LÖSNINGAR – TENTAMEN 2006-10-20

① $f(z) = z^5 - 3z^2 + z + 2$

Sätt $g(z) = -z^5$

$$|f(z) + g(z)| = |-3z^2 + z + 2| \leq 3|z|^2 + |z| + 2 \overset{|z|=2}{=} 16$$

$$|g(z)| = |z|^5 \overset{|z|=2}{=} 32$$

$$16 < 32$$

Rouchés sats $\Rightarrow$ $f(z)$ och $g(z)$ har lika många nollställen i $|z| < 2$, dvs. 5 stycken.

② $y(n+1) - a\,y(n) = 0 \qquad y(0) = 1$

Z-transformera:

$$z(Y(z) - y(0)) - a\,Y(z) = 0$$

$$(z-a)Y(z) = z$$

$$Y(z) = \frac{z}{z-a}$$

$\Rightarrow \; y(n) = a^n \quad$ om $a \neq 0$

$a = 0$: $\quad Y(z) = \frac{z}{z} = 1$

$\Rightarrow \; y(n) = \delta(n)$

3. $$\int_{C_2(2)} \frac{z^2}{z^3-1}\,dz$$

poler: $z^3-1=0 \iff z=1,\ z=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

endast $z=1$ ligger innanför $C_2(2)$

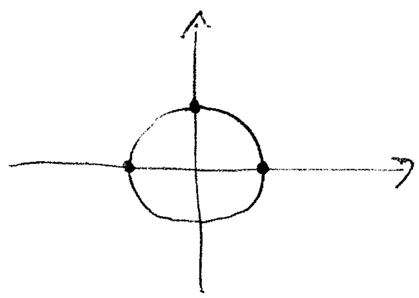
Residuesatsen $\Rightarrow$

$$\int_{C_2(2)} \frac{z^2}{z^3-1}\,dz = 2\pi i \cdot \mathrm{Res}\left[\frac{z^2}{z^3-1}, 1\right]$$

$$\mathrm{Res}\left[\frac{z^2}{z^3-1}, 1\right] = \lim_{z\to 1} (z-1)\frac{z^2}{z^3-1} = \lim_{z\to 1}\frac{z^2}{z^2+z+1} = \frac{1}{3}$$

$$\Rightarrow \int_{C_2(2)} \frac{z^2}{z^3-1}\,dz = \frac{2\pi i}{3}$$

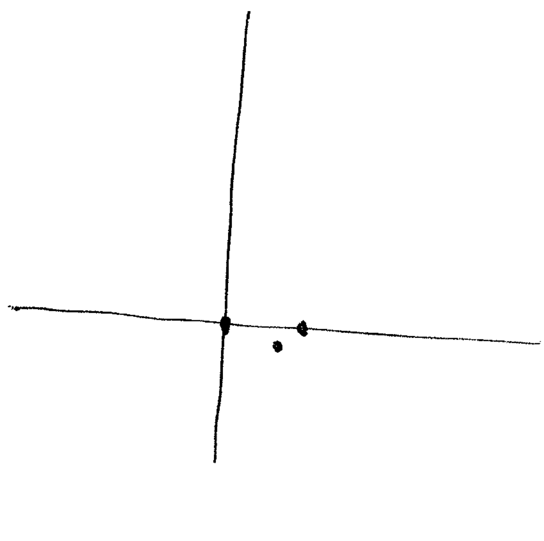
4. $$T(z) = \frac{z+1}{3z+1}$$



$T(1) = \frac{1}{2}$

$T(i) = \frac{i+1}{3i+1} = \frac{2}{5} - i\cdot\frac{1}{5}$

$T(-1) = 0$



Reella axeln avbildas på reella axeln, och enhetscirkeln står reella axeln i rät vinkel i 1 och −1. $T(z)$ är en Möbiusavbildning och således vinkelbevarande. Alltså står bildcirkeln reella axeln i rät vinkel, dvs. $[0, 1/2]$ är diameter till bildcirkeln, som alltså är $C_{1/4}(1/4)$. $T(0)=1 \Rightarrow T(D_1(0)) =$ det yttre av $D_{\frac{1}{4}}(\frac{1}{4})$

