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CLASSIFICATION OF PROJECTIVE MONTESINOS LINKS

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ABSTRACT. Links in the three-dimensional projective space analogous to Montesinos links in the three-dimensional sphere are classified up to isotopy and up to homeomorphism.

§1. INTRODUCTION

1.1. **Projective Montesinos links.** The classification problem for links in the three-dimensional sphere has been effectively solved for only some special classes of links. One of such classes is the class of Montesinos links (which contains links with two bridges, or four-braidings). In this paper we introduce an analogous class of links in the projective space $\mathbb{R}P^3$ and solve the classification problems up to isotopy and homeomorphism for this class. Before stating the main results we give several definitions.

A *tangle* is a one-dimensional compact smooth submanifold t of the standard ball D^3 with boundary ∂t consisting of four distinct points p_1, p_2, p_3, and p_4 in ∂D. Two tangles are *isotopic* if they are mapped onto each other by an isotopy of the ball fixed on ∂D^3. We will consider only such tangles whose boundary consists of two pairs of diametrically opposite points lying on orthogonal diameters. It is clear that two tangles are isotopic if and only if their diagrams can be connected by a sequence of Reidemeister transformations Ω_1–Ω_3 (see, for example, [1], [2]).

The *product* $(t_1 t_2)$ of two tangles t_1 and t_2 is a tangle obtained as the union $\Gamma \cup t_1' \cup t_2'$, where Γ are the arcs in $D^3 \backslash (\operatorname{Int} D_1 \cup \operatorname{Int} D_2)$ shown in Figure 1, t_2' is the image of the tangle t_2 under the obvious homothety $D^3 \to D_2$, and t_1' is the image of the tangle t_1 under the composition of the symmetry with respect to the plane orthogonal to the plane of the diagram and passing through the points p_1 and p_3, and the homothety $D^3 \to D_1$. An example of product is shown in Figure 2.

Integer tangles n, $-n$ $(n \in \mathbb{N})$, and 0 are the tangles shown in Figure 3, a–c, respectively. The tangle shown in Figure 3,d, is denoted by ∞.

The product $(\cdots((i_1 i_2) i_3) \cdots i_n)$ of integer tangles $i_1, \dots, i_n$ is called the *rational tangle* $i_1 \cdots i_n$. Conway [3] has shown that two rational tangles $i_1 \cdots i_n$ and $j_1 \cdots j_m$ are isotopic if and only if the corresponding continued fractions

$$i_n + 1/i_{n-1} + 1/i_{n-2} + \cdots + 1/i_1$$

and

$$j_m + 1/j_{m-1} + 1/j_{m-2} + \cdots + 1/j_1$$

are equal (we set $1/0 = \infty$, $1/\infty = 0$, $\infty + k = \infty$, where $k \in \mathbb{Z}$). Thus any rational tangle different from 0, ∞, 1, and -1 can be reduced to the standard form which is either $i_1 \cdots i_n$ or $i_1 \cdots i_n 0$, where $|i_1| \geq 2$, all the numbers $i_1, \dots, i_n$ are nonzero

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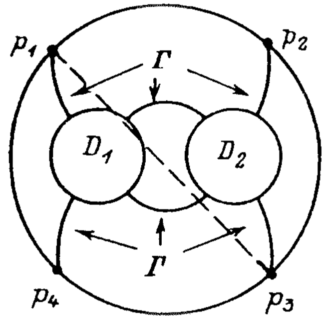

FIGURE 1

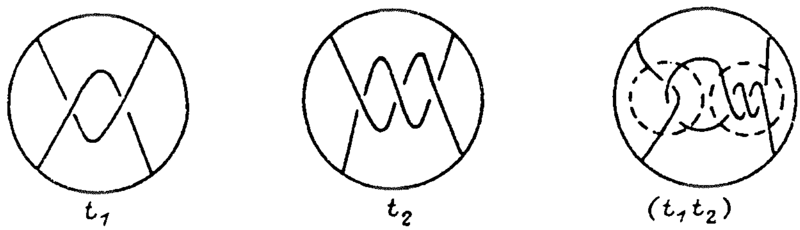

FIGURE 2

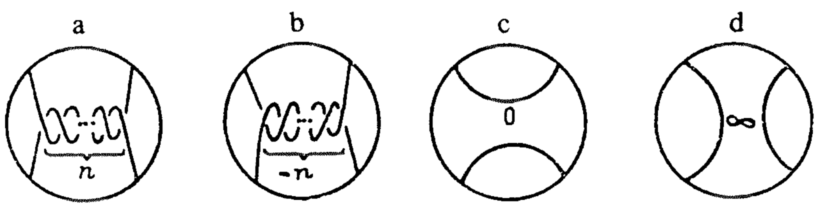

FIGURE 3

and have the same sign. A rational tangle $i_1 \cdots i_n$ with

$$i_n + 1/i_{n-1} + 1/i_{n-2} + \cdots + 1/i_1 = p/q \in \mathbb{Q}$$

is called the rational p/q tangle. The rational p/q tangle will be denoted by $t_{p/q}$.

