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Matematik för signalbehandling 1IT080
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2005-10-25

Solutions to Exam Problems from 2005-10-21

1. If $|x|, |y| \le 1$, then

$$f_Y(y) = \int_{\infty}^{\infty} f_{X,Y}(x,y)dx = \int_{-1}^{1} \frac{3}{8}(1-x^2)dx = \frac{1}{2},$$

so Y has uniform distribution in the interval $[-1,1]$. Hence

$$\begin{aligned} \mathrm{Var}[X \mid Y = 1/3] &= \mathrm{E}[X^2 \mid Y = 1/3] - \mathrm{E}[X \mid Y = 1/3]^2 \\ &= \int_{-1}^{1} x^2 \frac{\frac{3}{8}(1-x^2)}{\frac{1}{2}} dx - \left(\int_{-1}^{1} x \frac{\frac{3}{8}(1-x^2)}{\frac{1}{2}} dx \right)^2 = \\ &= \frac{1}{5}. \end{aligned}$$

(The last integral is zero because the integrated function is odd.)

2. Since we are dealing with an IID sequence, it follows that $R_X[n] = \mathrm{Var}[X]\delta[n] + \mathrm{E}[X]^2 = 5\delta[n] + 4$. Therefore

$$\begin{aligned} R_Y[n] &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h[i]h[j]R_X[n+i-j] = \sum_{i=0}^{2} \sum_{j=0}^{2} \frac{1}{3} \cdot \frac{1}{3} \cdot (5\ \delta[n+i-j] + 4) \\ &= 4 + \frac{5}{9} \left\{ \begin{array}{ll} 3 & \text{if } n = 0, \\ 2 & \text{if } |n| = 1, \\ 1 & \text{if } |n| = 2, \\ 0 & \text{otherwise .} \end{array} \right\} = \left\{ \begin{array}{ll} 17/3 & \text{if } n = 0, \\ 46/9 & \text{if } |n| = 1, \\ 41/9 & \text{if } |n| = 2, \\ 4 & \text{otherwise .} \end{array} \right. \end{aligned}$$

3. Directly from the definition of convolutions we have

$$\mathrm{rect} * \mathrm{rect}(t) = \int_{-1/2}^{1/2} \mathrm{rect}(t-\tau)d\tau = \left\{ \begin{array}{ll} 1 - |t| & \text{if } |t| \le 1, \\ 0 & \text{otherwise .} \end{array} \right.$$

Since the Fourier transform of $\text{rect}(t)$ if $\text{sinc}(f)$, the convolution property implies that the Fourier transform of $\text{rect} * \text{rect}(t)$ is $\text{sinc}^2(f)$.

4. If $f(z) = 3z^3 - 2z^2 - z - 7$ and $g(z) = 7$, then if $|z| = 1$,

$$|f(z) + g(z)| = |3z^3 - 2z^2 - z| \leq 3 + 2 + 1 = 6 < 7 = |g(z)|.$$

By Rouche's theorem f has no zeros in $D_1(0)$ because g has no zeros there. Now, if $h(z) = -3z^3$, then if $|z| = 2$,

$$|f(z) + h(z)| = |-2z^2 - z - 7| \leq 8 + 2 + 7 = 17 < 24 = |h(z)|.$$

By Rouche's theorem f has 3 zeros in $D_2(0)$ because h has 3 zeros there.

5. Since $z^4 + 1 = (z^2 - i)(z^2 + i)$ the roots of the equation $z^4 + 1 = 0$ are simply the square roots of i and $-i$, that is:

$$\pm\frac{1+i}{\sqrt{2}}, \ \pm\frac{1-i}{\sqrt{2}}.$$

Only two of them are in the upper half-plane: $(\pm 1 + i)/\sqrt{2}$. The function $f(z) = z^2/(z^4 + 1)$ has poles of order 1 at these points. Hence

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= 2\pi i \left\{ \text{Res}\left[f(z), (1+i)/\sqrt{2}\right] + \text{Res}\left[f(z), (-1+i)/\sqrt{2}\right] \right\} \\ &= 2\pi i \left(\frac{1-i}{2\sqrt{2}} + \frac{-1-i}{2\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}. \end{aligned}$$

6. Since $|z| > 1$, we have

$$\begin{aligned} X(z) &= \frac{3}{(2z-i)(z+i)} = \frac{i}{z+i} - \frac{2i}{2z-i} = \frac{i}{z} \cdot \frac{1}{1 - \frac{-i}{z}} - \frac{i}{z} \cdot \frac{1}{1 - \frac{i}{2z}} \\ &= -\sum_{n=1}^{\infty} (-i)^n z^{-n} - \sum_{n=1}^{\infty} 2 \left(\frac{i}{2}\right)^n z^{-n} \end{aligned}$$

Therefore

$$x[n] = \begin{cases} -(-i)^n - 2^{1-n} i^n & \text{if } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

7. Let $S = \{z \in \mathbb{C} : \pi < \operatorname{Im} z < 2\pi\}$. Since

$$T(z) = i + \frac{1}{e^z - i},$$

finding of the image of S through T can be split into the following sequence of operations:

- $z = x + iy \mapsto e^z = e^x e^{iy}$ maps S onto $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$;
- $z \mapsto z - i$ maps $\{z \in \mathbb{C} : \operatorname{Im} z < 0\}$ onto $\{z \in \mathbb{C} : \operatorname{Im} z < -1\}$;
- $z \mapsto 1/z$ maps $\{z \in \mathbb{C} : \operatorname{Im} z < -1\}$ onto $D_{1/2}(i/2)$;
- $z \mapsto z + i$ maps $D_{1/2}(i/2)$ onto $D_{1/2}(3i/2)$.

8. Applying the unilateral Laplace transform to both sides of the differential equation gives:

$$s^2 X(s) - x'(0) - sx(0) + sX(s) - x(0) - 2X(s) = 0.$$

Substituting the values of x and x' at 0 and solving for $X(s)$ gives

$$X(s) = \frac{5 + 2s}{s^2 + s - 2}.$$