CATEGORIFICATION USING DUAL PROJECTION FUNCTORS

ANNA-LOUISE GRENSING AND VOLODYMYR MAZORCHUK

Abstract. We study finitary 2-categories associated to dual projection functors for finite dimensional associative algebras. In the case of path algebras of admissible tree quivers (which includes all Dynkin quivers of type A) we show that the monoid generated by dual projection functors is the Hecke-Kiselman monoid of the underlying quiver and also obtain a presentation for the monoid of indecomposable subbimodules of the identity bimodule.

1. Introduction

Study of 2-categories of additive functors operating on a module category of a finite dimensional associative algebra is motivated by recent advances and applications of categorification philosophy, see [CR, Ro, KL, Ma] and references therein. Such 2-categories appear as natural 2-analogues of finite dimensional algebras axiomatized via the notion of *finitary* 2-categories as introduced in [MM1]. The series [MM1, MM2, MM3, MM5, MM6] of papers develops basics of the structure theory and the 2-representation theory for the so-called *fiat 2-categories*, that is finitary 2-categories having a weak involution and adjunction morphisms. Natural examples of such fiat 2-categories are 2-categories generated by *projective* functors, that is functors given by tensoring with projective bimodules, see [MM1, Subsection 7.3]. Fiat 2-categories also naturally appear as quotients of 2-Kac-Moody algebras from [KL, Ro, We], see [MM2, Subsection 7.1] and [MM5, Subsection 7.2] for detailed explanations. There are also many natural constructions which produce new fiat 2-categories from known ones, see e.g. [MM6, Section 6].

Despite of some progress made in understanding fiat 2-categories in the papers mentioned above, the general case of finitary 2-categories remains very mysterious with the only general result being the abstract 2-analogue of the Morita theory developed in [MM4]. One of the major difficulties is that so far there are not that many natural examples of finitary 2-categories which would be “easy enough” for any kind of sensible understanding. In [GrMa], inspired by the study of the so-called *projection* functors in [Gr, Pa], we defined a finitary 2-category which is a natural 2-analogue of the semigroup algebra of the so-called *Catalan monoid* of all order-decreasing and order-preserving transformations of a finite chain. This 2-category is associated to the path algebra of a type A Dynkin quiver with a fixed uniform orientation (meaning that all edges are oriented in the same direction).

The main aim of the present paper is to make the next step and consider a similarly defined 2-category for an arbitrary orientation of a type A Dynkin quiver and, more generally, for any admissible orientation of an arbitrary tree quiver. There is one important difference, which we will now explain, between this general case and the case of a uniform orientation in type A. The basic structural properties of a finitary 2-category are encoded in the so-called *multisemigroup* of a 2-category as defined in [MM2, Subsection 3.3]. Elements of this multisemigroup are isomorphism classes of *indecomposable* 1-morphisms in our 2-category. It turns out that for a uniform orientation of a type A Dynkin quiver any composition of projection functor is either indecomposable or zero. This fail in all other cases in which the orientation is not uniform as well as for all admissible tree quivers outside type A. This is the principal added difficulty of the present paper compared to [GrMa].

For technical reasons it turns out that it is more convenient to work with a dual version of projection functors, which we simply call *dual projection functors*. Roughly speaking these are the right exact functors given by maximal subfunctors of the identity functor. The first part of the paper is devoted to some basic structure theory for such functors. This is developed in Section 3 after various preliminaries collected in Section 2. In particular, in Proposition 8 we make the connection between projection and dual projection functors very explicit. This, in particular, allows us to transfer, for free, many results of [Gr, Pa] to our situation.

Section 4 contains basic preliminaries on 2-categories. In Section 5 we define finitary 2-categories given by dual projection functors and also finitary 2-categories given by non-exact ancestors of dual projection functors which we call *idealization functors*. Section 6 is the main part of the paper and contains several results. This includes classification of indecomposable dual projection functor in Theorem 21 and also the statement that composition of indecomposable dual projection functors for any admissible orientation of a tree quiver is indecomposable, see Proposition 23. Our classification is based on generalization of the Dyck path combinatorics in application to subbimodules of the identity bimodule for admissible tree quivers as described in Subsections 6.5, 6.6, 6.7 and 6.8.

Proposition 23 mentioned above implies that the multisemigroup of the 2-category of dual projection functors associated to any admissible orientation of a tree quiver is, in fact, an ordinary semigroup. This observation automatically makes this semigroup into an interesting object of study. In Section 7 we give a presentation for this semigroup in Theorem 30 and also for the semigroup of all idealization functors in Theorem 29. Our proof of Theorem 29 is rather elegant, it exploits the idea of decategorification: the canonical action of our 2-category on the underlying module category gives rise to a linear representation of a certain Hecke-Kiselman monoid from [GM]. Proof of Theorem 29 basically reduces to verification that this representation is effective (in the sense that different elements of the monoid are represented by different linear transformations). This effectiveness was conjectured in [GM] and proved in [Fo]. Theorem 30 requires more technical work as the monoid of indecomposable dual projection functors is not a Hecke-Kiselman monoid on the nose, but after some preparation it also reduces to a similar argument.

Acknowledgment. The first author is supported by priority program SPP 1388 of the German Science Foundation. The second author is partially supported by the Swedish Research Council, Knut and Alice Wallenbergs Stiftelse and the Royal Swedish Academy of Sciences.

2. Preliminaries

2.1. **Notation and setup.** In this paper we work over a fixed field $\Bbbk$ which for simplicity is assumed to be algebraically closed. All categories and functors considered in this paper are supposed to be $\Bbbk$-linear, that is enriched over $\Bbbk$-Mod. If not explicitly stated otherwise, by a module we always mean a *left* module.

For a finite dimensional associative $\Bbbk$-algebra A we denote by A-mod the (abelian) category of all finitely generated A-modules. By A-Mod we denote the (abelian) category of all A-modules. We also denote by A-proj the (additive) category of all finitely generated projective A-modules and by A-inj the (additive) category of all finitely generated injective A-modules.

We denote by mod-A the category of all finitely generated right A-modules and define proj-A and Mod-A respectively.

We denote by A-mod-A the category of all finitely generated A-A–bimodules. Denote by $\mathcal{AF}_A$ the category of all additive $\Bbbk$-linear endofunctors of A-mod. This is an abelian category since A-mod is abelian.

Abusing notation, we write $*$ for both the $\Bbbk$-duality functors

$$\mathrm{Hom}_{-\Bbbk}({}_-,\Bbbk) : A\text{-mod} \to \text{mod-}A \quad \text{and} \quad \mathrm{Hom}_{\Bbbk\text{-}}({}_-,\Bbbk) : \text{mod-}A \to A\text{-mod}.$$

Let $L_1, L_2, \dots, L_n$ be a complete and irredundant list of representatives of isomorphism classes of simple A-modules. Then $L_1^*, L_2^*, \dots, L_n^*$ is a complete and irredundant list of representatives of isomorphism classes of simple right A-modules. For $i, j = 1, 2, \dots, n$, set $L_{ij} := L_i \otimes_\Bbbk L_j^*$. This gives a complete and irredundant list of representatives of isomorphism classes of simple A-A–bimodules. For $i = 1, 2, \dots, n$ we denote by P_i and I_i the indecomposable projective cover and injective envelope of L_i, respectively.

When working with the opposite algebra, we will add the superscript $^{\mathrm{op}}$ to all notation.

2.2. **Trace functors.** With each $N \in A$-mod one associates the corresponding *trace functor* $\mathrm{Tr}_N : A\text{-mod} \to A\text{-mod}$ defined in the following way:

- For every $M \in A$-mod, the module $\mathrm{Tr}_N(M) \in A$-mod is defined as the submodule $\displaystyle\sum_{f:N\to M} \mathrm{Im}(f)$ of M.
- For every $M, M' \in A$-mod and every $f : M \to M'$, the corresponding morphism $\mathrm{Tr}_N(f) : \mathrm{Tr}_N(M) \to \mathrm{Tr}_N(M')$ is defined as the restriction of f to $\mathrm{Tr}_N(M)$.

Directly from the definition it follows that Tr_N is a subfunctor of the identity functor for every N. We denote by $\iota_N : \mathrm{Tr}_N \hookrightarrow \mathrm{Id}_{A\text{-mod}}$ the corresponding injective natural transformation.

Lemma 1. *Let $N \in A$-mod.*

(*i*) *The functor Tr_N preserves monomorphisms.*

(*ii*) *If N is projective, then Tr_N preserves epimorphisms.*

(*iii*) *We have $\mathrm{Tr}_N \circ \mathrm{Tr}_N \cong \mathrm{Tr}_N$.*

Proof. Let $f : M \to M'$ be a monomorphism. In the commutative diagram

$$\begin{tikzcd}
M \arrow[r, hook, "f"] & M' \\
\mathrm{Tr}_N(M) \arrow[u, hook, "\iota_M"] \arrow[r, "\mathrm{Tr}_N(f)"] & \mathrm{Tr}_N(M') \arrow[u, hook, "\iota_{M'}"']
\end{tikzcd}$$

we have $f \circ \iota_M$ is a monomorphism. Hence $\mathrm{Tr}_N(f)$ is a monomorphism as well. This proves claim (i).

Let $f : M \to M'$ be an epimorphism and $g : N \to M'$ any map. If N is projective, then there is $h : N \to M$ such that $g = f \circ h$. Hence $\mathrm{Im}(g) = f(\mathrm{Im}(h))$ showing that $\mathrm{Tr}_N(M)$ surjects onto $\mathrm{Tr}_N(M')$. This proves claim (ii).

Claim (iii) follows directly from the definition of Tr_N. This completes the proof of the lemma. □

We remark that, in general, Tr_N is neither left nor right exact (even if N is projective). Indeed, let A be the path algebra of the quiver $1 \longrightarrow 2$, P_1 be the indecomposable projective A-module $\Bbbk \xrightarrow{\mathrm{Id}} \Bbbk$, L_1 be the simple A-module $\Bbbk \longrightarrow 0$ and L_2 be the simple A-module $0 \longrightarrow \Bbbk$. For $N = P_1$, applying Tr_N to the short exact sequence

$$0 \to L_2 \to P_1 \to L_1 \to 0,$$

gives the sequence

$$0 \to 0 \to P_1 \to L_1 \to 0$$

which has homology in the middle position.