After identification of the diametrically opposite points of the boundary sphere ∂D^3 a tangle $t \subset D^3$ becomes a link $L \subset \mathbb{R}P^3$. Such links obtained from rational tangles will be called *projective four-braidings* (see the explanation of this term in Appendix).

The *projective Montesinos link of type* $(e, p_1/q_1, \ldots, p_r/q_r)$ is the link shown in Figure 4, where $e \in \mathbb{Z}$, t_{p_i/q_i} is the rational p_i/q_i tangle, $p_i \geq 2$, and p_i, q_i are relatively prime for all $i = 1, \ldots, r$.

1.2. **Classification of projective four-braidings.**

Theorem 1. *Two links in the real projective space $\mathbb{R}P^3$ obtained from rational p/q and r/s tangles (with relatively prime p, q and r, s) are isotopic if and only if either $p/q = r/s$ or $p/q = -s/r$.*

Theorem 2. *Two links in the real projective space $\mathbb{R}P^3$ obtained from rational p/q and r/s tangles (with relatively prime p, q and r, s) are homeomorphic if and only if either $|p/q| = |r/s|$ or $|p/q| = |s/r|$.*

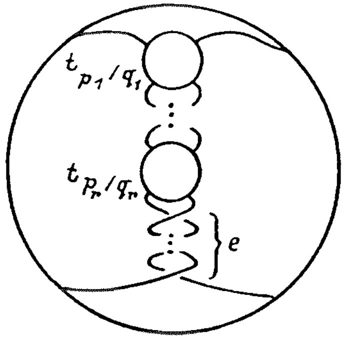

FIGURE 4

Proof of Theorem 2. *Sufficiency.* It is easy to see that multiplication of the fraction p/q by -1 corresponds to the symmetry of the tangle with respect to the projection plane, and inversion of the fraction p/q (i.e. changing places of p and q) corresponds to the symmetry with respect to the plane orthogonal to the projection plane and lassing through the points p_1 and p_3 (Figure 1). Both of these symmetries generate autohomeomorphisms of the space $\mathbb{R}P^3$.

Necessity of the condition follows from Theorem 3, which will be proved in §2, and Theorem 4 on the topological classification of prism manifolds. ●

Theorem 3. *There are precisely two double coverings of the projective space* $\mathbb{R}P^3$ *branched over the link which is obtained from the rational* p/q *tangle. They are homeomorphic to the prism manifolds* $Q(|p|, |q|)$ *and* $Q(|q|, |p|)$ (*under assumption that* p *and* q *are relatively prime*).

Theorem 4 (see [4], [5]). *The manifolds* $Q(m_1, n_1)$ *and* $Q(m_2, n_2)$ *are homeomorphic if and only if* $|m_1| = |m_2|$ *and* $|n_1| = |n_2|$.

Theorem 1 is derived below in 2.7 from Theorem 2 and the well-known fact that there is no orientation reversing autohomeomorphism of $Q(m, n)$ for $m \neq 1$ and $n \neq 0$.

1.3. **Classification of projective Montesinos links.**

Theorem 5. *Two projective Montesinos links* L *and* L' *of types*

$$(e, p_1/q_1, \dots, p_r/q_r) \qquad \text{and} \qquad (e', p'_1/q'_1, \dots, p'_{r'}/q'_{r'})$$

respectively, with $r, r' \geq 2$ *are isotopic if and only if*

(1) $e - \sum_{i=1}^{r} p_i/q_i = e' - \sum_{i=1}^{r'} p'_i/q'_i$;

(2) *the sequences*

$$(p_1/q_1 \bmod 1, \dots, p_r/q_r \bmod 1) \qquad \text{and} \qquad (p'_1/q'_1 \bmod 1, \dots, p'_{r'}/q'_{r'} \bmod 1)$$

can be obtained one from the other by a cyclic permutation and (*or*) *reversing the order.*

