LINKING NUMBER IN A PROJECTIVE SPACE VIA THE DEGREE OF A MAP

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Introduction

As is well-known, the linking number of disjoint oriented circles $C_1, C_2 \subset \mathbb{R}^3$ can be presented as the degree of the map

$$C_1 \times C_2 \to S^2 : (x, y) \mapsto \frac{y - x}{|y - x|}. \tag{1}$$

The linking number is defined in a more general situation for two disjoint oriented circles realizing homotopy classes of finite order in any oriented 3-manifold. In particular, for disjoint oriented circles in the projective space $\mathbb{R}P^3$. However, even in $\mathbb{R}P^3$, with its rich geometry, the interpretation of the linking number as the degree of a map similar to (1) does not exist. The reasons for this are presented below in Section 1.

In this paper for any two disjoint oriented circles in $\mathbb{R}P^3$, we construct a 3-dimensional oriented configuration space and its map to $\mathbb{R}P^3$ such that the degree of this map is the linking number of the circles multiplied by 2. This construction seems to be the closest possible replacement for (1). A diagramatical formula for the linking number, which emerges in evaluation of this degree by counting pre-images of a regular value, is similar to the well-known diagramatical formula for the degree of (1). In fact, a local ingredient of the diagramatical formula for the linking number of circles in $\mathbb{R}P^3$, the local writhe of a link diagram at a crossing point, has appeared in my paper [1] and inspired the present work.

The construction of the configuration space and its map to $\mathbb{R}P^3$ discussed above is generalized to the case of a pair of disjoint oriented closed submanifolds (or even sub-pseudo-manifolds) of $\mathbb{R}P^n$ such that the sum of their dimensions is $n - 1$.

1. The classical construction and impossibility of its straightforward generalization

The Gauss map (1) can be viewed as a composition of the natural inclusion

$$C_1 \times C_2 \to (\mathbb{R}^3 \times \mathbb{R}^3) \smallsetminus \Delta = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid x \neq y\}$$

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and the projection

$$(\mathbb{R}^3 \times \mathbb{R}^3) \smallsetminus \Delta \to S^2 : (x, y) \mapsto \frac{y - x}{|y - x|}.$$

This composition induces a homomorphism

$$\mathbb{Z} = H_2(C_1 \times C_2) \to H_2((\mathbb{R}^3 \times \mathbb{R}^3) \smallsetminus \Delta)) \to H_2(S^2) = \mathbb{Z}.$$

The degree of the Gauss map is an integer *deg* such that $[C_1 \times C_2] \mapsto deg \cdot [S^2]$ for fundamental classes $[C_1 \times C_2]$ and $[S^2]$. Since

$$H_2((\mathbb{R}^3 \times \mathbb{R}^3) \smallsetminus \Delta)) \to H_2(S^2) = \mathbb{Z}$$

is an isomorphism as a map induced by a homotopy equivalence, the degree of the Gauss map is actually defined by the inclusion homomorphism $H_2(C_1 \times C_2) \to H_2((\mathbb{R}^3 \times \mathbb{R}^3) \smallsetminus \Delta))$.

Consider now the case of two oriented disjoint circles C_1, C_2 in $\mathbb{R}P^3$. The classical case suggests to consider the inclusion

$$C_1 \times C_2 \to (\mathbb{R}P^3 \times \mathbb{R}P^3) \smallsetminus \Delta = \{(x, y) \in \mathbb{R}P^3 \times \mathbb{R}P^3 : x \neq y\}.$$

A natural hope is to find an expression for the linking number in terms of the homomorphism induced by this map in the second homology. However, this hope is ruined by the fact that the second homology of the target space is $\mathbb{Z}_2$.

