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# AN ANALOGUE OF THE JONES POLYNOMIAL FOR LINKS IN $\mathbb{R}P^3$ AND A GENERALIZATION OF THE KAUFFMAN-MURASUGI THEOREM

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ABSTRACT. We define analogues of the Jones polynomial for links in the projective space $\mathbb{R}P^3$. We prove corresponding generalizations of the Kauffman-Murasugi theorem on the connection between the combinatorial properties of the diagram of a link and the properties of the Jones polynomial. Finally, we study criteria for isotopy of a link in the space $\mathbb{R}P^3$ to a link lying in an affine part of the space.

## §1. Introduction

The polynomial invariant of a link in the 3-sphere discovered by Jones in 1985 has turned out to be closely related to the combinatorial properties of the diagram of the link. In particular, Kauffman [1] and Murasugi [2] have used it to verify two old conjectures of Tait concerning the diagrams of alternating links. The question naturally arises of carrying over the Jones polynomial to the case of links in 3-manifolds other than the sphere. In the present paper we study links in real projective 3-space $\mathbb{R}P^3$. The main results are listed below.

**1.1. Diagrams.** Links in $\mathbb{R}P^3$ can be specified by diagrams that differ from the usual diagrams of links in $\mathbb{R}^3$ in that they are given not on a plane, but in a disk, and the endpoints of arcs that go out to the boundary of the disk are divided into pairs of diametrically opposite points. In §2.4 we define five diagram transformations generalizing the Reidemeister transformations of diagrams of ordinary links and having the property that two links in $\mathbb{R}P^3$ are isotopic if and only if their diagrams can be joined by a sequence of such transformations.

**1.2. Polynomials.** By means of Kauffman's approach [1], the bracket polynomial of a framed link and the Jones polynomial of an oriented link are generalized from the case of a link in $\mathbb{R}^3$ to that of a link in $\mathbb{R}P^3$. (Here the word "generalized" is understood in the sense that for a link lying in $\mathbb{R}^3 \subset \mathbb{R}P^3$, the new polynomials coincide with the old.) The Jones polynomial of an oriented link $L$ in $\mathbb{R}P^3$ will be denoted by $V_L$.

**1.3. Bounds on the number of double points of a diagram.** By a *net* we mean the image in $\mathbb{R}P^2$ of a set of circles under a general-position immersion, i.e., an immersion for which the inverse image of any point consists of at most two points and all double points are points of transversal intersection. Every diagram determines a net in the projective plane obtained from the disk of the diagram by identifying opposite points of the boundary. Let $D$ be the diagram of a link $L$. We denote by $c(D)$ the number of double points in $D$, and by

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$r(D)$ the number of connected components of the corresponding net. For a Laurent polynomial $V$ in one variable, we denote by $\operatorname{span}(V)$ the difference between the highest and lowest exponents of the terms of $V$.

THEOREM 1. *Let $D$ be the diagram of a link $L$ in $\mathbb{R}P^3$. Then*

$$4(c(D) + r(D) - 1) \geq \operatorname{span}(V_L).$$

This generalizes the Kauffman-Murasugi inequality; see, e.g., [3], Theorem 1, (i). For diagrams of a special form the inequality can be strengthened.

A net is said to be *separating* if it is the common boundary of two subsets of its complement in $\mathbb{R}P^2$. It is called *contractible* if its imbedding into $\mathbb{R}P^2$ is homotopic to a constant mapping. Clearly, a net is contractible if and only if its complement contains a one-sidedly imbedded circle in $\mathbb{R}P^2$. Furthermore, obviously, every contractible net is separating. The net corresponding to a link diagram separates $\mathbb{R}P^2$ if and only if the link is homologous to zero in $\mathbb{R}P^3$. If the net corresponding to the diagram is contractible, then obviously the link is contractible in $\mathbb{R}P^3$ and, moreover, is isotopic to a link that lies in an affine part of $\mathbb{R}P^3$ (i.e., that fails to intersect some projective plane). The converse is false.

THEOREM 2. *For a diagram whose net is separating and noncontractible,*

$$4(c(D) + r(D)) - 6 \geq \operatorname{span}(V_L).$$

THEOREM 3. *For a diagram whose net is nonseparating and contains $2p$ one-sidedly imbedded circles without common edges (but of course with common vertices),*

$$4(c(D) + r(D) - 1 - p) \geq \operatorname{span}(V_L).$$

We note that the number $\operatorname{span}(V_L)$ is always even (see §3.2).

**1.4. Extremal properties of the bounds.** We recall that the diagram of a link in $\mathbb{R}^3$ is said to be alternating if, along it, underpasses and overpasses alternate. For a link in $\mathbb{R}P^3$, the diagram is called *alternating* if, along it, underpasses and overpasses alternate when and only when the arc between successive double points either fails to intersect the boundary of the disk of the diagram or intersects it in $4k$ points, $k = 1, 2, 3, \ldots$. As shown below in §4.1, the net of an alternating diagram is separating.

A diagram with a separating net is called *weakly alternating* if it is the connected sum (in the sense explained in §3.3) of alternating diagrams. A diagram with nonseparating net is called *weakly alternating* if it is the connected sum of diagrams that are all alternating except one, and for this exceptional one the alternating condition fails on exactly one edge.

A diagram is called *reduced* if there is no two-sidedly imbedded circle in $\mathbb{R}P^2$ that intersects the net of the diagram in exactly two points near a double point, as in Figure 1.

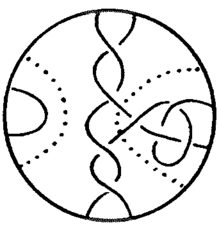


FIGURE 1

The following Theorems 4 and 5 describe the extremal properties of the inequality of Theorem 1. Theorem 4 generalizes the Kauffman-Murasugi theorem; see [3], Theorem 1, (ii).

THEOREM 4. *For a diagram with a separating net,* $4(c(D)+r(D)-1) = \mathrm{span}(V_L)$ *if and only if* $D$ *is a weakly alternating reduced diagram with a contractible net.*

THEOREM 5. *For a diagram with a nonseparating net,* $4(c(D)+r(D)-1) = \mathrm{span}(V_L)$ *if and only if* $D$ *is a weakly alternating reduced diagram.*

The extremal properties of the inequality of Theorem 2 are described by Theorem 6:

THEOREM 6. *For a diagram with a separating net,* $4(c(D)+r(D))-6 = \mathrm{span}(V_L)$ *if and only if* $D$ *is a weakly alternating reduced diagram with a noncontractible net.*

COROLLARY TO THE THEOREMS. *Two weakly alternating reduced diagrams of isotopic links in* $\mathbb{R}P^3$ *have the same number of double points. This number is the smallest number of double points for all diagrams of links of the given isotopy type. Any diagram of a link of this isotopy type with this smallest number of double points is weakly alternating and reduced.*

However, the corollary is also easily derived from the Kauffman-Murasugi theorem (cf., e.g., [3], p. 207) by passing to the diagrams of the inverse images of the links under the covering $S^3 \to \mathbb{R}P^3$.

The Kauffman-Murasugi theorem, of which Theorem 4 is a generalization, has also been generalized by Lickorish and Thistlethwaite [4], in the case of links in $S^3$, to a larger class of diagrams—that of the so-called adequate diagrams. A similar generalization is given below in §6.6 for links in $\mathbb{R}P^3$.