Proof. Necessity. First we construct the preimage of the projective Montesinos link L under the covering $S^3 \to \mathbb{R}P^3$. To do this (see [2]) place the diagram D of the link L on the plane. Right under D place its image under the slide symmetry with respect to the line passing through the center of the disc of diagram D; in this image replace all undercrossings by overcrossings. Connect by a simple path each end of an arc in diagram D lying on the boundary circle with the point obtained from this end via the composition of the symmetry about the center of the disc of diagram D and the slide

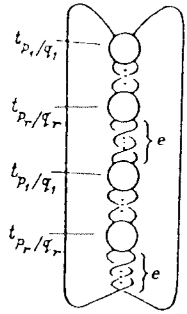

FIGURE 5

symmetry. Choose pairwise disjoint connecting paths. It is easy to see that, under the described slide symmetry and the subsequent replacement of undercrossings by overcrossings, a rational tangle is mapped onto itself. Therefore, the preimage of the projective Montesinos link L of type $(e, p_1/q_1, \dots, p_r/q_r)$ under the covering $S^3 \to \mathbb{R}P^3$ is the regular Montesinos link $\widetilde{L}$ of type

$$(2e, p_1/q_1, \dots, p_r/q_r, p_1/q_1, \dots, p_r/q_r);$$

see Figure 5. If the links L and L' are isotopic, then their preimages $\widetilde{L}$ and $\widetilde{L}'$ in S^3 are also isotopic. From the isotopy classification of Montesinos links in S^3 (see, for example, [6]) it follows that

$$2e - 2\sum_{i=1}^{r} p_i/q_i = 2e' - 2\sum_{i=1}^{r'} p'_i/q'_i$$

and that the sequences

$$(p_1/q_1 \bmod 1, \dots, p_r/q_r \bmod 1, p_1/q_1 \bmod 1, \dots, p_r/q_r \bmod 1)$$

and

$$(p'_1/q'_1 \bmod 1, \dots, p'_{r'}/q'_{r'} \bmod 1, p'_1/q'_1 \bmod 1, \dots, p'_{r'}/q'_{r'} \bmod 1)$$

can be obtained from each other by a cyclic permutation and (or) reversing the order. Therefore, conditions (1) and (2) of the theorem are satisfied.

Sufficiency. Suppose that conditions (1) and (2) are satisfied. It is clear that by the isotopy of the link L generated by a cyclic permutation of tangles and (or) reversing their order we can make $p_i/q_i = p'_i/q'_i \bmod 1$ for all $i = 1, \dots, r$. The obvious isotopy (Figure 6,a) allows us to move some of the e crossings into the space between tangles.

Do it in such a way that there are $e_i = p_i/q_i - p'_i/q'_i$ crossings between the tangles t_{p_i/q_i} and $t_{p_{i+1}/q_{i+1}}$, see Figure 6,b. Adjoining e_i crossings to the tangle t_{p_i/q_i} (Figure 6,b) gives the tangle $t_{p'_i/q'_i}$. Thus we get the Montesinos link of type

$$\left(e - \sum_{i=1}^{r}(p_i/q_i - p'_i/q'_i), p'_1/q'_1, \dots, p'_r/q'_r\right),$$

i.e. link L'. ●

Corollary. *Two projective Montesinos links* L *and* L' *of types*

$$(e, p_1/q_1, \dots, p_r/q_r) \quad \textit{and} \quad (e', p'_1/q'_1, \dots, p'_{r'}/q'_{r'})$$

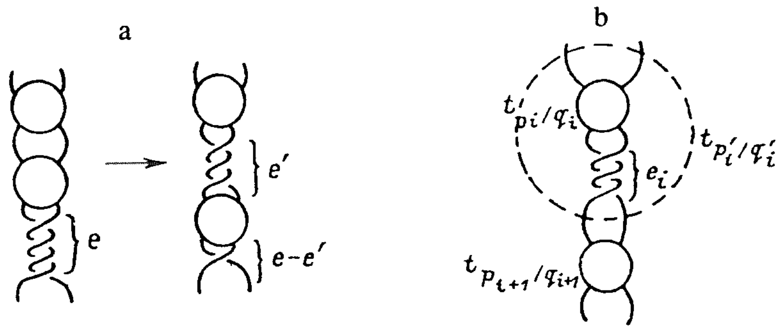

FIGURE 6

with $r, r' \geq 2$ *are homeomorphic if and only if there exists an* ε *equal to either* 1 *or* -1, *such that*

(1) $e - \sum_{i=1}^{r} p_i/q_i = \varepsilon(e' - \sum_{i=1}^{r'} p_i'/q_i')$;
(2) *the sequences*

$$(p_1/q_1 \bmod 1, \dots, p_r/q_r \bmod 1) \quad \textit{and} \quad (\varepsilon p_1'/q_1' \bmod 1, \dots, \varepsilon p_{r'}'/q_{r'}' \bmod 1)$$

can be obtained one from the other by a cyclic permutation and (or) reversing the order.