To prove that $H_2((\mathbb{R}P^3 \times \mathbb{R}P^3) \smallsetminus \Delta) = \mathbb{Z}_2$, consider the natural projection

$$(\mathbb{R}P^3 \times \mathbb{R}P^3) \smallsetminus \Delta \to \mathbb{R}P^3, (x, y) \mapsto x.$$

This is a locally trivial fibration over $\mathbb{R}P^3$ with a fiber $\mathbb{R}P^3 \smallsetminus \{x\}$. The fiber is homotopy equivalent to $\mathbb{R}P^2$ and the fibration can be trivialised using a trivialization of the tangent bundle of $\mathbb{R}P^3$. Hence $H_2((\mathbb{R}P^3 \times \mathbb{R}P^3) \smallsetminus \Delta) = H_2(\mathbb{R}P^3 \times \mathbb{R}P^2)$. By Künneth formula, $H_2(\mathbb{R}P^3 \times \mathbb{R}P^2) = \mathbb{Z}_2$.

Thus the inclusion $C_1 \times C_2 \to (\mathbb{R}P^3 \times \mathbb{R}P^3) \smallsetminus \Delta$ induces in the second homology a homomorphism $\mathbb{Z} \to \mathbb{Z}_2$ and the linking number of C_1 and C_2 (which is a half-integer) cannot be expressed in terms of this homomorphism.

2. Configuration space

An appropriate configuration space for a homology definition of the linking number can be found in the next dimension. Let C_1, C_2 be two oriented disjoint circles in $\mathbb{R}P^3$. Consider a space K_0 of all triples (x, t, y), where $x \in C_1$, $y \in C_2$ and t $(x \neq t \neq y)$ is a point on the line passing through x and y.

This space is naturally equipped with a structure of a smooth oriented 3-manifold. We introduce this structure by describing an appropriate class of local coordinate systems. One of the coordinates is defined globally in the following way. For $(x, t, y) \in K_0$, consider a straight segment I containing t and bounded by x, y. Denote by J the segment on I bounded by x and t. Put $\tau = \frac{|J|}{|I|}$, where $| \cdot |$ denotes the length with respect to the Fubini-Study metric on $\mathbb{R}P^3$. This defines a function $\tau : K_0 \to (0, 1)$.

To construct a local coordinate system in a neighbourhood of an arbitrary point (x_0, t_0, y_0) of K_0, choose a local coordinate ξ in a neighbourhood of x_0

on C_1, a local coordinate η in a neighbourhood of y_0 on C_2. Functions ξ, τ, η restricted to a neighborhood of (x_0, t_0, y_0) form a local coordinate system in K_0.

The coordinate systems provided by this construction define a differential structure on K_0. By choosing only those of these coordinate systems, in which the first coordinate defines the orientation of C_1 and the third coordinate, the orientation of C_2, we define an orientation on K_0. This orientation is canonical in the sense that it depends only on the orientation of C_1 and C_2 and the ordering of them (the orientation changes, if we exchange C_1 and C_2).

In order to make this configuration space compact, we allow $t = x$ or $t = y$. In other words, consider

$$K_1 = \{(x, t, y) \in C_1 \times \mathbb{R}P^3 \times C_2 \mid t \in Span(x, y)\}.$$

This is a compact smooth 3-manifold containing K_0 as the complement of two disjoint copies of $C_1 \times C_2$. These copies are $S_1 = \{(x, t, y) \in K_1 \mid t = x\}$ and $S_2 = \{(x, t, y) \in K_1 \mid t = y\}$. Denote $S_1 \cup S_2$ by S.

The orientation of K_0 does not extend over K_1: it reverses when one goes transversally through the surface S at any point. Moreover, if at least one of the circles C_1 and C_2 is not homologous to zero in $\mathbb{R}P^3$, then K_1 is not orientable. Indeed, K_1 is fibered over the torus $C_1 \times C_2$ with fiber S^1, and the first Stiefel-Whitney class is zero iff both C_1 and C_2 are zero-homologous in $\mathbb{R}P^3$. (To describe the fibration completely, observe that the Euler number of this fibration is 0, since it has disjoint sections S_1 and S_2.)

3. Mapping and final factorizing of the configuration space

There is a natural map $F_1 : K_1 \to \mathbb{R}P^3$ defined by $F(x, t, y) = t$. Under this map the whole surface S_i is mapped to curve C_i, $i = 1, 2$. The preimage of a point $x \in C_1$ is the circle formed by triples (x, x, y) with $y \in C_2$, while the preimage of $y \in C_2$ consists of (x, y, y) with $x \in C_1$.