2.3. **Projection functors.** For $N \in A$-mod we define the corresponding *projection functor* $\mathrm{Pr}_N : A\text{-mod} \to A\text{-mod}$ as the cokernel of the natural transformation ι_N. Let $\pi_N : \mathrm{Id}_{A\text{-mod}} \twoheadrightarrow \mathrm{Pr}_N$ denote the corresponding surjective natural transformation. The following properties of projection functors appear in [Pa, Gr]:

- For any N, the functor Pr_N preserves epimorphisms.
- If N is simple, then the functor Pr_N preserves monomorphisms.
- If N is simple and $\mathrm{Ext}^1_A(N, N) = 0$, then $\mathrm{Pr}_N \circ \mathrm{Pr}_N \cong \mathrm{Pr}_N$.
- If N and K are simple and $\mathrm{Ext}^1_A(K, N) = 0$, then
$$\mathrm{Pr}_N \circ \mathrm{Pr}_K \circ \mathrm{Pr}_N \cong \mathrm{Pr}_K \circ \mathrm{Pr}_N \circ \mathrm{Pr}_K \cong \mathrm{Pr}_N \circ \mathrm{Pr}_K.$$
- If N and K are simple and $\mathrm{Ext}^1_A(N, K) = \mathrm{Ext}^1_A(K, N) = 0$, then
$$\mathrm{Pr}_N \circ \mathrm{Pr}_K \cong \mathrm{Pr}_K \circ \mathrm{Pr}_N.$$

For the record, we also point out the following connection between the functors Pr_N and Tr_N.

Lemma 2. *For any fixed N, the functor Tr_N is exact if and only if the functor Pr_N is exact.*

Proof. For an exact sequence $X \overset{f}{\hookrightarrow} Y \overset{g}{\twoheadrightarrow} Z$ in A-mod consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Tr}_N(X) \arrow[r, "\mathrm{Tr}_N(f)"] \arrow[d, hook, "\iota_N(X)"'] & \mathrm{Tr}_N(Y) \arrow[r, "\mathrm{Tr}_N(g)"] \arrow[d, hook, "\iota_N(Y)"'] & \mathrm{Tr}_N(Z) \arrow[d, hook, "\iota_N(Z)"] \\
X \arrow[r, hook, "f"] \arrow[d, two heads, "\pi_N(X)"'] & Y \arrow[r, two heads, "g"] \arrow[d, two heads, "\pi_N(Y)"'] & Z \arrow[d, two heads, "\pi_N(Z)"] \\
\mathrm{Pr}_N(X) \arrow[r, "\mathrm{Pr}_N(f)"] & \mathrm{Pr}_N(Y) \arrow[r, "\mathrm{Pr}_N(g)"] & \mathrm{Pr}_N(Z)
\end{tikzcd}$$

Here all columns are exact by construction and the middle row is exact by assumption. Therefore the Nine Lemma (a.k.a. the 3×3-Lemma) says that the first row is exact if and only if the third row is exact. $\square$

3. Dual projection functors

3.1. **Idealization functors.** The algebra A is an A-A–bimodule, as usual. Tensoring with this bimodule (over A) is isomorphic to the identity endofunctor of A-mod. We identify subbimodules of ${}_AA_A$ and two-sided ideals of A. For each two-sided ideal $I \subset A$ denote by Su_I the endofunctor of A-mod defined in the following way:

- For every $M \in A$-mod, the module $\mathrm{Su}_I(M)$ is defined as IM.
- For every $M, M' \in A$-mod and $f : M \to M'$, the morphism $\mathrm{Su}_I(f)$ is defined as the restriction of f to IM.

We will call Su_I the *idealization functor* associated to I, where the notation Su stands for "Sub".

Let $\gamma : \mathrm{Su}_I \hookrightarrow \mathrm{Id}_{A\text{-mod}}$ denote the injective natural transformation given by the canonical inclusion $IM \hookrightarrow M$. Directly from the definition we obtain that for any two two-sided ideals I and J in A we have

$$\mathrm{Su}_I \circ \mathrm{Su}_J = \mathrm{Su}_{IJ}. \tag{1}$$

Furthermore, if $I \subset J$, then we have the canonical inclusion $\mathrm{Su}_I \hookrightarrow \mathrm{Su}_J$.

3.2. **Exactness of idealization.** Here we prove the following property of idealization functors.

Lemma 3. *Let I be a two-sided ideal in A.*

(*i*) *The functor* $\mathrm{Su}_I(M)$ *preserves monomorphisms.*

(*ii*) *The functor* $\mathrm{Su}_I(M)$ *preserves epimorphisms.*

Proof. Claim (i) follows from the definition of Su_I and the fact that the restriction of a monomorphism is a monomorphism. To prove claim (ii), consider an epimorphism $f : M \twoheadrightarrow M'$, $v \in M'$ and $a \in I$. Then there is $w \in M$ such that $f(w) = v$ and hence $af(w) = f(aw) = av$. As $aw \in \mathrm{Su}_I(M)$, we obtain that av belongs to the image of $\mathrm{Su}_I(f)$, completing the proof. $\square$

We note that Su_I is neither left nor right exact in general. Indeed, consider the algebra $A = \Bbbk[x]/(x^2)$, let L be the simple A-module and set $I := \mathrm{Rad}(A)$. Applying Su_I to the short exact sequence

$$0 \to L \to {}_AA \to L \to 0,$$

we obtain the sequence

$$0 \to 0 \to L \to 0 \to 0$$

which has homology in the middle position.

3.3. **Idealization functors versus trace functors.** Let I be a two-sided ideal of A. Since both Su_I and Tr_N are subfunctors of the identity functor, it is natural to ask when they are isomorphic. In this subsection we would like to present some examples showing that, in general, these families of functors are really different.

Lemma 4. *If A is not semi-simple, then $\mathrm{Su}_{\mathrm{Rad}(A)}$ is not isomorphic to any trace functor.*

Proof. We have $\mathrm{Su}_{\mathrm{Rad}(A)}(A) = \mathrm{Rad}(A) \neq 0$ as A is not semi-simple. Hence $\mathrm{Su}_{\mathrm{Rad}(A)}$ is not the zero functor, in particular, it is not isomorphic to Tr_0. At the same time, let $L := A/\mathrm{Rad}(A)$. Then $\mathrm{Su}_{\mathrm{Rad}(A)}(L) = 0$. On the other hand, for any non-zero $N \in A$-mod the module N surjects onto some simple A-module. As every simple A-module is a summand of L, we have $\mathrm{Tr}_N(L) \neq 0$. The claim follows. □

Lemma 5. *If N is simple and not projective, then Tr_N is not isomorphic to any idealization functor.*

Proof. Let $f : P \twoheadrightarrow N$ be a projective cover of N. As N is not projective, $\mathrm{Tr}_N(f)$ is the zero map. We also have the obvious isomorphism $\mathrm{Tr}_N(N) \cong N$. At the same time, each idealization functor preserves epimorphisms by Lemma 3(ii). The obtained contradiction proves the statement. □

3.4. **Definition of dual projection functors.** Recall that, for any additive functor $\mathrm{F} : A\text{-proj} \to A\text{-mod}$, there is a unique, up to isomorphism, right exact functor $\mathrm{G} : A\text{-mod} \to A\text{-mod}$ such that the restriction of G to A-proj is isomorphic to F. As ${}_AA$ is an additive generator of A-proj, the condition that the restriction of G to A-proj is isomorphic to F is equivalent to the condition that the A-A–bimodules $\mathrm{F}(A)$ and $\mathrm{G}(A)$ are isomorphic. The functor G is isomorphic to the functor $\mathrm{F}(A) \otimes_A {}_-$, see [Ba, Chapter 2] for details.

For an ideal I in A define a *dual projection functor* corresponding to I as a functor isomorphic to the functor

$$\mathrm{Dp}_I := \mathrm{Su}_I(A) \otimes_A {}_- : A\text{-mod} \to A\text{-mod}.$$

Directly from the definition we have that Dp_I is right exact.

Lemma 6. *If A is hereditary, then the functor Dp_I is exact for any I.*

Proof. As $\mathrm{Su}_I(A) \subset A$ and A is hereditary, the right A-module $\mathrm{Su}_I(A)$ is projective. This means that Dp_I is exact. □

Corollary 7. *If A is hereditary, then $\mathrm{Dp}_I \circ \mathrm{Dp}_J \cong \mathrm{Dp}_{IJ}$ for any two two-sided ideals I, J in A.*

Proof. Note that for hereditary A the functor Dp_I preserves A-proj. Because of exactness, established in Lemma 6, it is thus enough to prove the isomorphism when restricted to A-proj where it reduces to formula (1). □

3.5. **Special dual projection functors.** The radical of A coincides with the radical of the A-A–bimodule ${}_AA_A$ and we have a short exact sequence

$$0 \to \mathrm{Rad}(A) \to {}_AA_A \to \bigoplus_{i=1}^{n} L_{ii} \to 0.$$

For every $i = 1, 2, \dots, n$, this gives, using canonical projection onto a component of a direct sum, an epimorphism ${}_AA_A \twoheadrightarrow L_{ii}$. Let J_i denote the kernel of the latter epimorphism. We will use the shortcut F_i for the corresponding dual projection functor Dp_{J_i}. Setting $n_i := \dim(L_i)$ for $i = 1, 2, \dots, n$, we have an isomorphism of left A-modules as follows:

$$J_i \cong \mathrm{Rad}(P_i)^{\oplus n_i} \oplus \bigoplus_{j \neq i} P_j^{\oplus n_j}.$$

3.6. **Dual projection functors versus projection functors.** In this subsection we explain the name *dual projection functors*.

For $i = 1, 2, \dots, n$ denote by G_i the unique, up to isomorphism, left exact endofunctor of A-mod satisfying the condition that

$$\mathrm{G}_i|_{A\text{-inj}} \cong \mathrm{Pr}_{L_i}|_{A\text{-inj}}.$$

For example, we can take

$$\mathrm{G}_i = \mathrm{Hom}_A((\mathrm{Pr}_{L_i}(A^*))^*, {}_-),$$

see [GrMa, Subsection 2.3] for details. In other words, G_i is the unique left exact extension of the projection functor corresponding to the simple module L_i.