Proof. Since every homeomorphism $\mathbb{R}P^3 \to \mathbb{R}P^3$ is isotopic to either the identity homeomorphism or the symmetry with respect to some plane, it suffices to note that all the terms in the sequence $(e, p_1/q_1, \dots, p_r/q_r)$ are multiplied by -1 under the symmetry of the link with respect to the projection plane. ●

§2. DOUBLE COVERINGS OF THE PROJECTIVE SPACE $\mathbb{R}P^3$ BRANCHED OVER A PROJECTIVE FOUR-BRAIDING

2.1. **The number of double branched coverings.** Recall the general classification theorem for double branched coverings.

Theorem. *Let* X *be an r-dimensional manifold, and let* $A \subset X$ *be a submanifold of codimension* 2 *with* $\partial A \subset \partial X$. *A double covering* $Y \to X$ *branched over* A *exists if and only if* $\text{in}_*[A] = 0 \in H_{r-2}(X, \partial X; \mathbb{Z}_2)$, *where* $\text{in}_*: H_{r-2}(A, \partial A; \mathbb{Z}_2) \to H_{r-2}(X, \partial X; \mathbb{Z}_2)$ *is the homomorphism induced by inclusion. Such coverings (considered up to equivalence) are in bijective correspondence with the classes* $w \in H_{r-1}(X, A \cup \partial X; \mathbb{Z}_2)$ *such that* $\partial w = [A]$, *where* ∂ *denotes the composition* $H_{r-1}(X, A\cup\partial X; \mathbb{Z}_2) \to H_{r-2}(A\cup\partial X, \partial X; \mathbb{Z}_2) \to H_{r-2}(A, \partial A; \mathbb{Z}_2)$ *of the boundary homomorphism from the sequence of the triple* $(X, A \cup \partial X, \partial X)$ *and the excision isomorphism. The covering corresponding to* w *is uniquely characterized by the fact that it is trivial over the complement of a cycle realizing* w. *Thus, the covering can be constructed by gluing together two copies of* X *split along such a cycle.*

This theorem implies that for any projective four-braiding $L \subset \mathbb{R}P^3$ (and, in general, for any link in $\mathbb{R}P^3$) there are exactly two (up to equivalence) double coverings $\tilde{k}_i: N_i \to \mathbb{R}P^3$, $i = 1, 2$, branched over L.

2.2. **Representation of the covering as a result of gluing the solid torus onto the Klein bottle.** Let $t \subset D^3$ be a rational tangle, $L \subset \mathbb{R}P^3$ be the corresponding projective four-braiding. The general theorem for double branched coverings shows that there is a unique double covering $B \to D^3$ of the ball D^3 branched over t. It is well known that the rational tangle t can be obtained from a tangle $t_0 \subset D^3$ of type 0 shown in

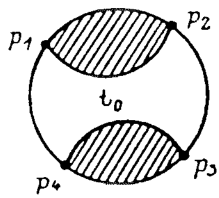

FIGURE 7

Figure 7 via a homeomorphism $\widetilde{\beta}: D^3 \to D^3$ with $\widetilde{\beta}(t_0) = t$. The total space B_0 of the double covering $B_0 \to D^3$ branched over t_0 is obtained as the result of gluing together two copies of the ball D^3 with two excised discs bounded by the tangle t_0 and the arcs of the great circle connecting the boundary points p_1, p_2, p_3, p_4 of the tangle t_0 (Figure 7). Hence B_0 is homeomorphic to the solid torus $D^2 \times S^1$. The map $\widetilde{\psi}: B_0 \to B$ covering $\widetilde{\beta}$ gives a homeomorphism $B \cong D^2 \times S^1$.