4.) Alt

$$w = \frac{z+1}{3z+1} = T(z)$$

$$\Rightarrow \quad z = \frac{1-w}{3w-1}$$

$$|z|^2 = 1 = \left|\frac{1-w}{3w-1}\right|^2 = \left|\frac{1-(u+iv)}{3(u+iv)-1}\right|^2 = \frac{(1-u)^2+v^2}{(3u-1)^2+9v^2}$$

$$(1-u)^2+v^2 = (3u-1)^2+9v^2$$

$$8u^2-4u+8v^2=0$$

$$8\left(u-\frac{1}{4}\right)^2+8v^2=\frac{1}{2}$$

$$\left(u-\frac{1}{4}\right)^2+v^2=\left(\frac{1}{4}\right)^2$$

$$\Rightarrow T(C_1(0)) = C_{\frac{1}{4}}\left(\tfrac{1}{4}\right)$$

$T(0)=1 \Rightarrow T(D_1(0)) =$ det ytre av $C_{\frac{1}{4}}\left(\frac{1}{4}\right)$

5.) $y''+4y'+3y=0 \qquad y(0)=0,\ y'(0)=1$

Laplacetransformerer:

$$s^2Y(s) - s\cdot \underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=1} + 4\left(sY(s)-y(0)\right) + 3Y(s) = 0$$

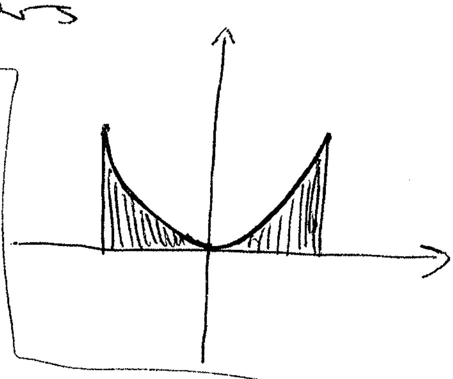
$$(s^2+4s+3)Y(s)=1$$

$$Y(s) = \frac{1}{(s+1)(s+3)} = \cdots = \frac{1}{2}\left(\frac{1}{s+1}-\frac{1}{s+3}\right)$$

$$\Rightarrow \quad y(t)=\frac{1}{2}\left(e^{-t}-e^{-3t}\right)$$

6. [redacted] $f_{X,Y}(x,y) = \begin{cases} \dfrac{4x+3}{2} & 0 \le y \le x^2 \le 1 \\ 0 & \text{altres} \end{cases}$

a) $\boxed{E(XY)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy =$

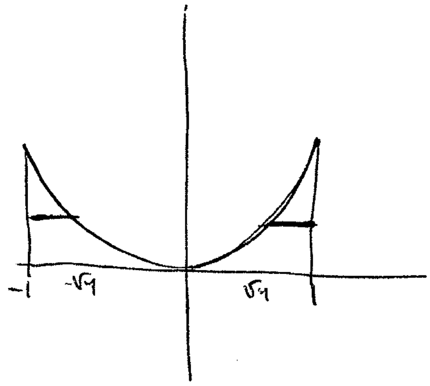


$= \int_{-1}^{1} \left(\int_{0}^{x^2} \frac{xy \cdot (4x+3)}{2}\, dy \right) dx =$

$= \int_{-1}^{1} \left[\frac{xy^2(4x+3)}{4} \right]_0^{x^2} dx = \int_{-1}^{1} \frac{4x^6+3x^5}{4} = \left[\frac{x^7}{7} + \frac{x^6}{8} \right]_{-1}^{1} = \frac{2}{7}$

b) $f_Y(y) = \int_{-1}^{-\sqrt{y}} \frac{4x+3}{2}\, dx + \int_{\sqrt{y}}^{1} \frac{4x+3}{2}\, dx =$

