UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
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Functional Analysis I, 5 hp
Spring 2010

Homework Assignment II

1. (a) Show that in an inner product space, it holds that $x \perp y$ if and only if $\|x + \alpha y\| \geq \|x\|$ for all scalars α.

(b) Let H be the Hilbert space $L^2[0,1]$ and Y the closed subspace

$$\left\{ f \in H \,\middle|\, \int_0^1 f(x)\,dx = \int_0^1 x f(x)\,dx = 0 \right\}.$$

Determine the orthogonal projection P_Y of H onto Y. Compute $P_Y(x^2)$ explicitly.

2. (a) Let H be a Hilbert space and let $u, v \in H$. Show that the operator $T : H \longrightarrow H$ given by

$$Tx = \langle x, u \rangle v$$

is bounded and compute its norm. Also compute its Hilbert-adjoint operator T^*.

(b) The operator $T : l^2 \longrightarrow l^2$ is given by

$$T(\xi_1, \xi_2, \xi_3, \cdots) = (0, \xi_1, \xi_2, \xi_3, \cdots).$$

Is T a normal operator?

3. (a) If (x_n) in a Banach space X is such that $(f(x_n))$ is bounded for all $f \in X'$, show that $(\|x_n\|)$ is bounded.

(b) If X and Y are Banach spaces and $T_n \in B(X, Y)$, $n = 1, 2, \cdots$, show that equivalent statements are:

(1) $(\|T_n\|)$ is bounded,
(2) $(\|T_n x\|)$ is bounded for all $x \in X$,
(3) $(|g(T_n x)|)$ is bounded for all $x \in X$ and all $g \in Y'$.

(Hint: Use the uniform boundedness theorem.)

4. (a) Let X and Y be normed spaces, $T \in B(X, Y)$ and (x_n) a sequence in X.
If $x_n \xrightarrow{w} x_0$, show that $Tx_n \xrightarrow{w} Tx_0$.

(b) Show that weak convergence in footnote 6 on page 266 implies weak* convergence. Show that the converse holds if X is reflexive.

Solutions should be handed in by Monday the 8th of March. You can either give them to me in class or leave them in my mailbox.