Let c be the antipodal involution of the sphere ∂D^3. Then we have a commutative diagram

$$\begin{array}{ccc} B & \longrightarrow & N_i \\ \downarrow & & \downarrow \tilde{k}_i \\ t \subset D^3 & \xrightarrow{\text{pr}} & \mathbb{R}P^3 \supset L \end{array}$$

It is clear that the manifolds N_i $(i = 1, 2)$ are obtained from B by factoring its boundary $T = \partial B$ by the involutions covering c. Note that there are two such involutions; they are related via composition with the nontrivial automorphism of the covering $\partial B \to \partial D^3$. As the involution c itself, the involutions $\tilde{c}_i$ that cover c $(i = 1, 2)$ act without fixed points and reverse orientation. Thus, the quotient space $X_i = T/\tilde{c}_i$ is a closed nonorientable manifold. Moreover, by the Riemann-Hurwitz formula $\chi(X_i) = \chi(\tilde{k}_i^{-1}(\widetilde{\text{pr}}(\partial D^3))) = 0$, where χ is the Euler characteristic. Hence X_i is homeomorphic to the Klein bottle K_i. Thus we get N_i as the result of gluing the solid torus B by its boundary onto the Klein bottle K_i.

2.3. **Prism manifolds** $Q(m, n)$. Let $S^1 = \{z \in \mathbb{C}: |z| = 1\}$ be the circle, and let $\sigma: S^1 \times S^1 \to S^1 \times S^1$ be the involution of the torus defined by the formula $\sigma(z_1, z_2) = (\overline{z}_1, -z_2)$, where $(z_1, z_2) \in S^1 \times S^1$. Then the quotient space of the torus $S^1 \times S^1$ by the involution σ is the Klein bottle K. Let $\pi: S^1 \times S^1 \to K$ be the natural projection.

Let $D^2 \times S^1$ be the solid torus and $\varphi: \partial(D^2 \times S^1) = S^1 \times S^1 \to S^1 \times S^1$ be some homeomorphism. Denote by $[\mu], [\lambda] \in H_1(S^1 \times S^1; \mathbb{Z})$ the homology classes of the meridian $\mu = S^1 \times i$ and parallel $\lambda = i \times S^1$ of the torus $S^1 \times S^1$ (here $i = \sqrt{-1}$). Clearly $\varphi_*[\mu] = m[\mu] + n[\lambda]$ for some relatively prime $m, n \in \mathbb{Z}$. The numbers m, n define the map φ up to isotopy and composition (on the right) with a homomorphism extending to a homeomorphism of the solid torus and so determine (up to homeomorphism) the manifold $Q(m, n) = D^2 \times S^1 \cup_{\pi\varphi} K$. It was shown in [7] that the result of gluing the solid torus $D^2 \times S^1$ onto K depends only on $|m|$, $|n|$ and that manifolds $Q(|m|, |n|)$ corresponding to distinct pairs $|m|, |n|$ are not homotopy equivalent.

2.4. **Geometry of the covering** π. Represent the torus $S^1 \times S^1$ as the union of two cylinders: $S^1 \times S^1 = C_+ \cup C_-$, where

$$C_+ = \{(z_1, z_2) \in S^1 \times S^1 : \operatorname{Re} z_1 \geq 0\},$$
$$C_- = \{(z_1, z_2) \in S^1 \times S^1 : \operatorname{Re} z_1 \leq 0\}.$$

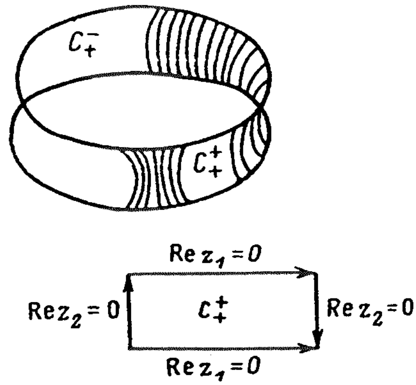

FIGURE 8

Since $C_+ = C_+^+ \cup C_+^-$ where

$$C_+^+ = \{(z_1, z_2) \in S^1 \times S^1 \colon \operatorname{Re} z_1 \geq 0, \operatorname{Re} z_2 \geq 0\},$$
$$C_+^- = \{(z_1, z_2) \in S^1 \times S^1 \colon \operatorname{Re} z_1 \geq 0, \operatorname{Re} z_2 \leq 0\}$$