**1.5. Affine links.** A link in $\mathbb{R}P^3$ is called *affine* if it is isotopic to a link in an affine part of $\mathbb{R}P^3$. In §7 we prove the following two theorems, which provide affineness criteria in terms of the Jones polynomial.

THEOREM 7. *Let* $L$ *be a link in* $\mathbb{R}P^3$ *with* $k$ *components. If there is a term in* $V_L$ *whose degree is not congruent to* $2(k-1) \bmod 4$, *then the link is nonaffine.*

THEOREM 8. *A link* $L$ *represented by an alternating diagram is affine if and only if* $\mathrm{span}(V_L) \equiv 0 \bmod 4$.

COROLLARY. *A link represented by an alternating diagram is affine if and only if the net corresponding to the diagram is contractible in* $\mathbb{R}P^2$.

In §7 we also discuss more elementary necessary conditions for affineness of knots in $\mathbb{R}P^3$ (in terms of the coefficient of self-linking of such a knot).

**1.6. Arrangement of subject matter.** In §2 we define the diagrams of links in $\mathbb{R}P^3$. In §3 we construct the polynomial of a framed link and the Jones polynomial of an oriented link. In §4 we study the properties of alternating diagrams. In §5 are collected the lemmas that form the basis of the proofs of Theorems 1–6. These theorems are proved in §6. In §7 we examine the affineness problem for links.

## §2. Links in $\mathbb{R}P^3$ and their diagrams

**2.1. Links in $\mathbb{R}P^3$ and their isotopies.** A *link* in $\mathbb{R}P^3$ is a 1-dimensional smooth closed submanifold $L \subset \mathbb{R}P^3$. An *isotopy* of a link $L$ is a smooth homotopy $H: L \times I \to \mathbb{R}P^3$ consisting of smooth imbeddings $h_t: L \to \mathbb{R}P^3$ with $h_t(x) = H(x, t)$, $t \in [0, 1]$, and $h_0 = \text{in}: L \hookrightarrow \mathbb{R}P^3$. Two links $L_1$, $L_2$ are *isotopic* if there exists an isotopy $h_t$ $(t \in [0, 1])$ of $L_1$ with $h_1(L_1) = L_2$.

**2.2. Diagrams of links.** As in the case of links in the sphere $S^3$, a link in $\mathbb{R}P^3$ can be specified by pictures in the plane—diagrams. To construct a diagram of a link, we make use of the standard model of $\mathbb{R}P^3$: its representation as a ball $D^3$ with diametrically opposite points of the bounding sphere identified. We choose the corresponding mapping $D^3 \to \mathbb{R}P^3$ so that the image of the poles of the ball does not belong to the link $L$. We denote by $L'$ the inverse image of $L$ in the ball. Let $p: L' \to D^2$ be the projection onto the equatorial disk $D^2 \subset D^3$ given by the formula $x \to c(x) \cap D^2$, where $c(x)$ is the (metric) circle in $D^3$ passing through the point $x \in L'$ and the poles of the ball $D^3$.

We assume that the link $L$ satisfies the following *conditions of general position*: 1) the image $p(L')$ contains no cusps, or 2) points of tangency, or 3) triple points; 4) $L'$ is a submanifold of the ball $D^3$, intersecting transversally the boundary $\partial D^3$; 5) no two points in $L'$ lie on the same arc of a great circle joining the poles of the ball in $\partial D^3$.

Any link can be made to satisfy these conditions 1)–5) by an arbitrarily small isotopy.

We orient compatibly the circular arcs along which the submanifold $L'$ is projected (for example, from north to south). This orientation determines an order on each pair of points constituting the inverse image of a double point under the projection $p$. The first (upper) point we call the *overpass* point; the second (lower), the *underpass*. For each underpass point we choose a sufficiently small connected neighborhood in $L'$, and denote by $U$ the union of these neighborhoods for all underpass points. The image $p(L' \setminus U) \subset D^2$ is then called *the diagram of the link* $L$.

**2.3. Connection with diagrams of links in the sphere $S^3$.** The diagram of a link $L$ in the projective space $\mathbb{R}P^3$ is connected with the diagram of its inverse image $L''$ in the sphere $S^3$ under the covering $S^3 \to \mathbb{R}P^3$. The latter diagram is obtained by projecting the link $L''$ from the poles of $S^3$ (these poles being the inverse images under the covering $S^3 \to \mathbb{R}P^3$ of the images of the poles of the ball $D^3$ under the factorization $D^3 \to \mathbb{R}P^3$) onto the equatorial sphere $S^2$. This diagram may be constructed in the following way. Place on a plane the diagram $D$ of the original link $L$; alongside it place its image under a slide symmetry with respect to a line passing through the center of the disk of the diagram $D$; in this image replace all underpasses by overpasses (and vice versa), and join by a simple arc every endpoint, on the bounding circle, of an arc of the diagram $D$ with the point obtained from it by applying, first, symmetry with respect to the center of the disk of $D$ and then the slide symmetry. The joining arcs are chosen so as to be pairwise disjoint; see Figure 2.

**2.4. Diagram transformations.** In the course of an isotopy of a link, the general-position conditions 1)–5) may be violated. The isotopy can always be adjusted so that at each moment $t \in (0, 1)$ at most one condition fails, and the failure is of the simplest form, as indicated in Figure 3. The corresponding

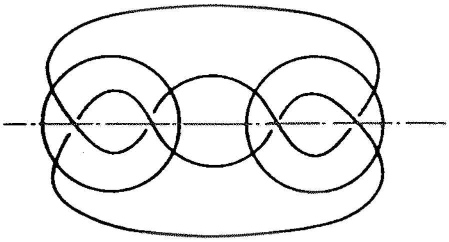


FIGURE 2

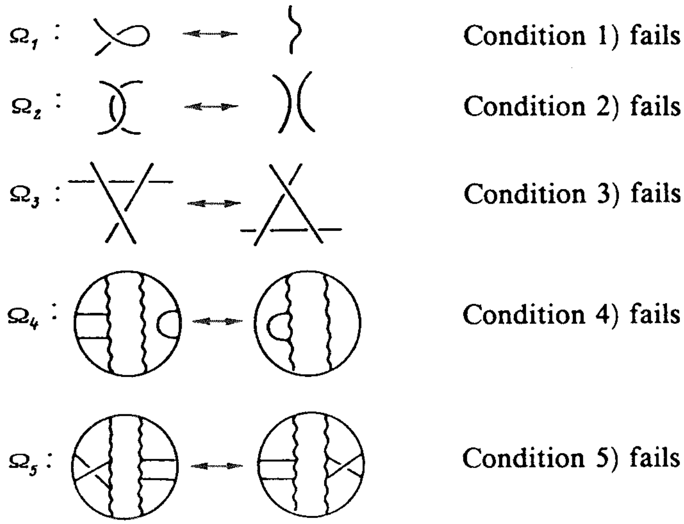



FIGURE 3

transformations of the diagram will be denoted by the symbols $\Omega_1$–$\Omega_5$. The transformations $\Omega_1$–$\Omega_3$ are the usual Reidemeister transformations.

