\begin{pmatrix}1 & -1\\ -2 & -2\end{pmatrix}.$$

The eigenvalues are $\frac{-1\pm\sqrt{17}}{2}$. Thus, $(1,1)$ is an (unstable) saddle point.

4. (a) Show: $r'>0$ when $r=1$ and $r'<0$ when $r=3$. Then use the Poincaré-Bendixson theorem.
(b) In polar coordinates the system becomes:

$$\begin{cases} r' &= 4r-r^3,\\ \theta' &= -4.\end{cases}$$

This gives

$$r(t)=\frac{2}{\sqrt{1+\frac{4-r_0^2}{r_0^2}e^{-8t}}},$$
$$\theta(t)=-4t+\theta_0.$$