and $\sigma(C_+^+) = C_+^-$, we see that $\pi(C_+) = \pi(C_+^+)$ is the Möbius band with boundary $\Delta = \pi(\{(z_1, z_2) \in S^1 \times S^1 \colon \operatorname{Re} z_1 = 0\})$ (Figure 8). Similarly, $\pi(C_-)$ is the Möbius band with the same boundary Δ. Thus, the Klein bottle $K = \pi(C_+ \cup C_-)$ can be viewed as the result of gluing together two Möbius bands $\pi(C_+)$ and $\pi(C_-)$ along their common boundary Δ. Since $\{(z_1, z_2) \in S^1 \times S^1 \colon \operatorname{Re} z_1 = 0\} = i \times S^1 \cup (-i) \times S^1$ and $\sigma(i \times S^1) = (-i) \times S^1$, we have $\Delta = \pi(i \times S^1)$. Note that the boundary Δ is isotopic to the doubled center line of the Möbius band, i.e. the doubled parallel of the Klein bottle K. The meridian of the Klein bottle is

$$\pi(\{(z_1, z_2) \in S^1 \times S^1 \colon \operatorname{Re} z_2 = 0\}) = \pi(S^1 \times i \cup S^1 \times (-i)) = \pi(S^1 \times i).$$

Thus the image of the parallel $i \times S^1$ of the torus $S^1 \times S^1$ under the covering π is the doubled parallel of the Klein bottle K, and the image of the meridian $S^1 \times i$ is a meridian.

2.5. **Simple closed curves on the Klein bottle.** Consider a restriction of the diagram we constructed in 2.2:

$$\begin{array}{ccc} T & \xrightarrow{\omega_i} & K_i \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle k_i} \\ S^2 & \xrightarrow{\text{pr}} & \mathbb{R}P^2 \end{array}$$

On the boundary sphere $S^2 = \partial D^3$ there are four marked points: p_1, p_2, p_3, p_4 which are the boundary of the tangle $t \subset D^3$. Denote by $d_{12}, d_{23}, d_{34}, d_{14}$ the arcs of the great circles on the sphere S^2 connecting the points p_1 and p_2, p_2 and p_3, p_3 and p_4, p_1 and p_4, respectively (Figure 9). The total space T of the double covering s branched over $\{p_1, p_2, p_3, p_4\}$ can be constructed, according to the general theorem for double branched coverings, by gluing together two copies of the sphere S^2 split along the arcs d_{12} and d_{34}. Put $\xi = s^{-1}(d_{34})$ and $\zeta = s^{-1}(d_{23})$.

The map k_i is a double covering branched over the points $q_1 = \text{pr}(p_1)[= \text{pr}(p_3)]$ and $q_2 = \text{pr}(p_2)[= \text{pr}(p_4)]$. Let $I_1 = \text{pr}(d_{34})[= \text{pr}(d_{12})]$ and $I_2 = \text{pr}(d_{23})[= \text{pr}(d_{14})]$ (Figure 10). Then one of the covering spaces K_i, say K_2, is obtained as the result of gluing together two copies of the plane $\mathbb{R}P^2$ split along the segment I_1, and the

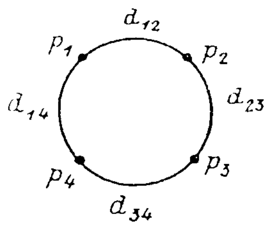

FIGURE 9

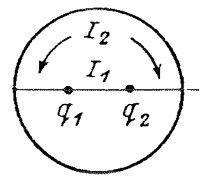

FIGURE 10

other covering space, K_1, as the result of gluing two copies of the plane $\mathbb{R}P^2$ split along the segment I_2. Clearly

$$\omega_i(\xi) = k_i^{-1}(\mathrm{pr}(s(\xi))) = k_i^{-1}(I_1),$$
$$\omega_i(\zeta) = k_i^{-1}(\mathrm{pr}(s(\zeta))) = k_i^{-1}(I_2).$$

From the two possible coverings k_i choose the one with the Klein bottle K_1 as the total space. Thus K_1 can be viewed as the result of gluing together two Möbius bands along their common boundary $k_1^{-1}(I_2)$. Then $k_1^{-1}(I_1)$ is obtained from the fibers of the Möbius bands fibered over the circle. Hence $\omega_1(\xi) = k_1^{-1}(I_1)$ is a meridian of the Klein bottle K_1, and $\omega_1(\zeta) = k_1^{-1}(I_2)$ is its doubled parallel. This allows us to construct a homeomorphism of the coverings π and ω_1

$$\begin{array}{ccc} S^1 \times S^1 & \longrightarrow & T \\ \pi \downarrow & & \downarrow \omega_1 \\ K & \longrightarrow & K_1 \end{array}$$

under which the meridian $S^1 \times i$ and the parallel $i \times S^1$ of the torus $S^1 \times S^1$ are mapped to the curves ξ and ζ, respectively, on the torus T.