Proposition 8. *There is an isomorphism of functors as follows:* $\mathrm{F}_i \cong * \circ \mathrm{G}_i^{\mathrm{op}} \circ *$.

Proof. Both F_i and $* \circ \mathrm{G}_i^{\mathrm{op}} \circ *$ are right exact functors and hence it is sufficient to prove that they are isomorphic on A-proj. For the additive generator A of the latter category we have

$$(\mathrm{G}_i^{\mathrm{op}}(A^*))^* \cong \mathrm{Hom}_{\text{-}A}(\mathrm{Pr}^{\mathrm{op}}_{L_i^*}(A^*)^*, A^*)^* \cong \mathrm{Hom}_{A\text{-}}(A, \mathrm{Pr}^{\mathrm{op}}_{L_i^*}(A^*))^* \cong (\mathrm{Pr}^{\mathrm{op}}_{L_i^*}(A^*))^*$$

and thus the claim of our proposition amounts to finding a natural isomorphism between $(\mathrm{F}_i(A))^* \cong J_i^*$ and $\mathrm{Pr}^{\mathrm{op}}_{L_i^*}(A^*)$.

Applying $*$ to the exact sequence $J_i \hookrightarrow A \twoheadrightarrow L_{ii}$ results in the exact sequence $L_{ii}^* \hookrightarrow A^* \twoheadrightarrow J_i^*$. As $L_{ii}^* \cong L_{ii}$ and all other simple subbimodules of A^* are of the form L_{jj} for some $j \neq i$, the submodule $\mathrm{Tr}^{\mathrm{op}}_{L_i^*}(A^*)$ coincides with L_{ii}^*. This implies that there is a bimodule isomorphism $J_i^* \cong \mathrm{Pr}^{\mathrm{op}}_{L_i^*}(A^*)$ which completes the proof. □

Proposition 8 allows us to freely transfer results for projection functors to dual projection functors and vice versa. For technical reasons in this paper we will mostly work with dual projection functors.

3.7. **Dual projection functors and coapproximation functors.** In some cases dual projective functors can be interpreted as partial coapproximation functors in the terminology of [KhMa, Subsection 2.4]. For $i = 1, 2, \dots, n$, set

$$Q_i := P_1 \oplus P_2 \oplus \cdots \oplus P_{i-2} \oplus P_{i-1} \oplus P_{i+1} \oplus P_{i+2} \oplus \cdots \oplus P_{n-1} \oplus P_n.$$

The functor C_i of *partial coapproximation* with respect to Q_i is defined as follows: Given $M \in A$-mod, consider a short exact sequence $K \hookrightarrow P \twoheadrightarrow M$ with projective P. Then

$$\mathrm{C}_i(M) := \mathrm{Tr}_{Q_i}(P/\mathrm{Tr}_{Q_i}(K))$$

and the action on morphisms is defined by first lifting them using projectivity and then restriction. From [KhMa, Lemma 9] it follows that C_i is right exact. The functor C_i comes together with a natural transformation $\kappa : \mathrm{C}_i \to \mathrm{Id}_{A\text{-mod}}$ which is injective on projective modules (note that, if M is projective in the above construction, then we may choose $K = 0$ and $\mathrm{C}_i(M) = \mathrm{Tr}_{Q_i}(M)$). In particular, if $\mathrm{Ext}^1_A(L_i, L_i) = 0$, then we have

$$\mathrm{Tr}_{Q_i}(P_j) \cong \begin{cases} P_j, & \text{if } i \neq j; \\ \mathrm{Rad}(P_i), & \text{otherwise.} \end{cases}$$

Lemma 9. *If* $\mathrm{Ext}^1_A(L_i, L_i) = 0$*, then* $\mathrm{C}_i \cong \mathrm{F}_i$*.*

Proof. As both functors are right exact, it is enough to check the bimodule isomorphism $\mathrm{C}_i(A) \cong \mathrm{F}_i(A)$. Since ${}_AA$ is projective, we have $\mathrm{C}_i(A) = \mathrm{Tr}_{Q_i}(A)$. At the same time, if $\mathrm{Ext}^1_A(L_i, L_i) = 0$, then $\mathrm{Tr}_{Q_i}(A) = J_i$. As the action of C_i on morphisms is defined via restriction, it follows that $\mathrm{C}_i(A) \cong J_i$ as a bimodule. This completes the proof. □

4. Some preliminaries on 2-categories

4.1. **Finite and finitary 2-categories.** We refer the reader to [Le, McL, Ma] for generalities on 2-categories. Denote by **Cat** the category of all small categories. A 2-*category* is a category enriched over **Cat**. A 2-category $\mathscr{C}$ is called *finite* if it has finitely many objects, finitely many 1-morphisms and finitely many 2-morphisms.

Recall from [MM1] that a 2-category $\mathscr{C}$ is called *finitary* over $\Bbbk$ provided that

- $\mathscr{C}$ has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is an idempotent split additive $\Bbbk$-linear category with finitely many isomorphism classes of indecomposable objects and finite dimensional spaces of morphisms;
- all compositions are biadditive and also $\Bbbk$-bilinear whenever the latter makes sense;
- all identity 1-morphisms are indecomposable.

For an object $\mathtt{i}$ of a 2-category we denote by $\mathbb{1}_{\mathtt{i}}$ the corresponding identity 1-morphism.

4.2. **The multisemigroup of a finitary 2-category.** For a finitary 2-category $\mathscr{C}$ denote by $\mathcal{S}_{\mathscr{C}}$ the set of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$ with an added external zero element 0. By [MM2, Subsection 3.3], the finite set $\mathcal{S}_{\mathscr{C}}$ has the natural structure of a *multisemigroup* given for $[\mathrm{F}], [\mathrm{G}] \in \mathcal{S}_{\mathscr{C}}$ by defining

$$[\mathrm{F}] \star [\mathrm{G}] := \begin{cases} \{[\mathrm{H}] \,:\, \mathrm{H} \text{ is isomorphic to a direct summand of } \mathrm{F} \circ \mathrm{G}\}, & \mathrm{F} \circ \mathrm{G} \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

We refer the reader to [KuMa] for more details on multisemigroups.

4.3. **$\Bbbk$-linearization of finite categories.** For a set X denote by $\Bbbk[X]$ the $\Bbbk$-vector space of all formal linear combinations of elements in X with coefficients in $\Bbbk$. Then X is naturally identified with a basis in $\Bbbk[X]$. Note that $\Bbbk[X] = \{0\}$ if $X = \varnothing$.

Let $\mathcal{C}$ be a finite category, that is a category with finitely many objects and morphisms. The $\Bbbk$-*linearization* of $\mathcal{C}$ is the category $\mathcal{C}_\Bbbk$ defined as follows:

- $\mathcal{C}_\Bbbk$ and $\mathcal{C}$ have the same objects;
- $\mathcal{C}_\Bbbk(\mathtt{i},\mathtt{j}) := \Bbbk[\mathcal{C}(\mathtt{i},\mathtt{j})]$;
- composition in $\mathcal{C}_\Bbbk$ is induced from composition in $\mathcal{C}$ by $\Bbbk$-bilinearity.

The *additive* $\Bbbk$-*linearization* $\mathcal{C}_\Bbbk^\oplus$ of $\mathcal{C}$ is then the "additive closure" of $\mathcal{C}_\Bbbk$ in the following sense:

- objects in $\mathcal{C}_\Bbbk^\oplus$ are all expressions of the form $\mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \dots \oplus \mathtt{i}_k$, where $k \in \{0,1,2,\dots\}$ and all $\mathtt{i}_i$ are objects in $\mathcal{C}_\Bbbk$;
- the set $\mathcal{C}_\Bbbk^\oplus(\mathtt{i}_1 \oplus \mathtt{i}_2 \oplus \dots \oplus \mathtt{i}_k, \mathtt{j}_1 \oplus \mathtt{j}_2 \oplus \dots \oplus \mathtt{j}_m)$ consists of all matrices of the form
$$\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1k} \\ f_{21} & f_{22} & \dots & f_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \dots & f_{mk} \end{pmatrix}$$
where $f_{st} \in \mathcal{C}_\Bbbk(\mathtt{i}_t, \mathtt{j}_s)$;
- composition in $\mathcal{C}_\Bbbk^\oplus$ is given by matrix multiplication.

4.4. **Finitarization of finite 2-categories.** Let $\mathscr{C}$ be a finite 2-category. Then the *finitarization* of $\mathscr{C}$ over $\Bbbk$ is the 2-category $\mathscr{C}_\Bbbk$ defined as follows:

- $\mathscr{C}_\Bbbk$ has the same objects as $\mathscr{C}$;
- $\mathscr{C}_\Bbbk(\mathtt{i},\mathtt{j}) := \mathscr{C}(\mathtt{i},\mathtt{j})_\Bbbk^\oplus$;
- composition in $\mathscr{C}_\Bbbk$ is induced from composition in $\mathscr{C}$ using biadditivity and $\Bbbk$-bilinearity.

Directly from the definition it follows that $\mathscr{C}_\Bbbk$ is finitary if and only if for each $\mathtt{i} \in \mathscr{C}$ the endomorphism algebra $\mathrm{End}_{\mathscr{C}_\Bbbk}(\mathbb{1}_\mathtt{i}) \cong \Bbbk[\mathrm{End}_{\mathscr{C}}(\mathbb{1}_\mathtt{i})]$ is local.

4.5. **Two 2-categories associated to an ordered monoid.** Let $(S, e, \cdot)$ be a finite monoid with a fixed admissible reflexive partial pre-order $\preceq$. Admissibility means that $s \preceq t$ implies both $sr \preceq tr$ and $rs \preceq rt$ for all $s,t,r \in S$. In this situation we may define a finite 2-category $\mathscr{C}^S$ as follows:

- $\mathscr{C}^S$ has one object $\clubsuit$;
- 1-morphisms in $\mathscr{C}^S(\clubsuit,\clubsuit)$ are elements in S and the horizontal composition of 1-morphisms is given by multiplication in S;
- for two 1-morphisms s and t, the set of 2-morphisms from s to t is empty if $s \not\preceq t$ and contains one element, denoted (s,t), otherwise (note that in this case all compositions of 2-morphisms are automatically uniquely defined).