$= \left[x^2 + \frac{3}{2}x \right]_{-1}^{-\sqrt{y}} + \left[x^2 + \frac{3x}{2} \right]_{\sqrt{y}}^{1} =$



$= \begin{cases} 3 - 3\sqrt{y} & 0 \le y \le 1 \\ 0 & \text{altres} \end{cases}$

c) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{6} \cdot \frac{4x+3}{1-\sqrt{y}}, & 0 \le y \le x^2 \le 1 \\ 0 & \text{altres} \end{cases}$

d) $\underline{E(X \mid Y=y)} = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx =$

$$= \frac{1}{6}\left(\int_{-1}^{-\sqrt{y}} \frac{4x^2+3x}{1-\sqrt{y}}\,dx + \int_{\sqrt{y}}^{1} \frac{4x^2+3x}{1-\sqrt{y}}\,dx\right) =$$

$$= \frac{1}{6(1-\sqrt{y})}\left(\left[\frac{4x^3}{3} + \frac{3x^2}{2}\right]_{-1}^{-\sqrt{y}} + \left[\frac{4x^3}{3} + \frac{3x^2}{2}\right]_{\sqrt{y}}^{1}\right) =$$

$$= \underline{\underline{\frac{4}{9}\left(\frac{1-y^{3/2}}{1-y^{1/2}}\right)}}$$

7) Betrakta $\underline{f_{Y(t_1+\tau), Y(t_2+\tau), ---, Y(t_m+\tau)}(y_1, -, y_m)} =$

$$= f_{X(a(t_1+\tau)+b), ---, X(a(t_m+\tau)+b)}(y_1, -, y_m) =$$

$$= f_{X(at_1+a\tau+b), ---, X(at_m+a\tau+b)}(y_1, -, y_m) \stackrel{X \text{ stationär}}{=}$$

$$= f_{X(at_1+b), ---, X(at_m+b)}(y_1, -, y_m) = \underline{f_{X(t_1), -, Y(t_m)}(y_1, -, y_m)}$$

$\Rightarrow Y$ stationär

8.) a) $S_{XY}(f) = H(f) S_X(f)$

$$H(f) = \frac{1}{7 + i2\pi f}$$

$$S_X(f) = \frac{8}{16 + (2\pi f)^2}$$

$$\underline{S_{XY}(f) = \frac{1}{7 + i2\pi f} \cdot \frac{8}{16 + (2\pi f)^2}}$$

b) $R_{XY}(\tau)$ - inverstransform av $S_{XY}(f)$

$$S_{XY}(f) = \frac{8}{(7 + i2\pi f)(4 + i2\pi f)(4 - i2\pi f)} = \{ \text{sätt } i2\pi f = x \} =$$

$$= \frac{8}{(7+x)(4+x)(4-x)} = \dots = \frac{-8/33}{7+x} + \frac{1/3}{4+x} + \frac{1/11}{4-x} =$$

$$= \frac{-8/33}{7 + i2\pi f} + \frac{1/3}{4 + i2\pi f} + \frac{1/11}{4 - i2\pi f} =$$

$$= \frac{-8/33}{7 + i2\pi f} + \frac{8/33}{4 + i2\pi f} + \frac{1/11}{4 + i2\pi f} + \frac{1/11}{4 - i2\pi f} =$$

$$= \frac{-8/33}{7 + i2\pi f} + \frac{8/33}{4 + i2\pi f} + \frac{8/11}{4^2 + (2\pi f)^2}$$

$$\Rightarrow \underline{\underline{R_{XY}(\tau) = \frac{-8}{33} e^{-7\tau} u(\tau) + \frac{8}{33} e^{-4\tau} u(\tau) + \frac{1}{11} e^{-4|\tau|}}}$$