Let us contract each of these circles. In other words, consider the quotient space

$$K = K_1/_{(x_1,y,y)\sim(x_2,y,y), \quad (x,x,y_1)\sim(x,x,y_2)}.$$

This space consists of an intact copy of K_0, and the images of S_1 and S_2, which are naturally identified with C_1 and C_2, respectively.

One can show that

- If both C_1 and C_2 are not zero-homologous in $\mathbb{R}P^3$ then K is homeomorphic to $\mathbb{R}P^3$.
- If one of C_1 and C_2 is zero-homologous, while the other is not, then K is homeomorphic to the quotient space of S^3 which can be obtained by factorizing of an unknot in S^3 by a fixed point free involution.
- If both C_1 and C_2 are zero-homologous in $\mathbb{R}P^3$, then K is homeomorphic to the result of attaching to each other two copies of the 3-sphere $C_1 * C_2$ by the identity homeomorphism between the copies of $C_1 \cup C_2 \subset C_1 * C_2$.

We do not prove this, since it is not needed in what follows.

Thus K is not necessarily a 3-manifold, but it is a stratified space which consists of a 3-stratum K_0 and 1-strata C_1 and C_2. Hence the orientation of K_0 defines a homology class $[K_0]$, which belongs to $H_3(K, C_1 \cup C_2)$. The latter group is naturally isomorphic to $H_3(K)$: the inclusion map $H_3(K) \to H_3(K, C_1 \cup C_2)$ is an isomorphism, since it is surrounded in the homology sequence of the pair $(K, C_1 \cup C_2)$ by trivial groups $H_3(C_1 \cup C_2)$ and $H_2(C_1 \cup C_2)$. Therefore we identify $H_3(K, C_1 \cup C_2)$ with $H_3(K)$. Denote the homology class corresponding to $[K_0]$ under this identification by $[K]$.

4. The Main Theorem

The map $F_1 : K_1 \to \mathbb{R}P^3$ defined above defines a map $F : K \to \mathbb{R}P^3$. The induced map $F_* : H_3(K) \to H_3(\mathbb{R}P^3)(= \mathbb{Z})$ maps $[K]$ to a class which can be presented as $d[\mathbb{R}P^3]$, since the orientation class $[\mathbb{R}P^3]$ of $\mathbb{R}P^3$ generates $H_3(\mathbb{R}P^3)$. The integer coefficient d depends only on the initial data of the construction, which provided K and F, that is C_1 and C_2. Let us denote d by $d(C_1, C_2)$.

Theorem 1. *Under the conditions above,* $d(C_1, C_2) = 2\,\mathrm{lk}(C_1, C_2)$.

Proof. To evaluate $d(C_1, C_2)$ geometrically, we choose a regular value $v \in \mathbb{R}P^3 \smallsetminus (C_1 \cup C_2)$ of the map F and take the sum of local degrees of F over the preimages of v.

We have to prove that this sum is equal to the doubled linking number of C_1 and C_2. To evaluate the doubled linking number, we take the intersection number of C_2 with a chain Σ such that $\partial\Sigma = 2C_1$. For Σ, we choose the projective cone $C_v(C_1)$ over C_1 with vertex at v, i.e. the union of all projective lines passing through v and intersecting C_1. Obviously, points of $C_2 \cap C_v(C_1)$ are in a natural one-to-one correspondents with points of $F^{-1}(v)$, and we need to check that the local intersection number of C_2 and $C_v(C_1)$ at a point of $C_2 \cap C_v(C_1)$ is equal to the local degree of F at the corresponding point of $F^{-1}(v)$. The comparison of these two local numbers are shown in Figure 1. There we see a line L intersecting oriented embedded circles C_1 and C_2 in points x and y, respectively. Vectors $\dot\xi = \frac{\partial}{\partial\xi}$ and $\dot\eta = \frac{\partial}{\partial\eta}$ are tangent to C_1 and C_2 at x and y, respectively.