The finitarization $\mathscr{C}_\Bbbk^S$ of $\mathscr{C}^S$ is then a finitary 2-category as the endomorphism algebra of each identity 1-morphism is just $\Bbbk$.

5. 2-categories of idealization functors and dual projection functors

5.1. **Monoid of two-sided ideals.** The set $\mathcal{I}$ of all two-sided ideals in A has the natural structure of a monoid given by multiplication of ideals $(I, J) \mapsto IJ$. The identity element of $\mathcal{I}$ is A and the zero element is the zero ideal. We note the following:

Lemma 10. *If* $\dim_{\Bbbk} \mathrm{Hom}_A(P_i, P_j) \leq 1$ *for all* $i, j \in \{1, 2, \dots, n\}$, *then* $|\mathcal{I}| < \infty$.

Proof. If $a, b \in A$ are idempotents, then, by adjunction, we have

$$\mathrm{Hom}_{A\text{-}A}(Aa \otimes_{\Bbbk} bA, A) \cong \mathrm{Hom}_{A\text{-}}(Aa, Ab) \cong \mathrm{Hom}_{\Bbbk}(\Bbbk, aAb) = aAb.$$

For $i, j \in \{1, 2, \dots, n\}$, the projective cover of the simple bimodule L_{ij} in A-mod-A is isomorphic to $P_i \otimes_{\Bbbk} I_j^*$ and hence from our assumptions it follows that the composition multiplicity of L_{ij} in ${}_AA_A$ is at most 1. This means that each subbimodule of ${}_AA_A$ is uniquely determined by its composition subquotients (and equals the sum of images of unique up to scalar nonzero homomorphisms from the projectives covers of these simple subquotients). Therefore $|\mathcal{I}| \leq 2^{\dim(A)}$. □

Corollary 11. *If A is the path algebra of a tree quiver or the incidence algebra of a finite poset, then* $|\mathcal{I}| < \infty$.

Proof. Both for the path algebra of a tree quiver and for the incidence algebra of a finite poset, the condition $\dim_{\Bbbk} \mathrm{Hom}_A(P_i, P_j) \leq 1$ for all $i, j \in \{1, 2, \dots, n\}$ is straightforward and thus the statement follows from Lemma 10. □

The monoid $\mathcal{I}$ is naturally ordered by inclusions, moreover, this order is obviously admissible.

5.2. **A 2-action of $\mathscr{C}^{\mathcal{I}}$ on A-mod by idealization functors.** We define a 2-action of the 2-category $\mathscr{C}^{\mathcal{I}}$ associated to the ordered monoid $(\mathcal{I}, A, \cdot, \subset)$ on A-mod as follows:

- the element $I \in \mathcal{I}$ acts as the functor Su_I;
- for $I \subset J$, the 2-morphism (I, J) acts as the canonical inclusion $\mathrm{Su}_I \hookrightarrow \mathrm{Su}_J$.

This is a strict 2-action because of (1).

This 2-action extends to a 2-action of $\mathscr{C}^{\mathcal{I}}_{\Bbbk}$ on A-mod in the obvious way. Note that this 2-action is clearly faithful both on the level of 1-morphisms and on the level of 2-morphisms. However, this 2-action is not full on the level of 2-morphisms in general. Indeed, in case the algebra A has a non-trivial center, the 1-dimensional endomorphism algebra of the identity 1-morphism in $\mathscr{C}^{\mathcal{I}}_{\Bbbk}$ cannot surject onto the non-trivial endomorphism algebra of the identity functor of A-mod.

5.3. **A 2-action of $\mathscr{C}^{\mathcal{I}}$ on A-mod by dual projection functors.** The main disadvantage of the 2-action defined in Subsection 5.2 is the fact that the functors Su_J are not exact from any side in general. In particular, they do not induce any reasonable maps on the Grothendieck group of A-mod. To overcome this problem one needs to define another action and dual projection functors are reasonable candidates. However, there is a price to pay. Firstly, to keep the action strong we will have to change A-mod to an equivalent category. Secondly, we will have to restrict to hereditary algebras.

Denote by $\overline{A\text{-proj}}$ the category whose objects are diagrams $P \xrightarrow{f} Q$ over A-proj and whose morphisms are equivalence classes of solid commutative diagrams

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g_2"'] & Q \arrow[d, "g_1"] \arrow[dl, dashed, "h"'] \\ P' \arrow[r, "f'"] & Q' \end{tikzcd}$$

modulo the equivalence relation defined as follows: the solid diagram is equivalent to zero provided that there exists a dashed map h as indicated on the diagram such that $g_1 = f'h$. The category $\overline{A\text{-proj}}$ is abelian and equivalent to A-mod, see [Fr]. This construction is called *abelianization* in [MM1, MM2].

If A is hereditary, then each Su_I preserves A-proj and hence the 2-actions of both $\mathscr{C}^{\mathcal{I}}$ and $\mathscr{C}^{\mathcal{I}}_{\Bbbk}$ defined in Subsection 5.2 extends component-wise to 2-actions of both these categories on $\overline{A\text{-proj}}$. By construction, this is not an action on A-mod but on a category which is only equivalent to A-mod. Moreover, the action is designed so that the ideal I acts by a right exact functor which is isomorphic to Su_I when restricted to A-proj. This means that this is a 2-action by dual projection functors.

5.4. **The 2-category of idealization functors.** The 2-action defined in Subsection 5.2 suggest the following definition. Fix a small category $\mathcal{C}$ equivalent to A-mod. Define the 2-category $\mathscr{Q} = \mathscr{Q}(A, \mathcal{C})$ in the following way:

- $\mathscr{Q}$ has one object $\clubsuit$ (which we identify with $\mathcal{C}$);
- 1-morphisms in $\mathscr{Q}$ are endofunctors of $\mathcal{C}$ which belong to the additive closure generated by the identity functor and all idealization functors;
- 2-morphisms in $\mathscr{Q}$ are all natural transformations of functors;
- composition in $\mathscr{Q}$ is induced from **Cat**.

Our main observation here is the following:

Proposition 12. *If A is connected and $|\mathcal{I}| < \infty$, then $\mathscr{Q}$ is a finitary 2-category.*

Proof. Connectedness of A ensures that the identity 1-morphism $\mathbb{1}_{\clubsuit}$ is indecomposable. Clearly, $\mathscr{Q}$ has finitely many objects. As $|\mathcal{I}| < \infty$, the 2-category $\mathscr{Q}$ has finitely many isomorphism classes of indecomposable 1-morphisms. It remains to check that all spaces of 2-morphisms are finite dimensional.

Let I and J be two ideals in A. Let $\eta : \mathrm{Su}_I \to \mathrm{Su}_J$ be a natural transformation. We claim that values of η on indecomposable injective A-modules determine η uniquely. Indeed, by additivity these values determine all values of η on all injective A-modules. For $M \in A$-mod choose some injective envelope $f : M \hookrightarrow Q$. Then, by Lemma 3(i), we have the commutative diagram:

$$\begin{tikzcd} \mathrm{Su}_I(M) \arrow[r, hook, "\mathrm{Su}_I(f)"] \arrow[d, "\eta_M"'] & \mathrm{Su}_I(Q) \arrow[d, "\eta_Q"] \\ \mathrm{Su}_J(M) \arrow[r, hook, "\mathrm{Su}_J(f)"] & \mathrm{Su}_J(Q) \end{tikzcd}$$

From this diagram we see that η_M is uniquely determined by η_Q. Consequently, all spaces of 2-morphisms in $\mathscr{Q}$ are finite dimensional. $\square$

The fact that Su_I is not left exact implies that, potentially, there might exist a natural transformation $\eta|_{A\text{-inj}} : \mathrm{Su}_I|_{A\text{-inj}} \to \mathrm{Su}_J|_{A\text{-inj}}$ which cannot be extended to a natural transformation $\eta : \mathrm{Su}_I \to \mathrm{Su}_J$. Note also that in the case when A has finite representation type the space of natural transformations between any two additive endofunctors on A-mod is finite dimensional.

5.5. **The 2-category of dual projection functors.** The 2-action defined in Subsection 5.3 suggest the following definition. Assume that A is hereditary. Fix a small category $\mathcal{C}$ equivalent to A-mod. Define the 2-category $\mathscr{P} = \mathscr{P}(A, \mathcal{C})$ in the following way:

- $\mathscr{P}$ has one object $\clubsuit$ (which we identify with $\mathcal{C}$);
- 1-morphisms in $\mathscr{P}$ are endofunctors of $\mathcal{C}$ which belong to the additive closure generated by the identity functor and all dual projection functors;
- 2-morphisms in $\mathscr{P}$ are all natural transformations of functors;
- composition in $\mathscr{P}$ is induced from **Cat**.

Our main observation here is the following:

Proposition 13. *If A is hereditary, connected and $|\mathcal{I}| < \infty$, then $\mathscr{P}$ is a finitary 2-category.*

Proof. Similarly to the proof of Proposition 12, the 2-category $\mathscr{P}$ has one object, finitely many isomorphism classes of indecomposable 1-morphisms thanks to the assumption $|\mathcal{I}| < \infty$, and indecomposable identity 1-morphism $\mathbb{1}_\clubsuit$ thanks to the assumption that A is connected. Spaces of 2-morphisms are finite dimensional as projection functors are right exact and hence are given by tensoring with finite dimensional bimodules which yields that spaces of 2-morphisms are just bimodule homomorphisms between these finite dimensional bimodules. $\square$

5.6. **Decategorification and categorification.** Let $\mathscr{C}$ be a finitary 2-category. Then the *decategorification* of $\mathscr{C}$ is the (1-)category $[\mathscr{C}]$ defined as follows.

- $[\mathscr{C}]$ has same objects as $\mathscr{C}$;
- for all $\mathtt{i}, \mathtt{j} \in \mathscr{C}$ the morphism set $[\mathscr{C}](\mathtt{i}, \mathtt{j})$ is defined to be the split Grothendieck group $[\mathscr{C}(\mathtt{i}, \mathtt{j})]_\oplus$ of the additive category $\mathscr{C}(\mathtt{i}, \mathtt{j})$;
- composition in $[\mathscr{C}]$ is induced from composition in $\mathscr{C}$.

Given a 2-functor Φ from $\mathscr{C}$ to the 2-category of additive categories, taking the split Grothendieck group for each $\Phi(\mathtt{i})$ induces a functor $[\Phi]$ from $[\mathscr{C}]$ to **Cat** which is called the *decategorification* of Φ.