2.6. **The attaching map.** Let $t_{p/q}$ be the rational tangle with $p/q = i_r + 1/i_{r-1} + 1/i_{r-2} + \cdots + 1/i_1$ (all the numbers i_k have the same sign and $|i_1| \geq 2$). If r is even, then decompose the quotient p/q again so that r becomes odd. It would suffice to represent i_1 in the form $(i_1 - 1) + 1/1$. Note that such decomposition of the quotient p/q is unique and allows us to represent the tangle $t_{p/q}$ by the diagram shown in Figure 11,a.

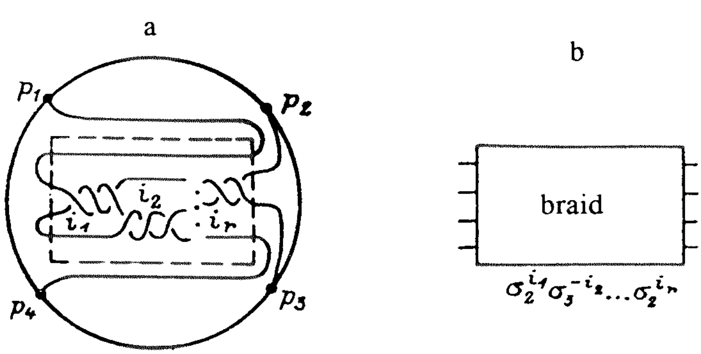

FIGURE 11

To the tangle $t_{p/q}$ corresponds a braid on four strings $\sigma_2^{i_1}\sigma_3^{-i_2}\sigma_2^{i_3}\cdots\sigma_2^{i_r} \in B_4$ (Figure 11,b). It has been remarked (in 2.2) that the tangle $t_{p/q} \subset D^3$ can be obtained from the standard tangle $t_0 \subset D^3$ shown in Figure 7 via some homeomorphism

β with $\beta(\partial t_0) = \partial t_{p/q}$. The homeomorphism β is constructed as the composition of homeomorphisms $\beta_3, \beta_2 : S^2 \to S^2$ as follows: $\beta = \beta_2^{i_r} \cdots \beta_3^{-i_2} \beta_2^{i_1}$, where β_k $(k = 2, 3)$ is the twisting corresponding to the elementary braid $\sigma_k \in B_4$. The homeomorphisms β_3, β_2 can be chosen so that their supports are contained in small neighborhoods of the arcs d_{34} and d_{23}. Thus, we can assume that the support of the homeomorphism β is contained in some neighborhood of the arc $d_{34} \cup d_{23}$. Denote by p_0 some point of the sphere S^2 not contained in the support of the homeomorphism β (so that $\beta(p_0) = p_0$).

Let $s: T \to S^2$ be a double covering branched over $\partial t_0 = \partial t_{p/q}$. Denote by ψ the homeomorphism of the torus T identical on $s^{-1}(p_0)$ and covering homeomorphism β:

$$\begin{array}{ccc} T & \xrightarrow{\psi} & T \\ s\downarrow & & \downarrow s \\ S^2 & \xrightarrow{\beta} & S^2 \end{array}$$

Let Homeo(T) be the set of all orientation preserving homeomorphisms of the torus T. The described construction of the isotopy class of homeomorphisms of the torus using a braid on four strings gives the homeomorphism $\gamma: B_4 \to \pi_0(\text{Homeo}(T))$; compare [8], 2.10. It is known that $\pi_0(\text{Homeo}(T)) \cong \text{SL}(2;\mathbb{Z})$, and this isomorphism is given by the induced automorphism of the group $H_1(T)$ and the isomorphism $H_1(T) \to \mathbb{Z} \oplus \mathbb{Z}$. Choose the latter isomorphism so that the images of the classes of curves ξ and ζ on the torus T are respectively the elements $\binom{0}{1}, \binom{1}{0} \in \mathbb{Z} \oplus \mathbb{Z}$. Then on the generators $\sigma_1, \sigma_2, \sigma_3$ of the braid group B_4 the homomorphism γ is given as follows

$$\gamma(\sigma_1) = \gamma(\sigma_3) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \qquad \gamma(\sigma_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The homeomorphism ψ corresponding to the tangle $t_{p/q}$ defines the automorphism of the homology group $H_1(T)$ with matrix

$$\Psi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{i_r} \cdots \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}^{-i_2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{i_1}.$$