By the definition of differential, $dF(\dot\xi)$ is the velocity of t at $t = v$ as x moves along C_1 with velocity $\dot\xi$, while y and the ratio, in which t divides $[y, x]$, stay fixed (the ratio is equal to the ratio, in which v divides $[y, x]$). Therefore L, $\dot\xi$, $dF(\dot\xi)$ are coplanar and $\dot\xi$ is contained in L iff $dF(\dot\xi)$ is contained in L. Similarly, $dF(\dot\eta)$ is the velocity of t at $t = v$ as y moves along C_2 with velocity $\dot\eta$, while x and the ratio in which t divides $[y, x]$ stay fixed. Therefore L, $\dot\eta$, $dF(\dot\eta)$ are coplanar and $\dot\eta$ is contained in L iff $dF(\dot\eta)$ is contained in L.

Therefore, if L, $\dot\xi$ and $\dot\eta$ are not coplanar, then dF is not degenerate at t. The local degree of F at (x, t, y) is the value of the standard orientation of $\mathbb{R}P^3$ on the frame $dF(\dot\xi), dF(\dot\tau), dF(\dot\eta)$. (Recall that we defined the orientation of K_0 as positive on the local coordinate system ξ, τ, η, see Section 2.)

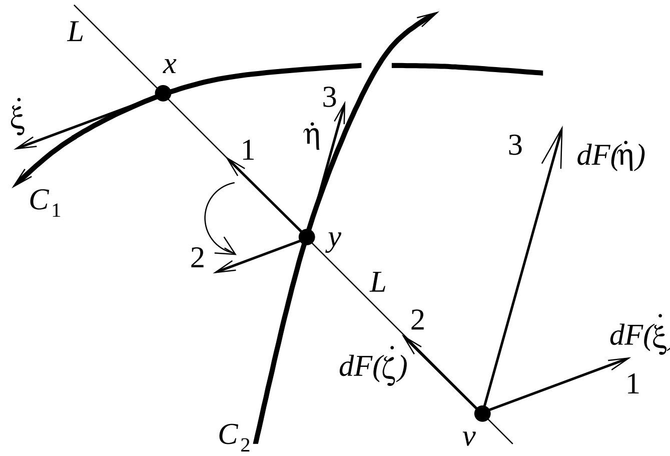

FIGURE 1

Let us compare this local degree to the local intersection number of C_2 and $C_t(C_1)$ at y. The local intersection number is the value taken by the same standard orientation of $\mathbb{R}P^3$ on the frame which consists of $\dot\eta$ and the basis of the tangent space of $C_t(C_1)$ at y positively oriented with respect to the orientation such that $\partial C_t(C_1) = 2C_1$. One of frames of this sort is shown in Figure 1. Its first vector is directed along L outwards the segment $[t, y]$ which does not contain x. The second vector is directed to the same side of the segment $[x, y]$ not containing t as $\dot\xi$.

In Figure 1 the orientations of these frames obviously coincide. (For reader's convenience the orderings of bases vectors are shown by numbers.) Figure 1 is necessarily affine, and, at first glance, quite special. One could guess that it does not represent all possible configurations. However, in fact, it is general: we get affine picture shown in Figure 1 for any $(x, t, y) \in K_0$, if we choose the infinity plane intersecting $L = span(x, y)$ in the segment bounded by x and y and not containing t. □

5. Versions and generalizations of Main Theorem

The number $d(C_1, C_2)$ involved in Theorem *1* can be introduced in a different way. Instead of F and K one can use F_1 and K_1. This would eliminate a step of the construction, which was the main contents of Section 3. However then instead of $[K] \in H_3(K)$ one should use relative homology $H_3(K_1, S)$ and the homology class $[K_1] \in H_3(K_1, S)$ defined by the orientation of $K_0 (= K_1 \smallsetminus S)$. The role of $F_* : H_3(K) \to H_3(\mathbb{R}P^3)$ would be played by $F_{1*} : H_3(K_1, S) \to H_3(\mathbb{R}P^3, C_1 \cup C_2)$. Since $H_3(\mathbb{R}P^3, C_1 \cup C_2)$ is isomorphic to $H_3(\mathbb{R}P^3) = \mathbb{Z}$, the image of $[K_1]$ under F_{1*} is equal to the d-fold multiple of the generator of $H_3(\mathbb{R}P^3, C_1 \cup C_2)$ defined by the standard orientation of $\mathbb{R}P^3$. Of course, this d is equal to $d(C_1, C_2)$.