Given a 2-functor Φ from $\mathscr{C}$ to the 2-category of abelian categories and exact functors, taking the usual Grothendieck group for each $\Phi(\mathtt{i})$ induces a functor $[\Phi]$ from $[\mathscr{C}]$ to **Cat** which is also called the *decategorification* of Φ.

Conversely, the 2-category $\mathscr{C}$ is called a *categorification* of the category $[\mathscr{C}]$ and the 2-functor Φ is called a *categorification* of the functor Φ. We refer to [Ma, Section 1] for more details and examples.

6. Indecomposable summands of dual projection functors for path algebras of admissible trees

In this section we study both the monoid $\mathcal{I}$ and the multisemigroup $\mathcal{S}_{\mathscr{P}}$ in case A is the path algebra of the quiver Q given by an admissible orientation of a tree.

6.1. **Categorification of the Catalan monoid.** To start with, we briefly recall the main results from [GrMa]. Let A be the path algebra of the following quiver

$$(2) \qquad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n.$$

The main result of [GrMa] asserts that the ring $[\mathscr{P}](\clubsuit,\clubsuit)$ in the corresponding decategorification is isomorphic to the integral monoid algebra of the monoid $\mathtt{C}_{n+1}$ of all order preserving and decreasing transformations of $\{1,2,\dots,n,n+1\}$, which is also known as the *Catalan monoid*. Moreover, the monoid $\mathtt{C}_{n+1}$ is an ordered monoid and the 2-category $\mathscr{P}$ is biequivalent to the corresponding 2-category $\mathscr{C}_{\Bbbk}^{\mathtt{C}_{n+1}}$. We can also observe that in this case the multisemigroup $\mathcal{S}_{\mathscr{P}}$ is a genuine monoid and is, in fact, isomorphic to $\mathtt{C}_{n+1}$.

A very special feature of this example is the fact that the bimodule ${}_AA_A$ has simple socle. Consequently, all ideals of A are indecomposable as A-A–bimodules. One observation in addition to the results from [GrMa] is the following.

Proposition 14. *If A is the path algebra of the quiver* (2)*, then the* 2*-categories $\mathscr{Q}$ and $\mathscr{P}$ are biequivalent.*

Proof. Note that in this situation A is hereditary and connected. From Lemma 10 it follows that $|\mathcal{I}|<\infty$. In particular, both $\mathscr{Q}$ and $\mathscr{P}$ are well-defined and finitary. For both of these 2-categories consider the restriction 2-functor to the category $\mathcal{C}_{\mathrm{proj}}$ of projective objects in $\mathcal{C}$. This is well defined as the action of Su_I preserves $\mathcal{C}_{\mathrm{proj}}$ for each I as A is hereditary. The restriction 2-functor is clearly faithful both on the level of 1-morphisms and on the level of 2-morphisms.

Now, for any non-zero I and J, the space $\mathrm{Hom}_{A\text{-}A}(I,J)$ is zero if $I\not\subset J$ and is one-dimensional otherwise since both I and J have simple socle which appears with multiplicity one in both of them. This means that $\mathrm{Hom}_{\mathscr{Q}}(\mathrm{Dp}_I,\mathrm{Dp}_J)$ is zero if $I\not\subset J$ and is one-dimensional otherwise. Similarly, from the proof of Proposition 12 it follows that $\mathrm{Hom}_{\mathscr{P}}(\mathrm{Dp}_I,\mathrm{Dp}_J)$ is zero if $I\not\subset J$ and is at most one-dimensional otherwise. However, the inclusion $I\subset J$ does give rise to a non-zero natural transformation in $\mathrm{Hom}_{\mathscr{P}}(\mathrm{Dp}_I,\mathrm{Dp}_J)$. Therefore $\mathrm{Hom}_{\mathscr{P}}(\mathrm{Dp}_I,\mathrm{Dp}_J)$ is one-dimensional if $I\subset J$. This implies that both restriction 2-functors are full and faithful. To complete the proof it is thus left to remark that, by construction of dual projection functors, the values of both these restrictions hit exactly the same isomorphism classes of endofunctors of $\mathcal{C}_{\mathrm{proj}}$. This completes the proof. $\square$

6.2. **Setup and some combinatorics.** For a vertex i of an oriented graph Γ we denote by $\deg_\Gamma(i)$ the degree of i, by $\deg^{\mathrm{in}}_\Gamma(i)$ the in-degree of i and by $\deg^{\mathrm{out}}_\Gamma(i)$ the out-degree of i. Clearly, $\deg_\Gamma(i)=\deg^{\mathrm{in}}_\Gamma(i)+\deg^{\mathrm{out}}_\Gamma(i)$.

For the rest of the paper we fix the following setup: Let Q be an oriented (connected) tree with vertex set $Q_0=\{1,2,\dots,n\}$, where $n>1$. Set

$$\mathbf{K}(Q)=\{i\in Q_0\,;\,\deg^{\mathrm{in}}_Q(i)\deg^{\mathrm{out}}_Q(i)=0\},\qquad \mathbf{K}'(Q)=\{i\in\mathbf{K}(Q)\,;\,\deg_Q(i)\geq 2\}.$$

In other words, $\mathbf{K}(Q)$ is the set of all sinks and sources in Q and $\mathbf{K}'(Q)$ is the set of all elements $i\in\mathbf{K}(Q)$ which are not leaves.

Following [Gr], we say that Q is *admissible* provided that all vertices of Q of degree at least 3 belong to $\mathbf{K}(Q)$. For example, the orientation of a D_4 diagram on the left hand side of the following picture is admissible while the one on the right hand side is not.

(3)
$$\begin{tikzcd} & 4 \arrow[d] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} & 4 \arrow[d] & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \end{tikzcd}$$

For $i \in Q_0$, we denote by $\bar{i}$ the set of all elements in Q_0 to which there is an oriented path (possibly empty) from i in the quiver Q.

A function $\alpha : Q_0 \to Q_0 \cup \{0\}$ is called a *path function* provided that $\alpha(i) \in \bar{i} \cup \{0\}$ for all i. A path function is called *monotone* provided that, for all i, j such that $i \in \bar{j}$ and $\alpha(i) \neq 0$, we have $\alpha(j) \neq 0$ and $\alpha(i) \in \overline{\alpha(j)}$. For a function $\alpha : Q_0 \to Q_0 \cup \{0\}$, the *support* $\mathrm{supp}(\alpha)$ of α is the set of all $i \in Q_0$ such that either $\alpha(i) \neq 0$ or there is an oriented path to i from some $\alpha(j)$ such that $\alpha(j) \neq 0$. We will identify $\mathrm{supp}(\alpha)$ with the full subtree in Q with vertex set $\mathrm{supp}(\alpha)$. A monotone path function $\alpha : Q_0 \to Q_0 \cup \{0\}$ will be called a *Catalan function* provided that the following conditions are satisfied:

(I) $\mathrm{supp}(\alpha)$ is connected;

(II) for any $i \in \mathbf{K}'(Q) \cap \mathrm{supp}(\alpha)$ we have $\deg_{\mathrm{supp}(\alpha)}(i) \in \{\deg_Q(i), 1\}$;

(III) $\alpha(i) \neq i$ for any $i \in \mathbf{K}'(Q)$ with $\deg_{\mathrm{supp}(\alpha)}(i) = 1$;

(IV) $\alpha(i) = i$ for any $i \in \mathbf{K}'(Q)$ with $\deg_{\mathrm{supp}(\alpha)}(i) = \deg_Q(i)$.

For example, if Q is the quiver given by the left hand side of (3), then possible supports for Catalan functions for Q are: Q_0, $\{1,2\}$, $\{2,3\}$, $\{2,4\}$ and $\varnothing$. The following is a complete list of Catalan function for Q with support Q_0 (a function α is written in the form $(\alpha(1), \alpha(2), \alpha(3), \alpha(4))$):

$$(1,2,3,4),\ (2,2,3,4),\ (1,2,2,4),\ (1,2,3,2),\ (2,2,2,4),$$
$$(1,2,2,2),\ (2,2,3,2),\ (2,2,2,2).$$

For the same Q, here is a full list of Catalan function for Q with support $\{1,2\}$:

$$(1,0,0,0),\ (2,0,0,0).$$

Finally, $(0,0,0,0)$ is the only Catalan function for Q with support $\varnothing$.

We denote by $\mathbf{C}$ the set of all Catalan functions for Q. A subtree Γ of Q is called a *Catalan subtree* if $\Gamma = \mathrm{supp}(\alpha)$ for some Catalan function α. We denote by $\mathbf{W}$ the set of all Catalan subtrees of Q. We write

$$\mathbf{C} = \bigcup_{\Gamma \in \mathbf{W}} \mathbf{C}(\Gamma)$$

where $\mathbf{C}(\Gamma)$ stands for the set of all Catalan functions with support Γ.

6.3. **Type A enumeration.** Here we enumerate Catalan functions for type A quivers which is supposed to motivate the name. In this subsection we let Q be the oriented quiver obtained by choosing some orientation of the following Dynkin diagram of type A_n:

(4)
$$\begin{tikzcd} 1 \arrow[r, dash] & 2 \arrow[r, dash] & \cdots \arrow[r, dash] & n \end{tikzcd}$$

As mentioned above, we assume $n > 1$. We write $\mathbf{K}(Q) = \{l_1, l_2, \dots, l_k\}$, where $1 = l_1 < l_2 < \cdots < l_k = n$.