Since there is a homeomorphism $\tilde{\psi}: B_0 \to B$ covering $\tilde{\beta}$ (see 2.2) and extending $\psi: \partial B_0 = T \to T = \partial B$, the covering space $N_1 = B \cup_{\omega_1} K_1$ can be viewed as $B_0 \cup_{\omega_1 \psi} K_1$. The curve $\xi = s^{-1}(d_{34}) \subset T$ is the meridian of the solid torus B_0, since it bounds a disc in B_0 which is the preimage under the double branched covering $B_0 \to D^3$ of the segment bounded by the arc d_{34} and the tangle t_0 (Figure 7). Let us compute $\Psi_*[\xi]$:

$$\begin{aligned} \Psi_*[\xi] = \Psi \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 & i_r \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ i_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & i_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} p \\ q \end{pmatrix} = q[\xi] + p[\zeta]. \end{aligned}$$

Hence the manifold N_1 is homeomorphic to $Q(q, p)$. It is not hard to see that the manifold N_2 is homeomorphic to $Q(p, q)$.

2.7. *Proof of Theorem* 1. It is known that any homeomorphism of the space $\mathbb{R}P^3$ is isotopic to either the identity homeomorphism or the symmetry with respect to some plane. Therefore, any two homeomorphic, but not isotopic, links are the reflections of each other.

Let L_1 be the projective four-braiding p/q, and L_2 be its reflection. (It has been remarked above, in the proof of Theorem 2, that L_2 corresponds to either the quotient $-p/q$ or q/p.) If the links L_1 and L_2 are isotopic, then there exists an orientation reversing homeomorphism $Q(p, q) \to Q(p, q)$. It is easy to show that the manifold $Q(p, q)$ is a Seifert fibration over $\mathbb{R}P^2$ with one singular fiber. It is known [9] that such manifolds admit an orientation reversing homeomorphism only if $p = 1$ and $q = 0$. The link obtained from the tangle $\infty = 1/0$ is an affine unknotted circle. Thus the projective four-braiding, considered up to isotopy, determines the pair of quotients $(p/q, -q/p)$.

APPENDIX: SPECIAL TYPES OF DIAGRAMS OF LINKS IN $\mathbb{R}P^3$

There is a number of special types of diagrams of links in the sphere: closed braids, $2n$-braidings, diagrams with n bridges. It is known that any link can be represented by a closed braid (Alexander's theorem), a closed braid on n strings is a $2n$-braiding, any $2n$-braiding is isotopic to a link with n bridges, and vice versa, any link with n bridges is isotopic to a $2n$-braiding.

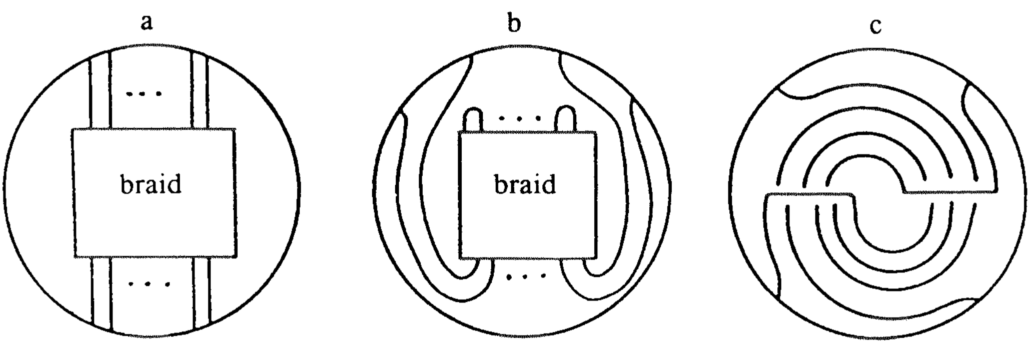

FIGURE 12

For links in the projective space we can select analogous special types of diagrams—closed braids, $2n$-braidings and links with n bridges. The general case of a closed braid and a $2n$-braiding is shown in Figure 12,a and b; an example of link with n bridges in Figure 12,c $(n = 2)$.

Such a translation of terminology from the case of links in the sphere to the case of links in $\mathbb{R}P^3$ is explained by the fact that the preimage of a link with a special diagram under the covering $S^3 \to \mathbb{R}P^3$ can be described by a diagram of the corresponding type. It is easy to see that among the special types of diagrams of links in $\mathbb{R}P^3$ there are relations similar to the relations mentioned above among the special types of diagrams of links in S^3.

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