Thus, *two links in $\mathbb{R}P^3$ are isotopic if and only if their diagrams can be joined by a sequence of transformations $\Omega_1$–$\Omega_5$ and diagram isotopies.*

**2.5. Diagrams and framed links.** As in the case of links in the sphere $S^3$, there are constructions forming, from the diagram of a link in $\mathbb{R}^3$, a framing (up to isotopy) of this link. Recall that to a diagram of a link in $S^3$ one assigns a "vertical" framing, all of whose vectors are directed vertically up (or all vertically down). For links in $\mathbb{R}P^3$ this procedure no longer applies, since under the formation of $\mathbb{R}P^3$ by gluing together the ball, the vectors tangent to meridians of the bounding sphere and directed north to south become identified with vectors directed south to north. However, the procedure does assign to a link in $\mathbb{R}P^3$ a framing by normal *lines.* It is easily seen that for every link component contractible in $\mathbb{R}P^3$ this line framing is induced by a vector framing, determined uniquely up to isotopy. In the case of a noncontractible component, if we want to obtain from the line framing a framing induced by a vector framing, we must change the former by a half-turn. For definiteness, let us agree to make the change so that any needed half-turns are by a left-handed screw.

Under all diagram transformations except $\Omega_1$, the framing determined by a diagram as described above remains unchanged. Under a transformation $\Omega_1$ the framing changes by one full turn (just as in the case of links in a sphere). Therefore, given a diagram and a framing for an arbitrary link in $\mathbb{R}P^3$ we can make the framing correspond to the diagram by means of several transformations $\Omega_1$. As in the case of links in $S^3$, it is easily shown that if two framed links with framings constructed in accordance with their diagrams are isotopic (as framed links), then their diagrams can be obtained one from the other solely by means of the transformations $\Omega_2$–$\Omega_5$.

## §3. Polynomials of Kauffman type for links in $\mathbb{R}P^3$

**3.1. States of a diagram and the polynomial of a framed link.** A *state* of a diagram is a choice of a pair of vertical angles at each double point. At each point this choice can be made in two ways; see Figure 4. The two chosen regions are usually indicated by joining them by a small line segment—a *marker*. Markers are of two types: type $A$ and type $B$ (Figure 4).

Let $s$ be a state of a diagram $D$. We denote by $a(s)$ and $b(s)$ the number of markers of types $A$ and $B$, respectively. At each double point we perform a *smoothing* in accordance with its marker; see Figure 5. This turns the diagram into a set of disjoint circles and arcs, and its image in the projective plane (obtained from the disk by identifying diametrically opposite boundary points) into a set of disjoint circles. The number of these circles we denote by $|s|$. We now define a polynomial in three variables $A$, $B$, $d$:

$$v(D) := \sum_s A^{a(s)} B^{b(s)} d^{|s|-1},$$

where the summation is over all states $s$ of the diagram $D$. It is easily verified that:

1) $v(\bigcirc) = 1$.

2) $v(D \coprod \bigcirc) = dv(D)$, where $D \coprod \bigcirc$ is the diagram obtained from $D$ by addition of one unknotted circle $\bigcirc$, disjoint from $D$.

3) $v(D) = Av(D_A) + Bv(D_B)$, where $D_A$ and $D_B$ are the diagrams obtained from $D$ by a smoothing at any one double point in accordance with a marker of type $A$ or type $B$.

As in the case of links in the sphere $S^3$, the requirement of invariance of the polynomial $v(D)$ with respect to the transformation $\Omega_2$ imposes on the

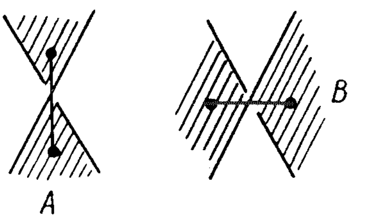



FIGURE 4

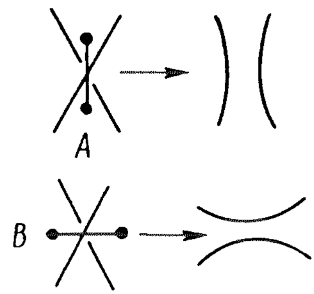



FIGURE 5

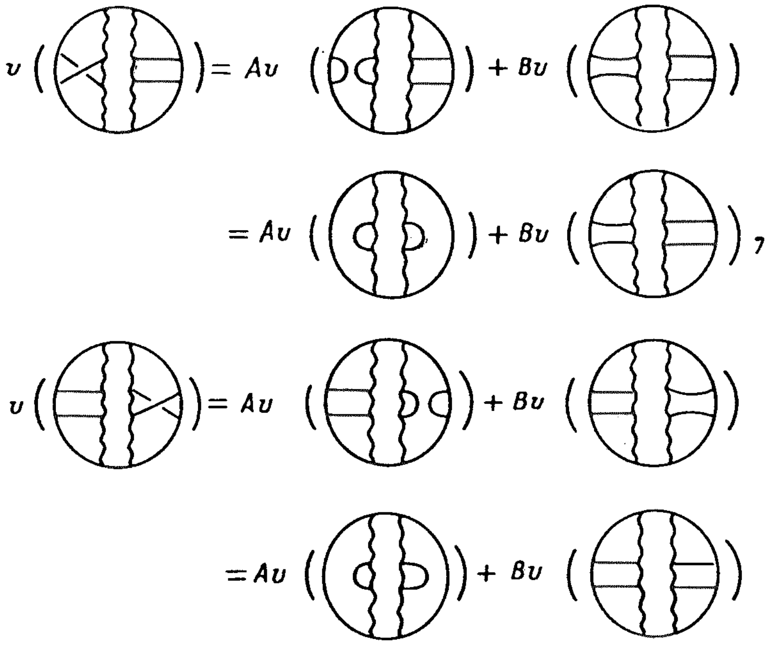


FIGURE 6

variables $A$, $B$, $d$ the restrictions $B = A^{-1}$, $d = -A^{-2} - A^2$; see [1]. The requirement of invariance of $v(D)$ with respect to the transformations $\Omega_3$–$\Omega_5$ gives no new relations. For $\Omega_4$ this is obvious; for $\Omega_3$, see [1]. To prove invariance with respect to $\Omega_5$, we use the "calculation" in Figure 6.

From the invariance of the polynomial $v(D)$ with respect to the transformations $\Omega_2$–$\Omega_5$ it follows, in view of the observations in §2.5, that our construction of the polynomial $v$ gives an invariant for framed links.

**3.2. The polynomial $V$ for oriented links.** Let $L$ be an oriented link in $\mathbb{R}P^3$, and $D$ its diagram. The orientation of $L$ determines an orientation of $D$, allowing us to define the number $w(D) = \sum_i \varepsilon_i$, where $\varepsilon_i = 1$ or $-1$ depending on the type of double point (see Figure 7), and where the summation is over all double points. It is obvious that $w(D)$ is invariant with respect to the transformations $\Omega_2$–$\Omega_5$.

For an oriented link $L$ in $\mathbb{R}P^3$ we define the polynomial

$$V_L(A) := (-A)^{-3w(D)} v(D) = (-A)^{-3w(D)} \sum_s A^{a(s)-b(s)} (-A^2 - A^2)^{|s|-1},$$

where $D$ is a diagram of the link. Repeating Kauffman's argument [1], we can show that the polynomial $V_L(A)$ is invariant with respect to the transformation $\Omega_1$ and that $w(D)$ and $v(D)$ are each invariant with respect to the transformations $\Omega_2$–$\Omega_5$. Consequently, the polynomial $V_L(A)$ is independent of the choice of the diagram $D$ and is invariant with respect to isotopy of the link $L$.

The polynomial $V_L(A)$ thus constructed generalizes the Jones polynomial (in the Kauffman form) for links in $\mathbb{R}^3$: for links in $\mathbb{R}^3 \subset \mathbb{R}P^3$ the two polynomials coincide.

