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Solutions to Exam Problems from 2005-01-12

IMPORTANT NOTICE: The Beta book is meant only as a computational aid during the exam. The definitions of theoretical concepts may differ slightly from those in the textbook we have used. The definitions adopted in our course are standard and take precedence of those in the Beta book.

1. If $x < 0$, then $f_X(x) = 0$. If $x \geq 0$, then

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y} dy = \lambda e^{-\lambda x}.$$

Therefore

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \lambda e^{-\lambda(y-x)}, & \text{if } y > x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

If $y \leq 0$, then $f_Y(y) = 0$. If $y > 0$, then

$$f_Y(y) = \int_0^y \lambda^2 e^{-\lambda y} dx = \lambda^2 y e^{-\lambda y}.$$

Therefore

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/y, & \text{if } y > x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

2. The average power of $X(t)$ is $E[X^2(t)] = 1$. Hence $Var[X(t)] = 1 - 4 = -3$.

Because of independence $E[X^2(t)\cos(2\pi t + Y)] = E[X^2(t)]E[\cos(2\pi t + Y)]$. But

$$\begin{aligned} E[\cos(2\pi t + Y)] &= \int_{-\infty}^\infty \cos(2\pi t + y) f_Y(y) dy = \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi t + y) dy \\ &= \frac{1}{2\pi} \sin(2\pi t + y) \Big|_0^{2\pi} = 0. \end{aligned}$$

3. Note that

$$R_X[n] = \begin{cases} 1/4, & \text{if } n = \pm 1, \\ 1/2, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$S_X(\phi) = \sum_{n=-1}^{1} R_X[n] e^{-2\pi i n\phi} = \frac{1}{4} e^{2\pi i\phi} + \frac{1}{2} + \frac{1}{4} e^{-2\pi i\phi} = \frac{1}{2} + \frac{1}{2}\cos(2\pi\phi).$$

Since the cosine function can take any value in the interval $[-1, 1]$, $S_X(\phi)$ can take any value in the interval $[0, 1]$

4. Let $f(z) = z^5 + 15z + 1$. If $g_1(z) = -z^5$ and $|z| = 2$, then $|f(z) + g_1(z)| = |15z + 1| \leq 31 < 32 = |g_1(z)|$. Hence, by Rouche's theorem f has exactly 5 roots in $D_2(0)$. If $g_2(z) = -15z$ and $|z| = 3/2$, then $|f(z) + g_2(z)| = |z^5 + 1| \leq 243/32 + 1 < 22.5 = |g_2(z)|$. Hence, by Rouche's theorem f has exactly 1 root in $D_{3/2}(0)$. Consequently f has exactly 4 roots in the annulus.

5. The function $f(z) = z^2/(z^4 + 4)$ has four poles of order one at the points $\pm(1 + i), \pm(-1 + i)$, since

$$z^4 + 4 = (z^2 - 2i)(z^2 + 2i) = (z - (1 + i))(z - (-1 - i))(z - (-1 + i))(z - (1 - i)).$$

Hence

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \left[\text{Res}_{1+i} f(z) + \text{Res}_{-1+i} f(z)\right] = 2\pi i \left[\frac{1}{8} - \frac{i}{8} - \frac{1}{8} - \frac{i}{8}\right] = \frac{\pi}{2}.$$

6. We have

$$\frac{2}{(z-1)(4-z)} = \frac{2/3}{z-1} + \frac{2/3}{4-z}.$$

Moreover, if $|z| > 1$, then

$$\frac{1}{z-1} = \frac{1}{z} \frac{1}{1 - 1/z} = \sum_{n=1}^{\infty} z^{-n}$$

and if $|z| < 4$, then

$$\frac{1}{4-z} = \frac{1}{4} \cdot \frac{1}{1-z/4} = \sum_{n=0}^{\infty} 4^{-(n+1)} z^n = \sum_{n=-\infty}^{0} 4^{n-1} z^{-n}.$$

Hence

$$x[n] = \begin{cases} (2/3)4^{n-1}, & \text{if } n \leq 0, \\ 2/3, & \text{if } n > 0. \end{cases}$$

7. We have $f(z) = T(e^z)$, where $T(z) = (z-i)/(z+i) = 1 - 2i/(z+i)$. The exponential function $e^z = e^{x+iy}$ maps the strip onto lower half-plane. T maps the lower half-plane onto the outside of the unit disc.

8. We have

$$s^2 X(s) - sx(0) - x'(0) + sX(s) - x(0) - 2X(s) = 0,$$

and thus

$$s^2 X(s) - 2s - 4 + sX(s) - 1 - 2X(s) = 0.$$

Hence

$$X(s) = \frac{2s+5}{s^2+s-2} = \frac{2s+5}{(s-1)(s+2)} = \frac{2}{s-1} - \frac{1}{s+2},$$

and so

$$x(t) = \mathcal{L}^{-1}[X(s)] = 2\mathcal{L}\left[\frac{1}{s-1}\right] - \mathcal{L}\left[\frac{1}{s+2}\right] = 2e^t - e^{-2t}.$$