UPPSALA UNIVERSITET
Matematiska institutionen
M. Klimek

Matematik för signalbehandling 1IT080
Informationsteknologi IT
2004-10-22

Solutions to Exam Problems from 2004–10–18

IMPORTANT NOTICE: The Beta book is meant only as a computational aid during the exam. The definitions of theoretical concepts may differ slightly from those in the textbook we have used. The definitions adopted in our course are standard and take precedence of those in the Beta book.

1. If $y < 0$ or $y > 1$, then $f_Y(y) = 0$. If $0 \le y \le 1$, then

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \frac{3}{4} dx = \frac{3}{2}\sqrt{1-y}.$$

Hence

$$E[X^2|Y=1/2] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|1/2)dx = \int_{-\infty}^{\infty} x^2 \frac{f_{X,Y}(x,1/2)}{f_Y(1/2)} dx = \int_{-\sqrt{1-y}}^{\sqrt{1-y}} x^2 \frac{1}{\sqrt{2}} dx = \frac{1}{6}.$$

2. If $n \neq m$, then X_m, X_n are independent and hence $E[X_m X_n] = E[X_m]E[X_n] = 1$. Moreover $E[X_m^2] = \mathrm{Var}[X_m] + E[X_m]^2 = 5$. Hence

$$R_X[k] = R_X[m,k] = E[X_m X_{m+k}] = \begin{cases} 5 & \text{if } k = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned} R_Y[m,k] &= E[Y_m Y_{m+k}] = 5R_X[k] - 2R_X[k-1] - 2R_X[k+1] = \\ &= \begin{cases} 1 & \text{if } |k| \ge 2, \\ -7 & \text{if } |k| = 1, \\ 21 & \text{if } k = 0. \end{cases} \end{aligned}$$

Moreover $C_Y[m,k] = R_Y[m,k] - 1$.

3.

$$\begin{aligned}
R_Y[n] &= \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} h[i]h[j]R_X[n+i-j] \\
&= \sum_{i=0}^{2}\sum_{j=0}^{2} R_X[n+i-j] \\
&= 3R_X[n] + 2R_X[n-1] + 2R_X[n+1] + R_X[n-2] + R_X[n+2] \\
&= \begin{cases} 14 & \text{if } n=0, \\ 12 & \text{if } |n|=1, \\ 6 & \text{if } |n|=2, \\ 2 & \text{if } |n|=3, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Note that because of symmetry it is enough to check only $n = 0, 1, 2, 3$.

4. It suffices to observe that $|z^5| > |-z+16|$ if $z \in C_2(0)$ and to use Rouche's theorem.

5. Within $C_{\sqrt{2}}(1+i)$ the function $f(z) = 1/[(z-1)^2(1+z^2)]$ has only two singularities: a double pole at 1 and a single pole at i. Hence the integral is equal to

$$2\pi i\,(\text{Res}[f(z), i] + \text{Res}[f(z), 1]) = 2\pi i\left(\frac{1}{4} - \frac{1}{2}\right) = -\frac{\pi i}{2}.$$

6. We have

$$X(z) = \frac{8/7}{2z-1} - \frac{4/7}{z+3} = \frac{4}{7}\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}z^{-n} - \frac{4}{21}\sum_{n=0}^{\infty}\left(-\frac{1}{3}\right)^n z^n.$$

Hence

$$x[n] = \begin{cases} \frac{4}{7}\frac{1}{2^{n-1}} & \text{if } n \geq 1, \\ -\frac{4}{21}(-3)^n & \text{if } n \leq 0. \end{cases}$$

7. The image is the set

$$\{z \in \mathbb{C} : \text{Im}(z) > 0,\ |z-(1+i)| > 1\}.$$

8. If X denotes the Laplace transform, then

$$s^2X(s) - sx(0) - x'(0) + 2(sX(s) - x(0)) + 5X(s) = \frac{4}{s+1}.$$

Hence

$$X(s) = \frac{s^2 + 4s + 7}{(s+1)(s^2+2s+5)}.$$

The poles are at -1 and $-1 \pm 2i$. Hence the solution is $x(t) = e^{-t}(1 + \sin 2t)$.