Observe that the degrees of the terms of the polynomial $V$ are all even. Indeed, let $c$ be the number of double points in a diagram $D$ of the link. Then

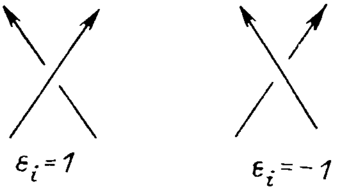



FIGURE 7

FIGURE 8

$a(s) - b(s) \equiv c \bmod 2$, $w(D) \equiv c \bmod 2$, and therefore $-3w(D) + a(s) - b(s) \equiv 0 \bmod 2$.

**3.3. Behavior of $V_L$ under link addition.** As analogues of the operations of connected and disconnected addition of links in $S^3$ we have the operations of connected and disconnected addition of a link in $\mathbb{R}P^3$ and a link in $S^3$. These operations are defined in the obvious fashion. To denote them we employ, as usual, the symbols $\#$ and $\coprod$.

Let $V_L$ be the polynomial of a link $L$ in $\mathbb{R}P^3$, and $V_K$ the Jones polynomial (in the Kauffman form) of a link $K$ in $S^3$. Then the polynomials $V_{L\coprod K}$ and $V_{L\#K}$ can be expressed in terms of $V_L$ and $V_K$ as follows:

$$V_{L\coprod K} = (-A^{-2} - A^2)V_L V_K, \qquad V_{L\#K} = V_L V_K.$$

To prove the first equality, it suffices to observe that every state $s$ of a diagram of the link $L\coprod K$ determines, in an obvious fashion, states $s_L$ and $s_K$ of the diagrams of the links $L$ and $K$. In turn, $s_L$ and $s_K$ determine $s$, with $|s| = |s_L| + |s_K|$. Similarly, to prove the second equality we observe that every state $s$ of a diagram of the link $L\#K$ determines a pair $s_L$, $s_K$ of states of the diagrams of $L$ and $K$, but now with $|s| = |s_L| + |s_K| - 1$.

## §4. Alternating diagrams and nets

**4.1. Alternating diagrams.** *If a link diagram is alternating, then the link has an even number of noncontractible components.*

PROOF. Consider a link with an odd number of noncontractible components. The portion $\alpha$ of the net of the diagram that corresponds to a noncontractible component of the link passes through double points an even number of times. Indeed, through every point of self-intersection it passes twice; it intersects the projection of every contractible component an even number of times, and the projection of every noncontractible component an odd number, but the number of these latter components is even. In addition, it intersects a projective line an odd number of times. Consequently, in passage along $\alpha$ the alternating condition cannot hold.

It follows from this that the net of an alternating diagram must be separating.

Any diagram of a link with an even number of noncontractible components can be made alternating by changing certain underpasses to overpasses and vice versa. Indeed, the projection of such a link into $\mathbb{R}P^2$ separates $\mathbb{R}P^2$ into two parts with a common boundary. Color the two parts in different colors. Then the underpasses and overpasses can be so chosen that all the markers joining regions of the same color are of type $A$. The diagram so obtained is alternating. This argument also shows that there exist exactly two ways of turning a connected projection of a link with an even number of noncontractible components into an alternating diagram.

Observe that if the diagram of a link in $\mathbb{R}P^3$ is alternating, then the corresponding diagram of the inverse image of the link under the covering $S^3 \to \mathbb{R}P^3$ (see §2.3) is also alternating (in the ordinary sense).

For this diagram of the inverse image of the link, the number $w$ is clearly equal to double the number $w$ for the original diagram. By Little's conjecture, proved by Murasugi [2] and Thistlethwaite [5], the number $w$ for an oriented weakly alternating reduced diagram of a link in $S^3$ is an isotopy invariant of the link. Consequently, the same is true of the number $w$ for an oriented weakly alternating reduced diagram of a link in $\mathbb{R}P^3$.

**4.2. Nets.** A *state* of a net is a choice at each vertex of a pair of vertical angles. As in the case of diagrams, a state of the net is described by a collection of markers, i.e., line segments joining the vertical angles selected.

If the net is separating, then by assigning two different colors to the two subsets of its complement in $\mathbb{R}P^2$ we obtain a checkerboard coloring of the plane $\mathbb{R}P^2$. A state of a separating net is called *alternating* if all its markers lie in regions of just one of the colors.

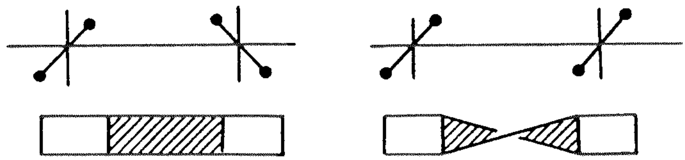
It is easily seen that a state of the net is alternating if and only if, for any edge, the markers at the endpoints of the edge both enter into the same region adjacent to the edge; see Figure 8.

To every state of a link diagram corresponds a state of the net of the diagram. A state of a diagram is called an *A-state* if at every double point the marker is of type $A$. Obviously, the diagram is alternating if and only if its $A$-state is alternating.

## §5. Preparation for the proofs of the theorems

**5.1. The Turaev surface.** (See [3], §2.) To every state $s$ of a diagram we assign a surface $M_s$, constructed as follows. To small rectangular neighborhoods of the vertices of the net attach narrow bands, one for each edge: if the markers at the endpoints of the edge both enter into the same region adjacent to the edge, what is to be attached is simply a regular neighborhood of the edge in $\mathbb{R}P^2$ (Figure 9a); in the opposite case, what is attached is the band obtained from a regular neighborhood by a half-turn twice (Figure 9b).

From the definition it is obvious that in the case of an alternating state $s$ the surface $M_s$ imbeds into $\mathbb{R}P^2$.



a b

FIGURE 9

**5.2. Lemmas on cycles on the surface $M_s$.**

5.2.1. LEMMA. *In the case of a separating net, the self-intersection index* (mod 2) *of any cycle on the surface $M_s$ is equal to the self-intersection index* (*in* $\mathbb{R}P^2$) *of the projection of the cycle.*

PROOF. Starting with an arbitrary cycle on $M_s$, displace it on $M_s$ so that the points of intersection of the two cycles (the original and the displaced) lie outside the bands of the surface. When the two cycles are projected onto $\mathbb{R}P^2$, there appear new points of intersection: as many as there are twisted bands traversed on $M_s$ by the original cycle (counting multiplicity of passage). But the number of these bands is even.

Indeed, the markers of the state $s$ are separated by the checkerboard coloring of $\mathbb{R}P^2$ into two classes: those lying in regions of the one color, and those in regions of the other. The twisted bands are precisely those that join markers of different colors. In moving along the cycle, the total number of passages from markers of the one color to markers of the other is even. This proves the lemma. ●

COROLLARY. *In the case of a separating net, the surface $M_s$ is orientable if and only if the net is contractible.*

5.2.2. LEMMA. *In the case of a nonseparating net, the self-intersection index of any cycle on $M_s$ is zero.*

PROOF. Choose an arbitrary cycle $z$ on $M_s$. As in the proof of Lemma 5.2.1, the self-intersection index of the projection $\tilde{z}$ of $z$ on $\mathbb{R}P^2$ is equal to the sum (mod 2) of the self-intersection index $z \circ_{M_s} z$ and the number $N$ of twisted bands on the surface $M_s$ along the cycle $z$:

$$\tilde{z} \circ_{\mathbb{R}P^2} \tilde{z} = z \circ_{M_s} z + N .$$

We now form a new net, by adding to the old one an arbitrary one-sidedly imbedded circle $\alpha$ that intersects the net at points on the edges, each edge being intersected transversally and no more than once. The new net is separating, and therefore determines a checkerboard coloring of the plane $\mathbb{R}P^2$. The markers of a state are separated by the coloring into two classes: those lying in regions of the one color, and those lying in regions of the other. A band on $M_s$ is twisted either when the markers at the endpoints of the corresponding edge lie in regions of different colors and the edge fails to intersect the circle $\alpha$, or when the markers at the endpoints of the edge lie in regions of the same color and the edge intersects $\alpha$.