Theorem *1* can be generalized by admitting more general C_1 and C_2. In the original setup, C_1 and C_2 are oriented smooth disjoint circles embedded in $\mathbb{R}P^3$. Of course, the assumption of their connectedness is not needed. If C_1, C_2 are oriented disjoint closed 1-submanifolds of $\mathbb{R}P^3$, one can repeat everything made above, besides the discussion of the topological type of K in Section 3. This discussion can be skeapt, it was included only to show that K may have singularities. In the case of multicomponent C_1 and C_2, K_1 consists of connected components which are of the same types as above, while K can be obtained from pieces of the same type by gluing them together along the components of C_1 and C_2. The formulation and proof of Theorem *1* do not change.

Moreover, Theorem *1* can be generalized to the case of cycles C_1, C_2 supported by disjoint smoothly stratified 1-dimensional spaces $|C_1|$ and $|C_2|$ (i.e., graphs). Each 1-stratum of $|C_i|$ is oriented and equipped with a coefficient so that the corresponding linear combination of the orientation cycles of the strata is C_i. Then we define

$$K_1 = \{(x,t,y) \in |C_1| \times \mathbb{R}P^3 \times |C_2| \mid t \in Span(x,y)\},$$

$S_1 = \{(x,t,y) \in K_1 \mid t = x\}$ and $S_2 = \{(x,t,y) \in K_1 \mid t = y\}$, put $S = S_1 \cup S_2$ and $K_0 = K_1 \smallsetminus S$. The stratifications of $|C_1|$ and $|C_2|$ define natural stratifications of these spaces. The orientations of 1-strata of $|C_i|$ define orientations of 3-strata of K_0 as above. Each of these 3-strata is defined by a pair consisting of a 1-stratum of $|C_1|$ and a 1-stratum of $|C_2|$. Assigning to each of the 3-strata the products of the corresponding coefficients, we get a 3-cycle of

$$K = K_1/_{(x_1,y,y)\sim(x_2,y,y) \quad (x,x,y_1)\sim(x,x,y_2)}.$$

The rest of the preparations to Theorem *1*, the theorem itself and its proof can be repeated literally.

Consider now a straightforward high-dimensional generalization. It is possible to consider arbitrary cycles, as above, but we restrict ourselves to the case of submanifolds. Let C_1 and C_2 be oriented closed disjoint smooth submanifolds of dimensions p_1 and p_2, respectively, of the real projective space $\mathbb{R}P^n$. Assume that $n = p_1 + p_2 + 1$ is an odd number and, if $p_i = 0$, then C_i is zero-homologous in $\mathbb{R}P^n$ (i.e., C_i consists of an even number of points, half of which are oriented positively and half, negatively).

As is well known, in this situation there is an integer or half-integer $\operatorname{lk}(C_1, C_2)$, which can be defined as follows. Take any oriented $(p_1 + 1)$-dimensional chain Σ with integer coefficients in $\mathbb{R}P^n$ with $\partial\Sigma = 2C_1$. If $p_1 > 0$, one can construct such Σ as a projective cone, see Proof of Theorem *1*. Put it to general position with respect to C_2. Then $\operatorname{lk}(C_1, C_2)$ is $\frac{1}{2}\Sigma \circ C_2$, where $\Sigma \circ C_2$ is the intersection number.