Lemma 15. *Let α be a non-zero Catalan function for Q. Then the set* $\mathrm{supp}(\alpha)$ *has the form $\{l_i, l_i + 1, \dots, l_j\}$ for some $i, j \in \{1, 2, \dots, k\}$ with $i < j$.*

Proof. By (I), $\mathrm{supp}(\alpha)$ is connected, so we need only to check that $\mathrm{supp}(\alpha)$ has more than one vertex and that both leaves of $\mathrm{supp}(\alpha)$ belong to $\mathbf{K}(Q)$. If $\mathrm{supp}(\alpha)$ would have only one vertex, it must be a sink, say i. As $n > 1$, there must be an arrow to i from some j. By monotonicity of α, j must be in $\mathrm{supp}(\alpha)$ too, a contradiction. The definition of $\mathrm{supp}(\alpha)$ and monotonicity of α readily implies that each leaf of $\mathrm{supp}(\alpha)$ is in $\mathbf{K}(Q)$. □

After Lemma 15, for $i, j \in \{1, 2, \dots, k\}$ with $i < j$ we denote by $\mathbf{C}(i, j)$ the set of all Catalan functions with support $\{l_i, l_i+1, \dots, l_j\}$. For $m = 0, 1, 2, \dots$, we denote by $\mathrm{cat}(m)$ the m-th Catalan number $\frac{1}{m+1}\binom{2m}{m}$. For $m = 1, 2, 3, \dots$, we set

$$\mathrm{cat}_1(m) := \mathrm{cat}(m) - \mathrm{cat}(m-1).$$

For $m = 2, 3, 4, \dots$, we set

$$\mathrm{cat}_2(m) := \mathrm{cat}(m) - 2\mathrm{cat}(m-1) + \mathrm{cat}(m-2).$$

Lemma 16.

(*i*) *If $k = 2$, then* $|\mathbf{C}(1, 2)| = \mathrm{cat}(n + 1)$.

(*ii*) *If $k > 3$ and $i \in \{2, 3 \dots, k-2\}$, then* $|\mathbf{C}(i, i+1)| = \mathrm{cat}_2(l_{i+1} - l_i + 2) + 1$.

(*iii*) *If $k > 4$ and $i, j \in \{2, 3 \dots, k-2\}$ with $j > i + 1$, then*

$$|\mathbf{C}(i, j)| = \mathrm{cat}_1(l_{i+1} - l_i + 1)\mathrm{cat}_1(l_j - l_{j-1} + 1) \prod_{s=i+1}^{j-2} \mathrm{cat}(l_{s+1} - l_s).$$

(*iv*) *If $k > 2$, then*

$$|\mathbf{C}(1, k)| = \mathrm{cat}(l_2)\mathrm{cat}(l_k - l_{k-1} + 1) \prod_{s=2}^{k-2} \mathrm{cat}(l_{s+1} - l_s).$$

(*v*) *If $k > 3$ and $j \in \{3, 4 \dots, k-1\}$, then*

$$|\mathbf{C}(1, j)| = \mathrm{cat}(l_2)\mathrm{cat}_1(l_j - l_{j-1} + 1) \prod_{s=2}^{j-2} \mathrm{cat}(l_{s+1} - l_s).$$

(*vi*) *If $k > 3$ and $i \in \{2, 3 \dots, k-2\}$, then*

$$|\mathbf{C}(i, k)| = \mathrm{cat}(l_k - l_{k-1} + 1)\mathrm{cat}_1(l_{i+1} - l_i + 1) \prod_{s=i+1}^{k-2} \mathrm{cat}(l_{s+1} - l_s).$$

(*vii*) *If $k > 2$, then*

$$|\mathbf{C}(1, 2)| = \mathrm{cat}_1(l_2 + 1) - 1 \quad \text{and} \quad |\mathbf{C}(k-1, k)| = \mathrm{cat}_1(l_k - l_{k-1} + 2) - 1.$$

Proof. If $k = 2$, then it is easy to check that elements of $\mathbf{C}(1, 2)$ correspond to order preserving and order increasing (or order decreasing, depending on orientation) transformations of $\{0, 1, \dots, n\}$ and hence their number is exactly $\mathrm{cat}(n + 1)$.

If $k > 2$, then the definition of a Catalan function implies that each segment $(x_{l_s}, x_{l_s+1}, \dots, x_{l_{s+1}})$ is given by an order preserving and order decreasing or an order preserving and order increasing transformation (depending on the orientation of this segment) with the condition that the values x_t for $t \in \{l_s, l_{s+1}\}$ are subject to the following rules:

- if $t \in \{i, j\} \cap \{l_2, l_3, \dots, l_{k-1}\}$, then $x_t \neq t$;
- if $i < t < j$, then $x_t = t$.

Using the inclusion-exclusion formula, it is easy to check that the number of possible values for this segment is given in terms of cat, cat_1 and cat_2 exactly as prescribed by the corresponding factors in the formulation of the lemma. The final formulae are then obtained using the multiplication principle. $\square$

6.4. **Some notation for the path algebra.** Let us go back to an admissible tree quiver Q as defined in Subsection 6.2. Let A be the path algebra of Q. For $i \in Q_0$ denote by e_i the trivial path in vertex i. Then $P_i = Ae_i$ and $L_i = P_i/\text{Rad}(P_i)$. For each $i, j \in \{1, 2, \dots, n\}$ such that $j \in \bar{i}$, denote by a_{ji} the unique path from i to j. Then $\{\mathsf{a}_{ji}\}$ is a basis in the one-dimensional vector space $e_j A e_i$.

From now on we assume that $\mathbf{K}'(Q) \neq \varnothing$. This is equivalent to the requirement that Q is not isomorphic to the quiver given by (2).

6.5. **Graph of the identity bimodule.** To study subbimodules of the identity bimodule, it is convenient to use a graphical presentation of the latter. For this we consider ${}_A A_A$ as a representation of a quiver $Q \times Q^{\text{op}}$ where we impose all possible commutativity relations, see e.g. [Sk] for details. These can be arranged into a graph with the left action of arrows in Q being depicted by solid arrow and the right action of arrows in Q being depicted by dashed arrows. For example, if Q is the quiver

$$1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5 \longrightarrow 6, \tag{5}$$

we obtain the following graphical presentation of ${}_A A_A$:

$$\begin{tikzcd}
\mathsf{a}_{11} \arrow[r] & \mathsf{a}_{21} & & & & \\
& \mathsf{a}_{22} \arrow[u, dashed] \arrow[d, dashed] & & & & \\
& \mathsf{a}_{23} \arrow[d, dashed] & \mathsf{a}_{33} \arrow[l] \arrow[d, dashed] & & & \\
& \mathsf{a}_{24} & \mathsf{a}_{34} \arrow[l] & \mathsf{a}_{44} \arrow[l] \arrow[r] & \mathsf{a}_{54} \arrow[r] & \mathsf{a}_{64} \\
& & & & \mathsf{a}_{55} \arrow[u, dashed] \arrow[r] & \mathsf{a}_{65} \arrow[u, dashed] \\
& & & & & \mathsf{a}_{66} \arrow[u, dashed]
\end{tikzcd} \tag{6}$$

If Q is the quiver given by the left hand side of (3), then we have the following graphical presentation of ${}_A A_A$:

(7)
$$\begin{tikzcd}
 & \mathtt{a}_{44} \arrow[d] & \\
\mathtt{a}_{11} \arrow[d] & \mathtt{a}_{24} & \mathtt{a}_{33} \arrow[d] \\
\mathtt{a}_{21} & \mathtt{a}_{22} \arrow[l, dashed] \arrow[r, dashed] \arrow[u, dashed] & \mathtt{a}_{23}
\end{tikzcd}$$

A *subchain* H in Q is a full subgraph of Q isomorphic to

(8)
$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow k$$

for some k. The set of all subchains in Q is partially ordered by inclusions. A *maximal subchain* is a maximal element of this poset. From the diagrammatic representation of ${}_A A_A$ described above it is clear that there is a bijection between maximal subchains in Q and simple subbimodules in the socle of ${}_A A_A$. Namely, each maximal subchain with source s and sink t contributes the simple subbimodule in the socle of ${}_A A_A$ with basis $\mathtt{a}_{ts}$.

6.6. **Diagram of a subbimodule in ${}_A A_A$.** Viewing ${}_A A_A$ as a representation of $Q \times Q^{\mathrm{op}}$ with all commutativity relations, as described in Subsection 6.5, subbimodules in ${}_A A_A$ are exactly subrepresentations. Let B be a subbimodule of ${}_A A_A$. From the proof of Lemma 10 it follows that the set of all $\mathtt{a}_{ij}$ contained in B is a basis of B. This yields a graphic presentation of B as a subgraph of the graphic presentation of ${}_A A_A$ discussed in Subsection 6.5.

For example, for the quiver (5) and the graph (6) of the corresponding identity bimodule, the graph of the subbimodule J_4 is given by:

$$\begin{tikzcd}
\mathtt{a}_{11} \arrow[r] & \mathtt{a}_{21} & & & & \\
 & \mathtt{a}_{22} \arrow[u, dashed] \arrow[d, dashed] & & & & \\
 & \mathtt{a}_{23} \arrow[d, dashed] & \mathtt{a}_{33} \arrow[l] \arrow[d, dashed] & & & \\
 & \mathtt{a}_{24} & \mathtt{a}_{34} \arrow[l] & 0 \arrow[l, dotted] \arrow[r, dotted] & \mathtt{a}_{54} \arrow[r] & \mathtt{a}_{64} \\
 & & & & \mathtt{a}_{55} \arrow[u, dashed] \arrow[r] & \mathtt{a}_{65} \arrow[u, dashed] \\
 & & & & & \mathtt{a}_{66} \arrow[u, dashed]
\end{tikzcd}$$

Here 0 stands on the place of $\mathtt{a}_{44}$ which is missing in J_4 from the identity bimodule and the dotted arrows depict the corresponding zero multiplication. This clearly shows that J_4 is a decomposable bimodule. In particular, we obtain that in this case the monoid $\mathcal{I}$ should be rather different from the multisemigroup $\mathcal{S}_{\mathscr{P}}$.

Note that, for example, the linear span of $\mathtt{a}_{44}$ and $\mathtt{a}_{54}$ is not a subrepresentation as it is not closed with respect to the action of the arrow $\mathtt{a}_{54} \to \mathtt{a}_{64}$. Therefore this linear span is not a subbimodule.