Thus, the number $N$ of twisted bands of the surface along the cycle $z$ is equal to the number of color alternations of markers along those edges of $z$ that fail to intersect $\alpha$ plus the number of edges that intersect $\alpha$ and have at their endpoints markers of the same color. Adding to this sum double the number of edges that intersect $\alpha$ and have at their endpoints markers of different colors, we see that the number $N (\mathrm{mod}\, 2)$ is equal to the number of color alternations of markers along $z$ $(= 0 \,\mathrm{mod}\, 2$, as in Lemma 5.2.1) plus the number of points of intersection of the projection $\tilde{z}$ with the circle $\alpha$, i.e., is equal to the intersection index $\tilde{z} \circ_{\mathbb{R}P^2} \alpha$.

Thus, $N = \tilde{z} \circ_{\mathbb{R}P^2} \alpha$. On the other hand, $\tilde{z} \circ_{\mathbb{R}P^2} \alpha = \tilde{z} \circ_{\mathbb{R}P^2} \tilde{z}$. Therefore $\tilde{z} \circ_{\mathbb{R}P^2} \tilde{z} = z \circ_{M_s} z + \tilde{z} \circ_{\mathbb{R}P^2} \tilde{z}$. This implies our assertion: $z \circ_{M_s} z = 0$. ●

COROLLARY. *In the case of a nonseparating net, the surface $M_s$ is orientable.*

**5.3. Lemmas on dual states.** To every state $s$ of a net corresponds a *dual* state $\check{s}$, obtained from $s$ by changing all markers simultaneously.

Let $c$ be the number of double points of the net, and $r$ the number of components; denote by $Q(F)$ the quadratic form of self-intersection indices in the homology of a surface $F$ with coefficients in $\mathbb{Z}_2$.

5.3.1. LEMMA. $|s| + |\check{s}| = 2r + c - \operatorname{rk} Q(M_s)$; *in particular,* $|s| + |\check{s}| \leq 2r + c$.

PROOF. The homology sequence

$$\cdots \to H_1(M_s; \mathbb{Z}_2) \overset{\text{rel}}{\to} H_1(M_s, \partial M_s; \mathbb{Z}_2) \overset{\partial}{\longrightarrow} H_0(\partial M_s; \mathbb{Z}_2) \overset{\text{in}}{\to} H_0(M_s; \mathbb{Z}_2) \to 0$$

of the pair $(M_s, \partial M_s)$ gives the relation

$$b_0(\partial M_s) = b_0(M_s) + b_1(M_s, \partial M_s) - \operatorname{rk}(\text{rel}).$$

The boundary $\partial M_s$ is obviously the disjoint union of the nets obtained by smoothing with respect to the states $s$ and $\check{s}$, respectively. Hence $b_0(\partial M_s) = |s| + |\check{s}|$.

But clearly

$$b_0(M_s) = r, \qquad b_1(M_s, \partial M_s) = b_1(M_s) = b_0(M_s) - \chi(M_s) = r + c,$$
$$\operatorname{rk}(\text{rel}) = \operatorname{rk} Q(M_s).$$

Therefore

$$|s| + |\check{s}| = r + r + c - \operatorname{rk} Q(M_s) = 2r + c - \operatorname{rk} Q(M_s). \quad \bullet$$

5.3.2. LEMMA. *For a nonseparating net containing* $2p$ *one-sidedly imbedded circles without common edges,*

$$|s| + |\check{s}| \leq 2r + c - 2p.$$

PROOF. We show that $\operatorname{rk} Q(M_s) \geq 2p$, and then use Lemma 5.3.1.

For each of the $2p$ cycles on the surface $M_s$, the self-intersection index is equal to zero (by Lemma 5.2.2). By assumption, any two of these cycles intersect only at vertices of the net; therefore their intersection index in $M_s$ is equal to the intersection index of their projections in $\mathbb{R}P^2$, i.e., to one. Thus, the value of the form $Q(M_s)$ on these cycles is given by the $2p \times 2p$-matrix

$$\begin{pmatrix} 0 & 1 & \cdots & & 1 \\ 1 & 0 & \cdots & 1 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix},$$

whose rank is $2p$. This implies our assertion. $\bullet$

A net will be called *prime* if for any circle imbedded in $\mathbb{R}P^2$ that intersects the net (transversally) in exactly two points and separates $\mathbb{R}P^2$ into a disk and a Möbius band, either the disk or the Möbius band intersects the net in a simple arc.

5.3.3. LEMMA. *In the case of a prime connected separating net,* $|s| + |\check{s}| = c + 2$ *if and only if the net is contractible and the state* $s$ *is alternating.*

PROOF. Suppose the net is noncontractible. Choose in it a cycle $z$ determined by a one-sidedly imbedded circle. By Lemma 5.2.1, $z \circ_{M_s} z = z \circ_{\mathbb{R}P^2} z = 1$. Consequently, $\operatorname{rk} Q(M_s) \geq 1$, and by Lemma 5.3.1, $|s| + |\check{s}| \leq 2 + c - 1 = c + 1$. Thus, $|s| + |\check{s}| \neq c + 2$.

Now suppose the net is contractible and the state $s$ is not alternating. Let $e$ be one of the edges on which the alternating condition fails. Adjoining this edge $e$ are two different (since the net is separating) components of the complement;

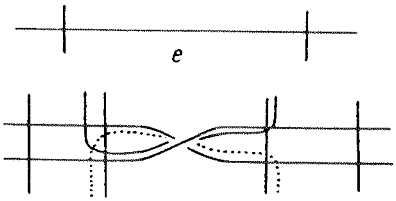



FIGURE 10

their closures intersect only along $e$ (since the net is prime). Hence $M_s$ contains two cycles whose intersection index is 1; see Figure 10.

This means that $\operatorname{rk} Q(M_s) \geq 1$ and $|s| + |\check{s}| \neq c + 2$.

As for the converse, suppose the net is contractible and $s$ is alternating. Then the surface $M_s$ imbeds into an affine plane, and the intersection index form on $M_s$ is trivial. So by Lemma 5.3.1, $|s| + |\check{s}| = c + 2$. ●

5.3.4. LEMMA. *If in a diagram with nonseparating net the alternating condition fails on exactly one edge, then this edge is adjoined on both sides by one and the same component of the complement of the net.*

PROOF. Consider in the net an arbitrary one-sidedly imbedded circle $\alpha$. We show that one of the edges that make up the circle $\alpha$ fails to satisfy the alternating condition. The net is a cycle modulo 2; since it is nonseparating, it realizes a nontrivial class in $H_1(\mathbb{R}P^2; \mathbb{Z}_2)$. If we delete from the net the cycle $\alpha$, which is not null-homologous, what remains *is* null-homologous. Hence the number of transversal intersection points of this remaining part with the cycle $\alpha$ is even. There may also be nontransversal intersection points; see Figure 11a. If now along $\alpha$ the alternating condition were to hold, then in passage along $\alpha$ the markers at the endpoints of each edge would enter into the same region adjacent to that edge—left or right (relative to direction of passage along $\alpha$); see Figure 11b. In the passage through a double point of transversal intersection the regions containing the markers switch (from right-hand to left, or vice versa). Since the number of transversal intersection points, as already shown, is even, this means that under a complete circuit of $\alpha$ the number of passings from a left- or right-hand region to a right- or left-hand is even. On the other hand, a neighborhood in $\mathbb{R}P^2$ of this one-sided circle $\alpha$ is a Möbius band, so that the number of such passings must be odd. This contradiction shows that on one of the edges that make up $\alpha$ the alternating condition must break down.

