Formelsamling till kursen i Transformmetoder

Z-transformen

$a = (a_n)_0^\infty$	$A(z) = \mathcal{Z}[a](z) = \sum_{n=0}^{\infty} a_n z^{-n}$
Allmänna regler	
$\alpha a_n + \beta b_n$	$\alpha A(z) + \beta B(z)$
$\lambda^n a_n$	$A(\frac{z}{\lambda})$
a_{n-k}, där $k \geq 1$ och $a_{-1} = \cdots = a_{-k} = 0$	$z^{-k} A(z)$
a_{n+k}, där $k \geq 1$	$z^k A(z) - a_0 z^k - a_1 z^{k-1} - \cdots - a_{k-1} z$
$n a_n$	$-zA'(z)$
$(a * b)_n = \sum_{k=0}^{n} a_k b_{n-k}$	$A(z)B(z)$
Speciella transformer	
$\begin{cases} 1 & \text{om } n = 0 \\ 0 & \text{om } n \geq 1 \end{cases}$	1
1	$\frac{z}{z-1}$
λ^n	$\frac{z}{z-\lambda}$
n	$\frac{z}{(z-1)^2}$
$n\lambda^n$	$\frac{\lambda z}{(z-\lambda)^2}$
n^2	$\frac{z^2+z}{(z-1)^3}$
$\binom{n}{k}\lambda^{n-k}$	$\frac{z}{(z-\lambda)^{k+1}}$
$\cos \alpha n$	$\frac{z^2 - z\cos\alpha}{z^2 - 2z\cos\alpha + 1}$
$\sin \alpha n$	$\frac{z\sin\alpha}{z^2 - 2z\cos\alpha + 1}$

Laplacetransformen

$f(t)$	$\tilde{f}(s) = F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt$
Allmänna regler	
$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
$e^{at}f(t)$	$F(s-a)$
$f(at), \quad a > 0$	$\frac{1}{a}F(\frac{s}{a})$
$f(t-a)H(t-a), \quad a > 0$	$e^{-as}F(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0)$ $-s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
$\int_0^t f(u)\,du$	$s^{-1}F(s)$
$f * g(t) = \int_0^t f(u)g(t-u)\,du$	$F(s)G(s)$
Speciella funktioner	
$\delta(t)$	1
$H(t)$	$\frac{1}{s}$
$t^n, \quad n = 0, 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sin at$	$\frac{a}{s^2+a^2}$
$t\cos at$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$t\sin at$	$\frac{2as}{(s^2+a^2)^2}$

Fouriertransformen

$f(t)$	$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$
Allmänna regler	
$\alpha f(t) + \beta g(t)$	$\alpha \hat{f}(\omega) + \beta \hat{g}(\omega)$
$e^{i\alpha t} f(t)$	$\hat{f}(\omega - \alpha)$
$f(t - t_0)$	$e^{-it_0\omega}\hat{f}(\omega)$
$f(-t)$	$\hat{f}(-\omega)$
$f(at) \quad (a \neq 0)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$
$tf(t)$	$i\,\frac{d\hat{f}}{d\omega}$
$f'(t)$	$i\omega\hat{f}(\omega)$
$\hat{f}(t)$	$2\pi f(-\omega)$
$f * g(t) = \int_{-\infty}^{\infty} f(u)g(t-u)\,du$	$\hat{f}(\omega)\hat{g}(\omega)$
Speciella funktioner	
$\chi_{[-a,a]}$	$\frac{2\sin a\omega}{\omega}$
$e^{-\lvert t\rvert}$	$\frac{2}{1+\omega^2}$
$\frac{1}{1+t^2}$	$\pi e^{-\lvert\omega\rvert}$
$e^{-t^2/2}$	$\sqrt{2\pi}e^{-\omega^2/2}$

Plancherels formler:

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\omega)|^2\,d\omega$$

$$\int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,d\omega$$

Fourierserier

Funktioner med period 2π

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{int} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$

där

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt$$

$$a_n = c_n + c_{-n}, \qquad b_n = i(c_n - c_{-n})$$

Parsevals formel:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Funktioner med period T

Sätt $\Omega = 2\pi/T$

$$f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{in\Omega t} = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t),$$

där

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-in\Omega t}\, dt$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\Omega t\, dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\Omega t\, dt.$$

Parsevals formel:

$$\frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2\, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (|a_n|^2 + |b_n|^2).$$

Några trigonometriska formler

$$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$$

$$2 \sin a \cos b = \sin(a-b) + \sin(a+b)$$

$$2 \cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2 \sin^2 t = 1 - \cos 2t, \qquad 2 \cos^2 t = 1 + \cos 2t$$