6.7. **Catalan function of an indecomposable subbimodule.** Let B be a subbimodule of ${}_AA_A$. Then

$$B = \bigoplus_{i,j=1}^{n} e_jBe_i$$

with each e_jBe_i being of dimension at most one. Moreover, $e_jBe_i \neq 0$ implies that $j \in \bar{i}$. For $i = 1, 2, \dots, n$, set

$$B_i := \bigoplus_{j=1}^{n} e_jBe_i.$$

Note that B_i is a submodule of ${}_AB$. We would like to point out the following properties:

- If $s \in \mathbf{K}'(Q)$ is a sink, then the full subtree $Q^{(s)}$ of Q with vertices $Q_0 \setminus \{s\}$ decomposes into a disjoint union of components $\Gamma^{(1)} \cup \Gamma^{(2)} \cup \dots \cup \Gamma^{(m)}$ and $m \geq 2$. If $B_s = 0$, then for each $j \in \{1, 2, \dots, m\}$ the space
$$\bigoplus_{t \in \Gamma^{(j)}} B_t$$
is a subbimodule of B. In particular, if B is indecomposable, then only one of these subbimodules is nonzero.
- If $s \in \mathbf{K}'(Q)$ is a source, then the full subtree $Q^{(s)}$ of Q with vertices $Q_0 \setminus \{s\}$ is a disjoint union of connected components $\Gamma^{(1)} \cup \Gamma^{(2)} \cup \dots \cup \Gamma^{(m)}$ and $m \geq 2$. Now, if $B_s \not\cong P_s$, then
$$B_s = \bigoplus_{j=1}^{m} B_s^{(j)} \qquad \text{where} \qquad B_s^{(j)} := \bigoplus_{t \in \Gamma^{(j)}} e_tBe_s.$$
Moreover, for each $j \in \{1, 2, \dots, m\}$ the space
$$B_s^{(j)} \oplus \bigoplus_{t \in \Gamma^{(j)}} B_t$$
is a subbimodule of B. In particular, if B is indecomposable, then only one of these subbimodules is nonzero.
- If $B_s \neq 0$ for some s and $t \in \bar{s}$ is such that $e_tBe_s \neq 0$, then $e_rBe_s \neq 0$ for any $r \in \bar{t}$.
- If $B_s \neq 0$ for some s and $s \in \bar{t}$, then $B_t \neq 0$.

From these observations we, in particular, obtain that, for an indecomposable subbimodule B of ${}_AA_A$, each B_i is an indecomposable projective A-module. This justifies the following definition.

For an indecomposable B define the function $\mathbf{x}_B : Q_0 \to Q_0 \cup \{0\}$, written

$$(x_1, x_2, \dots, x_n) = (\mathbf{x}_B(1), \mathbf{x}_B(2), \dots, \mathbf{x}_B(n)),$$

in the following way:

- if $B_i = 0$ for $i \in Q_0$, then set $x_i = 0$;
- if $B_i \neq 0$ for $i \in Q_0$, then x_i is defined as the unique element in Q_0 such that $B_i \cong P_{x_i}$.

We also define the *support* of B as

$$\mathrm{supp}(B) := \{s \in Q_0 \,:\, e_sBe_t \neq 0 \text{ or } e_tBe_s \neq 0 \text{ for some } t\}.$$

Proposition 17. *Let B be an indecomposable subbimodule of ${}_AA_A$. Then $\mathbf{x}_B$ is a Catalan function and* $\mathrm{supp}(\mathbf{x}_B) = \mathrm{supp}(B)$.

Proof. If P_j is a submodule of P_i, then there is an oriented path from i to j. This implies that $\mathbf{x}_B$ is a path function. Monotonicity of $\mathbf{x}_B$ is a consequence of the fact that B is closed with respect to both the left and the right multiplications by arrows from Q. Conditions (I)–(IV) follow from indecomposibility of B and the first two remarks above which led to the definition of $\mathbf{x}_B$.

We have $\mathbf{x}_B(s) \neq 0$ if and only if $e_t B e_s \neq 0$ for some t. Therefore, $e_x B e_y \neq 0$ if and only if $\mathbf{x}_B(y) \neq 0$ and $x \in \overline{\mathbf{x}_B(y)}$. Together this implies $\mathrm{supp}(\mathbf{x}_B) = \mathrm{supp}(B)$. □

Proposition 18. *Let B be an indecomposable subbimodule of ${}_AA_A$ and assume that*

$$\mathrm{soc}(B) \cong \bigoplus_{i=1}^{m} L_{t_i s_i}.$$

Then $\mathrm{supp}(B)$ *is the full subgraph of Q obtained as the union of maximal chains with sources s_i and sinks t_i for $i = 1, 2, \dots, m$.*

Proof. This follows easily from the definitions. □

6.8. **Subbimodules of ${}_AA_A$ associated to Catalan functions.** For a Catalan function $\mathbf{x} : Q_0 \to Q_0 \cup \{0\}$ denote by $B_{\mathbf{x}}$ the subspace in ${}_AA_A$ obtained as the linear span of all a_{ts} for which $x_s \neq 0$ and $t \in \overline{x_s}$. The fact that $\mathbf{x}$ is a path function ensures that this definition does make sense.

Example 19. For the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longrightarrow 6 \longrightarrow 7 \longrightarrow 8 \longleftarrow 9$$

and the Catalan function given by $(0, 0, 0, 3, 5, 7, 8, 8, 8)$ we have the subbimodule of ${}_AA_A$ given by the bold and solid part of the diagram in Figure 1, where the regular dotted part shows the rest of ${}_AA_A$.

Proposition 20. *For every Catalan function $\mathbf{x}$, the subspace $B_{\mathbf{x}}$ of ${}_AA_A$ is an indecomposable subbimodule and* $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(B_{\mathbf{x}})$.

Proof. Directly from the definition we see that $B_{\mathbf{x}}$ is closed with respect to the left multiplication with arrows from Q. Monotonicity of $\mathbf{x}$ implies that $B_{\mathbf{x}}$ is closed with respect to right multiplication with arrows from $\mathrm{supp}(\mathbf{x})$. Conditions (II) and (IV) say that the only non-trivial possible right multiplication of an element in $B_{\mathbf{x}}$ with an arrow outside $\mathrm{supp}(\mathbf{x})$ could be if this arrow goes in or comes out of $i \in \mathrm{supp}(\mathbf{x}) \cap \mathbf{K}'(Q)$ with $\deg_Q(i) > \deg_{\mathrm{supp}(\mathbf{x})}(i)$. However, in the latter situation condition (III) and monotonicity of $\mathbf{x}$ guarantee that the corresponding product is zero. Therefore $B_{\mathbf{x}}$ is a subbimodule in ${}_AA_A$.

Indecomposability of $B_{\mathbf{x}}$ is equivalent to the statement that the full subgraph of the graph of ${}_AA_A$ generated by all a_{ts} spanning $B_{\mathbf{x}}$ is connected. This follows from construction and conditions (I), (II) and (IV).

The statement $\mathrm{supp}(\mathbf{x}) = \mathrm{supp}(B_{\mathbf{x}})$ follows by construction. □

$$\begin{tikzcd}
\mathtt{a}_{11} \arrow[r, dotted] & \mathtt{a}_{21} \arrow[r, dotted] & \mathtt{a}_{31} & & & & & & \\
& \mathtt{a}_{22} \arrow[u, dotted] \arrow[r, dotted] & \mathtt{a}_{32} \arrow[u, dotted] & & & & & & \\
& & \mathtt{a}_{33} \arrow[u, dotted] \arrow[d, dotted] & & & & & & \\
& & \mathtt{a}_{\mathbf{34}} \arrow[d, dashed] & \mathtt{a}_{44} \arrow[l, dotted] \arrow[d, dotted] & & & & & \\
& & \mathtt{a}_{\mathbf{35}} & \mathtt{a}_{\mathbf{45}} \arrow[l] & \mathtt{a}_{55} \arrow[l] \arrow[r] & \mathtt{a}_{\mathbf{65}} \arrow[r] & \mathtt{a}_{\mathbf{75}} \arrow[r] & \mathtt{a}_{\mathbf{85}} & \\
& & & & & \mathtt{a}_{66} \arrow[u, dotted] \arrow[r, dotted] & \mathtt{a}_{\mathbf{76}} \arrow[u, dashed] \arrow[r] & \mathtt{a}_{\mathbf{86}} \arrow[u, dashed] & \\
& & & & & & \mathtt{a}_{77} \arrow[u, dotted] \arrow[r, dotted] & \mathtt{a}_{\mathbf{87}} \arrow[u, dashed] & \\
& & & & & & & \mathtt{a}_{\mathbf{88}} \arrow[u, dashed] \arrow[d, dashed] & \\
& & & & & & & \mathtt{a}_{89} & \mathtt{a}_{99} \arrow[l, dotted]
\end{tikzcd}$$

FIGURE 1. Diagram used in Example 19.

6.9. **Classification of indecomposable subbimodules of $_AA_A$.** We can now collect the above facts into the following statement.

Theorem 21. *The maps $B \mapsto \mathbf{x}_B$ and $\mathbf{x} \mapsto B_{\mathbf{x}}$ are mutually inverse bijections between the set of all indecomposable subbimodules of $_AA_A$ and the set of all Catalan functions.*

Proof. From the definitions it is straightforward to check that these two maps are inverses to each other. $\square$

For $\Gamma \in \mathbf{W}$ we denote by $\mathbf{B}(\Gamma)$ the set of all indecomposable subbimodules $B \subset {}_AA_A$ for which $\mathrm{supp}(B) = \Gamma$.

6.10. **Partial order.** We identify $\mathcal{S}_{\mathscr{P}}$ with the subset $\mathcal{I}^{\mathrm{ind}}$ of $\mathcal{I}$ consisting of all indecomposable subbimodules (our convention is that 0 is an indecomposable subbimodule). To this end, we do not know whether $\mathcal{I}^{\mathrm{ind}}$ is a submonoid of $\mathcal{I}$, that is, whether $IJ \in \mathcal{I}^{\mathrm{ind}}$ for any $I, J \in \mathcal{I}^{\mathrm{ind}}$. The original multivalued operation in $\mathcal{I}^{\mathrm{ind}}$ sends (I, J) to the set of all indecomposable direct summands of IJ up to isomorphism.

The set $\mathcal{I}^{\mathrm{ind}}$ inherits from $\mathcal{I}$ the partial order given by inclusions. Clearly, $_AA_A$ is the maximum element with respect to this order both in $\mathcal{I}^{\mathrm{ind}}$ and in $\mathcal{I}$ (note that $_AA_A \in \mathcal{I}^{\mathrm{ind}}$ as we assume Q to be connected).