Thus, any one-sidedly imbedded circle in the net contains an edge on which the alternating condition fails. Hence, since the given state is alternating everywhere except on a single edge, all the one-sidedly imbedded circles in the net pass through this edge. This means that, when the edge is removed, what remains of the net is a contractible set, so that there exists a one-sidedly imbedded circle in $\mathbb{R}P^2$ that this set fails to intersect. Then obviously this circle intersects the original net only in a point of the deleted edge. Consequently, this edge is adjoined on both sides by one and the same component of the complement. ●

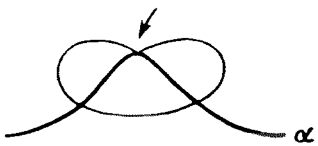



FIGURE 11a

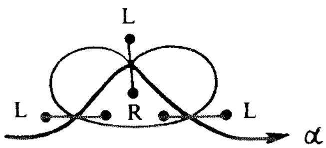



FIGURE 11b

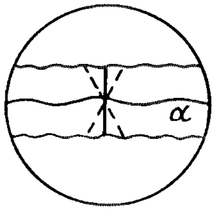



FIGURE 12

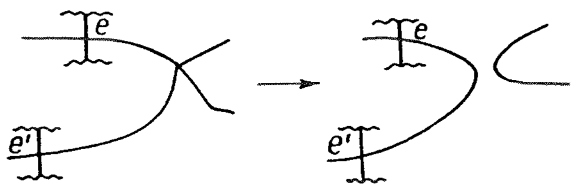



FIGURE 13

5.3.5. LEMMA. *In the case of a prime connected nonseparating net,* $|s| + |\check{s}| = c+2$ *if and only if the state* $s$ *is alternating everywhere except on a single edge.*

PROOF. If $s$ is alternating everywhere except on a single edge, then the surface $M_s$ imbeds in an annulus. Indeed, by Lemma 5.3.4, there exists a one-sidedly imbedded circle $\alpha$ that intersects the net only in a point of the edge on which the alternating condition fails; see Figure 12. Hence the part of $M_s$ that corresponds to the alternating part of the net imbeds into the complement of a neighborhood of $\alpha$, i.e., into a disk, and all of $M_s$ imbeds into a surface $M$ obtained from this disk by attaching a band appropriately. This $M$ is orientable, since $M_s$ is orientable (see the Corollary to Lemma 5.2.2) and the inclusion homomorphism $H_1(M_s; \mathbb{Z}_2) \to H_1(M; \mathbb{Z}_2)$ is surjective. Consequently, $M$ is an annulus, and $\operatorname{rk} Q(M) = 0$. This means that $\operatorname{rk} Q(M_s) = 0$, and by Lemma 5.3.1, $|s| + |\check{s}| = c+2$.

To prove the converse, assume the alternating condition fails on at least two edges $e$ and $e'$. Suppose the components of the complement of the net that adjoin at least one of the edges are different. Then, as in the proof of Lemma 5.3.3, the surface $M_s$ contains two cycles with intersection index 1. In that case, $\operatorname{rk} Q(M_s) \geq 2$ and $|s| + |\check{s}| \neq c+2$.

On the other hand, if each of the two edges has just one component adjoining it, then the components in question are the same. Indeed, if we consider in each component a one-sidedly imbedded circle intersecting the corresponding edge, the two circles must have at least one point in common. That being the case, there exists a two-sidedly imbedded circle intersecting the net in exactly two points on different edges. Such a circle is obtained by perturbing the union of the two one-sidedly imbedded circles; see Figure 13. But this contradicts the assumption that the net is prime. ●

5.3.6. LEMMA. *In the case of a noncontractible prime connected separating net,* $|s| + |\check{s}| = c+1$ *if and only if the state* $s$ *is alternating.*

PROOF. If $s$ is alternating, the surface $M_s$ imbeds into $\mathbb{R}P^2$; since on $\mathbb{R}P^2$ the intersection index form has rank 1, we have $\operatorname{rk} Q(M_s) \leq 1$. Consequently, by Lemma 5.3.1, $c+2 \geq |s| + |\check{s}| \geq c+1$. By Lemma 5.3.2, $|s| + |\check{s}| \neq c+2$. Therefore $|s| + |\check{s}| = c+1$.

To prove the converse, suppose $s$ is not alternating. Let $e$ be an edge on which the alternating condition fails. Adjoining this edge $e$ are two different components of the complement. Since the net is prime, their closures intersect either only along the edge $e$ or along two edges $e$ and $e'$ such that the union of the components contains a noncontractible loop intersecting the net only in these edges. In either case, consider on the surface $M_s$ two cycles $c_1$ and $c_2$, as in Figure 14. In the first case, $c_1 \circ_{M_s} c_2 = 1$, $c_i \circ_{M_s} c_i = \tilde{c}_i \circ_{\mathbb{R}P^2} \tilde{c}_i = 0$ (see

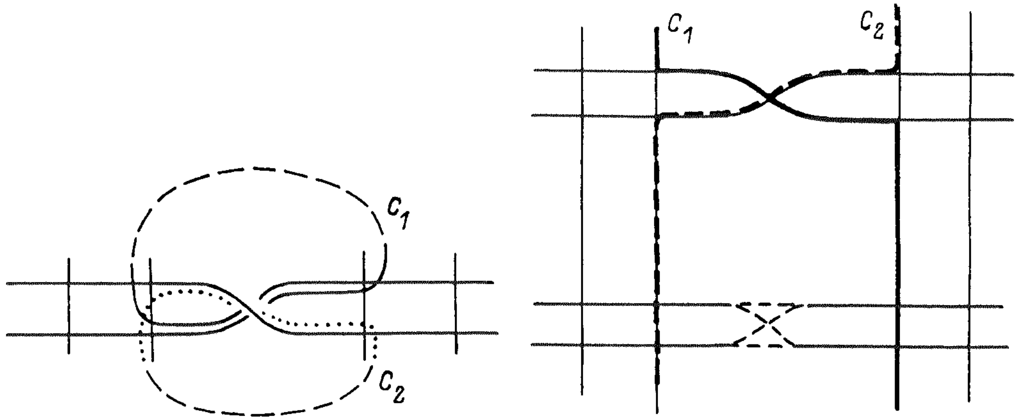



FIGURE 14

Lemma 5.2.1), where $\tilde{c}_i$ is the projection of $c_i$ on $\mathbb{R}P^2$, $i = 1, 2$. Therefore $\operatorname{rk} Q(M_s) \geq 2$. By Lemma 5.3.1, $|s| + |\check{s}| \leq 2 + c - 2 \neq c + 1$.

