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Solutions to Exam Problems from 2006-01-10

1. We have

$$f_Y(y) = \begin{cases} \int_0^y 9e^{-3y}dx = 9ye^{-3y} & \text{if } y \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{X|Y}(y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/y & \text{if } 0 \leq x < y, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\mathrm{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x,y)dx = \begin{cases} y/2 & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently $\mathrm{E}[X|Y] = Y/2$.

2. Obviously $\mathrm{E}[S] = 1$ and $\mathrm{E}[S^2] = 2$. Thus $\mu_X(t) = t - 1$. The autocorrelation function is

$$\begin{aligned} R_X(t,\tau) &= \mathrm{E}[X(t)X(t+\tau)] = \mathrm{E}[(t-S)(t+\tau-S)] \\ &= t^2 + t\tau - 2t\mathrm{E}[S] - \tau\mathrm{E}[S] + \mathrm{E}[S^2] = t^2 + t\tau - 2t - \tau + 2 \\ &= (t-1)(t+\tau-1) + 1 \end{aligned}$$

and the autocovariance function is

$$C_X(t,\tau) = R_X(t,\tau) - \mu_X(t)\mu_X(t+\tau) = 1.$$

3. The discrete time Fourier transform of h_n is

$$H(\phi) = \sum_{n=-\infty}^{\infty} h_n e^{-2\pi i\phi n} = \frac{1}{3}(1 + e^{-2\pi i\phi} + e^{-4\pi i\phi})$$

Since the random variables X_n are IID, $R_X[n] = 9\delta[n]$ and so $S_X(\phi) = 9$. Therefore

$$S_Y(\phi) = |H(\phi)|^2 S_X(\phi) = (1 + e^{-2\pi i\phi} + e^{-4\pi i\phi})(1 + e^{2\pi i\phi} + e^{4\pi i\phi}).$$

4. If $f(z) = z^5$, then if $|z| = 1$ and $x = \operatorname{Re} z$ we have

$$|f(z) + g(z)| = |e^{z-2}| = e^{x-2} < 1 = |z^5| = |f(z)|.$$

Rouche's theorem implies the required conclusion as f has a zero of order 5 at $z = 0$.

5. In the upper half-plane the function

$$f(z) = \frac{1}{(z^2+1)^2(z^2+4)} = \frac{1}{(z+i)^2(z-i)^2(z+2i)(z-2i)}$$

has a pole of order 2 at $z = i$ and a pole of order 1 at $z = 2i$. The integrai we seek is equal to

$$2\pi i \left\{ \left(\frac{1}{(z+i)^2(z^2+4)} \right)' \Big|_{z=i} + \left(\frac{1}{(z^2+1)^2(z+2i)} \right) \Big|_{z=2i} \right\} = 2\pi i \left(\frac{-i}{36} + \frac{-i}{36} \right) = \frac{\pi}{9}.$$

6.

$$\begin{aligned} X(z) &= \frac{1}{z}\left(\frac{1}{z-i}\right)' = \frac{1}{z}\left(\frac{i}{1-(-iz)}\right)' = \frac{i}{z}\left(\sum_{n=0}^{\infty}(-i)^n z^n\right)' \\ &= \frac{i}{z}\left(\sum_{n=1}^{\infty} n(-i)^n z^{n-1}\right) = \sum_{n=1}^{\infty} -n(-i)^{n+1} z^{n-2} \\ &= \sum_{m=-\infty}^{1} -(2-m)(-i)^{3-m} z^{-m} \end{aligned}$$

Hence $x(m) = (2-m)(-i)^{1-m}$ for $m \le 1$ and $x(m) = 0$ for $m > 1$.

7. Since

$$T(z) = \left(\frac{-(1-z)+2}{1-z}\right)^2 = \left(-1 + \frac{2}{1-z}\right)^2$$

it is easy to chech that the image is the upper half-plane.

8. As $s^2X(s) - x'(0) - sx(0) + 2(sX(s) - x(0)) + X(s) = 4!/s^5$, it follows that $X(s) = (4!s^{-5} + s + 4)/(s^2 + 2s + 1)$.