Define

$$K_1 = \{(x,t,y) \in C_1 \times \mathbb{R}P^n \times C_2 \mid t \in Span(x,y)\},$$

$S_1 = \{(x,t,y) \in K_1 \mid t = x\}$ and $S_2 = \{(x,t,y) \in K_1 \mid t = y\}$, put $S = S_1 \cup S_2$ and $K_0 = K_1 \smallsetminus S$. The smooth structures and orientations of C_1 and C_2 define a smooth structure and orientation of K_0 exactly as in Section 2. In particular, the orientation is positive on the local coordinate system, in which the first p_1 coordinates are taken from the positively oriented local coordinates on C_1, the $p_1 + 1^{\text{st}}$ coordinate is directed along the line connecting C_1 to C_2 towards C_2, and the last p_2 coordinates are taken from the positively oriented local coordinates on C_2. The natural map

$$F_1 : K_1 \to \mathbb{R}P^n : (x,t,y) \mapsto t$$

can be factored through the quotient space

$$K = K_1 /_{(x_1,y,y)\sim(x_2,y,y) \quad (x,x,y_1)\sim(x,x,y_2)}.$$

Denote, as above, the resulting map $K \to \mathbb{R}P^n$ by F. The orientation of K_0 defines a class $[K] \in H_n(K)$. The image $F_*[K] \in H_n(\mathbb{R}P^n)$ is $d(C_1,C_2)[\mathbb{R}P^n]$. As in Theorem *1*, under these conditions $d(C_1,C_2) = 2\,\mathrm{lk}(C_1,C_2)$.

6. The self-linking coefficient

In [1] I introduced a numerical invariant of a homologous to zero oriented knot in RP^3. The preimage of such a knot in the universal covering space S^3 of $\mathbb{R}P^3$ is a two-component link. The self-linking coefficient is defined to be the linking number of the components of this link. The self-linking coefficient was used in [1] as an obvious obstruction for a knot to be isotopic to a knot contained in an affine part of $\mathbb{R}P^3$.

Here the self-linking coefficient $sl(k)$ of a knot k is represented via the degree of a map. The source space of the map is constructed as follows.

For a triple (x,t,y) consisting of $x,y \in k$, $x \neq y$ and $t \in Span(x,y)$, consider a loop composed of a half of k, which is bounded by x and y, and the segment of the line $Span(x,y)$ which is bounded by x and y and does not contain t. The homology class of this loop does not depend on the choice of the half of k, since k is zero-homologous in $\mathbb{R}P^3$. Denote by K_0 the set of triples (x,t,y) with $x,y \in k$, $x \neq y$ and $t \in Span(x,y)$ such that the homology class of the loop constructed above for (x,t,y) is not zero.

The orientation of k and the direction of the segment of $Span(x,y)$ from x to y define an orientation of K_0.

The space K_0 is fibered over $k \times k \smallsetminus \Delta$ with fiber $\mathbb{R}$. Consider a larger space K_1 which is the closure of K_0 in the space of all triples (x,t,y) with $x,y \in k$ and $t \in Span(x,y)$. Let K be the quotient space of K_1:

$$K = K_1 /_{(x,x,y_1)\sim(x,x,y_2) \quad (x_1,y,y)\sim(x_2,y,y)}.$$

Obviously, $K \smallsetminus K_0$ can be identified with k. Hence K is a stratified space with a 3-dimensional stratum K_0 and 1-dimensional stratum k. The orientation of K_0 defines a homology class $[K] \in H_3(K)$.

Define $F_1 : K_1 \to \mathbb{R}P^3$ by $F_1(x,t,y) = t$. Denote by F the quotient map $K \to \mathbb{R}P^3$. Let $d(k)$ be an integer such that $F_*[K] = d(k)[\mathbb{R}P^3]$.

Theorem 2. *Under the conditions above*

$$d(k) = 2sl(k).$$

The proof of Theorem *2* is similar to the proof of Theorem *1*. □

References

[1] Julia Drobotukhina, *An analogue of the Jones polynomial for links in* $\mathbb{R}P^3$ *and a generalization of the Kauffman-Murasugi theorem* Algebra i analiz **2:3** (1990) (Russian, English translation in Leningrad Math. J. **2:3** (1991) 613–630.)

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