In the second case, $c_1 \circ_{M_s} c_2 = 0$, $c_i \circ_{M_s} c_i = \tilde{c}_i \circ_{\mathbb{R}P^2} \tilde{c}_i = 1$, $i = 1, 2$. Therefore, $\operatorname{rk} Q(M_s) \geq 2$ and $|s| + |\check{s}| \neq c + 1$. ●

## §6. Proofs of the theorems

**6.1.** PROOF OF THEOREM 1. This is essentially a repetition of the proof of the corresponding result for links in a sphere; see [3], §4. To each state $s$ of the diagram $D$ corresponds the polynomial $A^{a(s)-b(s)}(-A^{-2} - A^2)^{|s|-1}$. Denote by $D_s$ the maximal degree of its terms, and by $d_s$ the minimal. Clearly, $D_s = a(s) - b(s) + 2|s| - 2$ and $d_s = a(s) - b(s) - 2|s| + 2$. For the $A$-state and $B$-state, which we denote by $s_A$ and $s_B$, the maximal and minimal exponents are

$$D_{s_A} = c + 2|s_A| - 2, \qquad d_{s_A} = -c - 2|s_B| + 2.$$

It is easily shown that for any state $s$,

$$D_s \leq D_{s_A} \quad \text{and} \quad d_s \geq d_{s_B}.$$

Hence

$$\deg_{\max} V_L \leq -3w(D) + D_{s_A}, \qquad \deg_{\min} V_L \geq -3w(D) + d_{s_B},$$

where $\deg_{\max} V_L$ and $\deg_{\min} V_L$ are the maximal and minimal degrees of the terms in $V_L$. Therefore

$$\operatorname{span}(V_L) \leq D_{s_A} - d_{s_B} = 2c(D) + 2(|s_A| + |s_B|) - 4. \qquad (*)$$

Using Lemma 5.3.1 (according to which $|s_A| + |s_B| \leq 2r(D) + c(D)$), we obtain:

$$\operatorname{span}(V_L) \leq 2c(D) + 4r(D) + 2c(D) - 4 = 4(c(D) + r(D) - 1). \quad ●$$

**6.2.** PROOF OF THEOREM 3. By Lemma 5.3.2, $|s_A| + |s_B| \leq 2r + c - 2p$. Using $(*)$, we obtain:

$$\operatorname{span}(V_L) \leq 4(c(D) + r(D) - 1 - p). \quad ●$$

**6.3.** PROOF OF THEOREM 4. If a link $L$ is a disconnected sum of $r$ links $L_i$ (one of which is a link in $\mathbb{R}P^3$; the remainder, links in $S^3$), then

$$\operatorname{span}(V_L) = 4r - 4 + \sum_{i=1}^{r} \operatorname{span}(V_{L_i}).$$

This is a consequence of the following property of the polynomial $V_L$:

$$V_L = V_{\amalg_{i=1}^r L_i} = (-A^{-2} - A^2)^{r-1} \prod_{i=1}^r V_{L_i}.$$

Let $D_1, \dots, D_r$ $(r = r(D))$ be diagrams of the links $L_1, \dots, L_r$. By Theorem 1,

$$4c(D) = 4\sum_{i=1}^r c(D_i) \geq \sum_{i=1}^r \operatorname{span}(V_{L_i}) = \operatorname{span}(V_L) - 4r + 4.$$

The equality $4(c(D) + r(D) - 1) = \operatorname{span}(V_L)$ is possible if and only if $4c(D_i) = \operatorname{span}(V_{L_i})$ for each $i$. For a link in $S^3$, this latter equality is equivalent to the diagram being weakly alternating and reduced; see [3]. Hence it suffices to prove the assertion of the theorem for a link in $\mathbb{R}P^3$, the net of whose diagram is connected. Furthermore, in view of the additivity of the numbers $c(D)$ and $\operatorname{span}(V_L)$ with respect to connected addition, we can assume that the net is prime. Thus, what remains to prove is that for a diagram $D$ with a prime connected separating net, $4c(D) = \operatorname{span}(V_L)$ if and only if $D$ is alternating reduced and its net is contractible.

Suppose $4c(D) = \operatorname{span}(V_L)$. Then from (*) (which in the present case becomes an equality) we find that $|s_A| + |s_B| = c(D) + 2$. By Lemma 5.3.3, the net of the diagram is contractible, and the $A$-state is alternating. This means that the diagram is alternating (see §4.2).

Furthermore, $D$ is reduced, since every diagram with a prime net is reduced except in two cases. The exceptions are indicated in Figure 15; but in these cases, $4 = 4c(D) \neq \operatorname{span}(V_L) = 0$.

Let us now prove the converse. Consider a diagram $D$ that is contractible, alternating and reduced. Its $A$-state is also alternating, and by Lemma 5.3.3, $|s_A| + |s_B| = c(D) + 2$. We prove that

$$\deg_{\max} V_L = -3w(D) + D_{s_A} \quad \text{and} \quad \deg_{\min} V_L = -3w(D) + d_{s_B}.$$

This will show that

$$\operatorname{span}(V_L) = D_{s_A} - d_{s_B} = 2c(D) + 2(|s_A| + |s_B|) - 4 = 4c(D).$$

To prove the equalities, we show that any state $s$ different from the $A$-state satisfies the strict inequality $D_s < D_{s_A}$, and any state $s$ different from the $B$-state, the strict inequality $d_s > d_{s_B}$.

Suppose that all markers in a state $s$ except for one are of type $A$, and that $s_A$ is the $A$-state. Then $a(s_A) - b(s_B) = a(s) - b(s) + 2$. Consider a checkerboard coloring of $\mathbb{R}P^2$ corresponding to the net of the diagram $D$. Since the diagram is reduced, around each vertex any two regions of the same color are different. Hence the number of regions of the same color that do *not* contain the markers of the state $s_A$ is equal to $|s_A|$. When a marker of type $A$ is changed to one

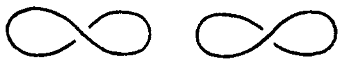


FIGURE 15

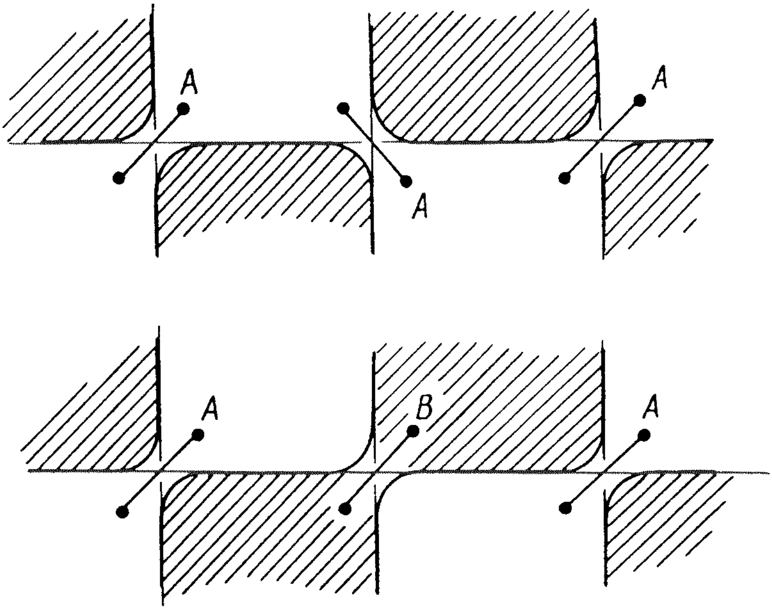



FIGURE 16

of type $B$, under the corresponding smoothing two regions combine into one; see Figure 16.

