UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Functional Analysis I, 5 hp
Spring 2010

Homework Assignment I

1. (a) If (x_n) is a Cauchy sequence in a metric space having a convergent subsequence, say, $x_{n_k} \longrightarrow x$, show that (x_n) is convergent with the limit x.
(b) Show that a Cauchy sequence in a metric space is bounded.

2. (a) Show that in a Banach space, an absolutely convergent series is convergent.
(b) If in a normed space X, any absolutely convergent series is convergent, show that X is complete. (Hint: Use (a) in the previous problem.)

3. (a) Let $A = (\alpha_{ij})$ be a complex $n \times n$ matrix. It defines an operator $A : \mathbb{C}^n \longrightarrow \mathbb{C}^n$, $x \longmapsto Ax$, through matrix multiplication. (Elements of $\mathbb{C}^n$ are considered as column vectors.) Compute the operator norm $||A||$ in terms of the matrix entries of A in case $\mathbb{C}^n$ is equipped with the norm

$$||x||_\infty = \max_{1 \le k \le n} |\xi_k| .$$

Do the same with respect to the norm

$$||x||_1 = \sum_{k=1}^{n} |\xi_k| .$$

(b) Find the norm of the operator $T : X \longrightarrow X$ given by $(Tf)(t) = tf(t)$, $0 \le t \le 1$, in the cases when $X = C[0,1]$ respectively $X = L^p[0,1]$ $(1 \le p < \infty)$.

4. Consider the subspace c_0 of l^∞ consisting of sequences of scalars converging to zero.
(a) Show that c_0 is a closed subspace of l^∞.
(b) Show that the dual space of c_0 is l^1.

Solutions should be handed in by Friday the 19th of February. You can either give them to me in class or leave them in my mailbox.