This means that $|s_A| = |s| + 1$. Thus, $D_A = D_{s_A} > D_s$. As already remarked (see §6.1), for any state $s'$ whose number of markers of type $A$ is 1 less than that of $s$, $D_{s'} \leq D_s$. Therefore for any state $s$ different from the $A$-state, $D_s < D_{s_A}$.

A similar proof gives the inequality $d_s > d_{s_B}$. ●

**6.4.** PROOF OF THEOREMS 6 AND 2. Observe that for an affine link, $\operatorname{span}(V_L) \equiv 0 \bmod 4$ (see [1]). Hence the equality $4(c(D) + r(D)) - 6 = \operatorname{span}(V_L)$ is possible only for noncontractible diagrams.

It suffices now to prove the following (cf. §6.3): in the case of a diagram with prime connected separating noncontractible net, $4c(D) - 2 = \operatorname{span}(V_L)$ if and only if $D$ is an alternating reduced diagram.

The proof is a repetition of that in §6.3, with the difference only that the reference to Lemma 5.3.3 must be replaced by one to Lemma 5.3.6.

As for Theorem 2, it is an obvious consequence of Theorems 1 and 6. ●

**6.5.** PROOF OF THEOREM 5. It suffices to prove the following: in the case of a diagram with a prime connected nonseparating net, $4c(D) = \operatorname{span}(V_L)$ if and only if $D$ is reduced and the alternating condition fails on just one edge.

Suppose $4c(D) = \operatorname{span}(V_L)$. Then $|s_A| + |s_B| = c(D) + 2$, and by Lemma 5.3.5 the $A$-state is alternating everywhere except for one edge. That means that the diagram is alternating everywhere except for that edge. That the diagram is reduced follows from the primeness of the net.

The converse is proved in the same way as in §6.4, with the difference only that the checkerboard coloring must be applied to a new net, obtained from the old by addition of a one-sidedly imbedded circle intersecting the net only at points of the edge on which the alternating condition fails. In counting the regions that do not contain markers of type $A$, we must observe that adjoining

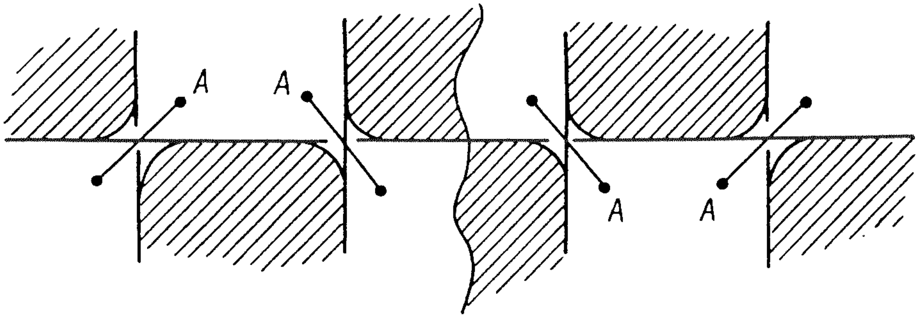



FIGURE 17

the one-sidedly imbedded circle there is exactly one such region; see Figure 17. ●

**6.6. Adequate diagrams.** Recall (see [4]) that a diagram $D$ of a link $L$ in $S^3$ is called *adequate* if $|s_A| > |s|$ for any state $s$ of $D$ with $b(s) = 1$ and $|s| > |s_B|$ for any state $s$ of $D$ with $a(s) = 1$. Lickorish and Thistlethwaite have proved ([4], Proposition 1) that if $D$ is an adequate diagram of a link $L \subset S^3$, then

$$\operatorname{span}(V_L) = 2c(D) + 2(|s_A(D)| + |s_B(D)|) - 4.$$

The definition of adequate diagrams, as well as this assertion about them, carries over verbatim to the case of links in $\mathbb{R}P^3$. The proofs above of Theorems 4, 5, and 6 are essentially based on this assertion and the fact that the diagrams involved in them are adequate.

## §7. Affine links

**7.1. Proofs of Theorems 7 and 8 and the Corollary.** Theorem 7 follows from the fact that for an affine link the generalized Jones polynomial coincides with the original Jones polynomial (in the Kauffman form). For this polynomial it is known that the degrees of all its terms are divisible by 4 if the number of components of the link is odd, and congruent to $2 \bmod 4$ in the opposite case. See [6]. ●

PROOF OF THEOREM 8. If $L$ is affine, then $\operatorname{span}(V_L) \equiv 0 \bmod 4$; see [6]. Conversely, the net of an alternating diagram is always separating (as noted in §4.1); hence if $\operatorname{span}(V_L) \equiv 0 \bmod 4$, then by Theorems 4 and 6 the net is contractible, and so the link is affine. ●

PROOF OF THE COROLLARY. Suppose the net of an affine alternating diagram is noncontractible. Then by Theorem 6, $\operatorname{span}(V_L) \not\equiv 0 \bmod 4$ but this is impossible in the case of an affine link. The converse is obvious. ●

**7.2. The self-linking coefficient.** The inverse image in $S^3$, under the mapping $S^3 \to \mathbb{R}P^3$, of a homologically trivial oriented knot in $\mathbb{R}P^3$ is an oriented two-component link. The linking coefficient of its components is an invariant of the original knot $K$ in $\mathbb{R}P^3$ (and in fact independent of the orientation of $K$). We call it the *self-linking coefficient* of $K$, and denote it by $\operatorname{sl}(K)$. It can be computed directly from the diagram of the knot, in the following way. The double points of the diagram of a homologically trivial oriented knot $K$ divide into two sets: those for which the two components obtained by smoothing in accordance with the orientation are both homologically nontrivial, and those for which they are both trivial. Summing up the numbers $\varepsilon_i$ (see §3.2) over all the double points of the first set gives, it is easily seen, the number $\mathrm{sl}(K)$.

For affine knots, clearly, the self-linking coefficient is zero.

**7.3. An example of a non-affine knot** $K$ **with** $\mathrm{sl}(K) = 0$ is exhibited in Figure 18. To give a simple proof that $K$ is non-affine, we show that the highest degree of the polynomial $V_K$ is not divisible by 4.

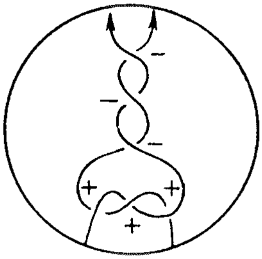


FIGURE 18

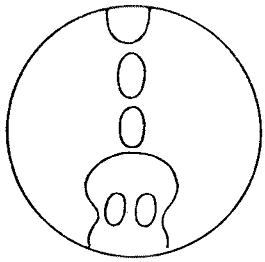


FIGURE 19

To each state $s$ of the diagram $D$ of the knot corresponds the polynomial $A^{a(s)-b(s)}(-A^{-2}-A^{2})^{|s|-1}$. Smooth the diagram in accordance with the markers of the $A$-state; see Figure 19. It is easily seen that only under this smoothing do we reach the maxima of the quantities $a(s)-b(s)$ [$= 6$] and $|s|$ [$= 5$]. (This follows from the adequateness of the diagram.) Hence the degree of the polynomial $V_K(A) = (-A)^{-3w(D)}v(D)$ (for our diagram, $w(D) = 0$) is $6 + 2\cdot(5-1) = 14 \not\equiv 0 \bmod 4$.

It is of interest to note that $\mathrm{span}(V_K) = 20 \equiv 0 \bmod 4$.

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