Consider the set $\mathbf{Q} := \{J_s : s = 1, 2, \dots, n\}$ and note that this is the set of maximal elements in $\mathcal{I} \setminus \{{}_AA_A\}$. The bimodule J_s is indecomposable if and only if we have $s \notin \mathbf{K}'(Q)$. For $s \in \mathbf{K}'(Q)$, let $t_1, t_2, \dots, t_{m_s}$ be the list of all $t \in Q_0$ for which there is an arrow $t \to s$ or an arrow $s \to t$. Let Γ be the full subgraph of Q with vertex set $Q_0 \setminus \{s\}$. Then

$$\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)} \cup \dots \cup \Gamma^{(m_s)}$$

where $\Gamma^{(q)}$ is the connected component containing t_q for $q = 1, 2, \dots, m_s$. We have the decomposition

$$J_s \cong \bigoplus_{q=1}^{m_s} J_s^{(q)} \tag{9}$$

where $J_s^{(q)}$ is the subbimodule of J_s defined as the direct sum of all $e_j(J_s)e_i$ with $i \in \Gamma^{(q)} \cup \{s\}$ and $j \in \Gamma^{(q)}$. Clearly, each $J_s^{(q)}$ is indecomposable since $\Gamma^{(q)}$ is connected.

Lemma 22.

(*i*) *The set $\{J_s : s \notin \mathbf{K}'(Q)\}$ is the set of maximal elements in $\mathbf{B}(Q) \setminus \{{}_AA_A\}$.*

(*ii*) *For $\Omega \in \mathbf{W} \setminus \{Q\}$, there is a unique maximal element, denoted B_Ω, in the set $\mathbf{B}(\Omega)$. Moreover, $B_\varnothing = 0$ and, for $\Omega \neq \varnothing$, we have*

$$B_\Omega = \prod_t J_t^{(p_t)} \prod_s J_s^{(q_s)} \tag{10}$$

where s runs through the set of sinks $i \in \mathbf{K}'(Q) \cap \Omega$ for which $\deg_\Omega(i) = 1$ and q_s is such that the corresponding $\Gamma^{(q_s)}$ has a common vertex with Ω, while t runs through the set of sources $i \in \mathbf{K}'(Q) \cap \Omega$ for which $\deg_\Omega(i) = 1$ and p_t is such that the corresponding $\Gamma^{(p_t)}$ has a common vertex with Ω.

Proof. Clearly each J_s with $s \notin \mathbf{K}'(Q)$ is maximal in $\mathbf{B}(Q) \setminus \{{}_AA_A\}$. Assume that $B \in \mathbf{B}(Q)$ is maximal in $\mathbf{B}(Q) \setminus \{{}_AA_A\}$. Then $B \subset J_s$ for some s. If $s \notin \mathbf{K}'(Q)$, then $B = J_s$ by maximality of B. If $s \in \mathbf{K}'(Q)$, then

$$B = \bigoplus_{q=1}^{m_s} (B \cap J_s^{(q)})$$

since all composition multiplicities in J_s are at most one. By indecomposability, we get $B = B \cap J_s^{(q)}$ for some q, which contradicts $B \in \mathbf{B}(Q)$. Therefore this case does not occur, which proves claim (i).

To prove claim (ii) we denote by B'_Ω the right hand side of (10). Note that $B_\varnothing = 0$ is clear and that for $\Omega \neq \varnothing$ the fact that $B'_\Omega \in \mathbf{B}(\Omega)$ follows by construction. The maximal element B_Ω in the set $\mathbf{B}(\Omega)$ is the sum of all subbimodules of ${}_AA_A$ with support Ω. Therefore to complete the proof of claim (ii) it remains to check that $B'_\Omega = B_\Omega$ for $\Omega \neq \varnothing$.

If there are s and t in (10) which are connected by an edge, then Ω must be the full subgraph of Q with vertices $\{s, t\}$ by connectedness. For such small Ω the equality $B'_\Omega = B_\Omega$ is checked by a direct computation. In the remaining case (no s and t in (10) are connected by an edge), all factors of (10) commute. Note that $B_\Omega \subset J_s$ and $B_\Omega \subset J_t$ for any s and t occurring in (10). From indecomposability, it follows that $B_\Omega \subset J_s^{(q_s)}$ and $B_\Omega \subset J_t^{(p_t)}$ for all s and t occurring in (10). From $J_s^2 = J_s$ and $J_t^2 = J_t$ it follows that $(J_s^{(q_s)})^2 = J_s^{(q_s)}$ and $(J_t^{(p_t)})^2 = J_t^{(p_t)}$. This implies $B_\Omega J_s^{(q_s)} = B_\Omega$ and $B_\Omega J_t^{(p_t)} = B_\Omega$ which yields $B_\Omega B'_\Omega = B_\Omega \subset B'_\Omega$. From the maximality of B_Ω we finally obtain that $B_\Omega = B'_\Omega$. $\square$

6.11. **Composition of indecomposable subbimodules.** The following is a crucial observation.

Proposition 23. *Let B and D be two indecomposable subbimodules in ${}_AA_A$. Then $B \otimes_A D \cong BD$ and the latter is either zero or an indecomposable subbimodule of ${}_AA_A$.*

Proof. As A is hereditary, $B \otimes_A {}_-$ is exact, in particular, it preserves inclusions. Hence, applying it to $D \hookrightarrow A$ gives $B \otimes_A D \hookrightarrow B \otimes_A A \cong B$ where the last inclusion is given by the multiplication map. Therefore the multiplication map $B \otimes_A D \to BD$ is an isomorphism.

It remains to prove indecomposability of BD in case the latter subbimodule is nonzero. Assume first that both B and D have support Q. If Q is as in (2), then BD is indecomposable or zero by [GrMa, Proposition 4]. If Q is not as in (2), then $\mathbf{K}'(Q) \neq \varnothing$ and hence for every $i \in \mathbf{K}'(Q)$ we have $e_i \in B$ and $e_i \in D$ by our assumption that both these bimodules have support Q. This implies $e_i \in BD$. From the last paragraph in Subsection 6.5 it thus follows that the bimodules B, D, BD and ${}_AA_A$ all have the same socle. Any non-trivial decomposition $BD = X_1 \oplus X_2$ of A-A-bimodules gives a non-trivial decomposition $\mathrm{soc}(BD) = \mathrm{soc}(X_1) \oplus \mathrm{soc}(X_2)$. From Proposition 18 it follows that supports of X_1 and X_2 intersect only in some vertices from $\mathbf{K}'(Q)$. Let s be a vertex of such intersection, then it is a leaf in both the support of X_1 and the support of X_2, which implies that $e_s \notin X_1$ and $e_s \notin X_2$, a contradiction. Therefore BD is indecomposable.

The general case reduces to the previous paragraph by considering the intersection of supports of B and D. It is easy to check that there exists an indecomposable subbimodule B' in B and an indecomposable subbimodule D' in D such that $B'D' = BD$ and the bimodules B' and D' have the same supports. We leave the technical details to the reader. This completes the proof. □

An immediate consequence of Proposition 23 is the following:

Corollary 24. *The multisemigroup $\mathcal{S}_{\mathscr{P}}$ is a monoid.*

We denote by $\mathcal{I}^{\mathrm{ind}}$ the submonoid of $\mathcal{I}$ consisting of indecomposable subbimodules in ${}_AA_A$. By the above, the monoids $\mathcal{S}_{\mathscr{P}}$ and $\mathcal{I}^{\mathrm{ind}}$ are isomorphic.

Problem 25. It would be interesting to know for which finite dimensional algebras the product of two indecomposable subbimodules of the identity bimodule is always indecomposable or zero.

7. Presentation for $\mathcal{I}$ and $\mathcal{I}^{\mathrm{ind}}$

The main aim of this section is to obtain presentations for both the monoid $\mathcal{I}$ and the monoid $\mathcal{I}^{\mathrm{ind}}$.

7.1. **Minimal generating systems.** We set

$$\mathbf{B} := \{J_s \,:\, s \notin \mathbf{K}'(Q)\} \cup \bigcup_{s \in \mathbf{K}'(Q)} \{J_s^{(q)} \,:\, q = 1, 2, \dots, m_s\}.$$

Proposition 26.

(*i*) *The set $\mathbf{Q}$ is the unique minimal generating system for the monoid $\mathcal{I}$.*

(*ii*) *The set $\mathbf{B}$ is the unique minimal generating system for the monoid $\mathcal{I}^{\mathrm{ind}}$.*

Proof. As $\mathbf{Q}$ is the set of all maximal elements in $\mathcal{I} \setminus \{{}_AA_A\}$, it must belong to any generating system. Therefore, to prove claim (i) it is enough to show that $\mathbf{Q}$ generates $\mathcal{I}$. Let S be the submonoid of $\mathcal{I}$ generated by $\mathbf{Q}$. Assume that $\mathcal{I} \setminus S \neq \varnothing$ and let B be a maximal element in $\mathcal{I} \setminus S$ with respect to inclusions. Certainly, $B \neq 0$ and $B \neq {}_AA_A$.

For the record we mention the following fact which follows directly from the definitions:

$$J_iP_j = \begin{cases} P_j, & i \neq j; \\ \mathrm{Rad}(P_i), & i = j. \end{cases} \tag{11}$$

Assume first that $B_i \in \{0, P_i\}$ for all i. As $B \neq {}_AA_A$, there is at least one i such that $B_i = 0$. As $B \neq 0$, there is at least one j such that $B_j = P_j$. Choose i such $B_i = 0$ and there is an arrow $j \to i$ for some j such that $B_j = P_j$. Then i is not a source, in particular, P_i has simple socle, say L_s. Then s is a sink. As B is a bimodule, it follows that $B_t = 0$ for all $t \in \bar{i}$, in particular, $B_s = 0$. The space $B' = B \oplus \Bbbk\{\mathsf{a}_{si}\}$ is easily checked to be a subbimodule of ${}_AA_A$. Moreover, using (11) and $B_s = 0$, we get $B = J_sB'$. As $B \subsetneq B'$, we have $B' \in S$ by maximality of B. Therefore $B \in S$, a contradiction.

Now we may assume that there is an i such that $B_i \notin \{0, P_i\}$. In this case we may choose $i \in \{1, 2, \dots, n\}$ such that $B_i \notin \{0, P_i\}$ and, additionally, for any j for which there is an arrow $j \to i$ we have $B_j = P_j$ (note that such B_j is automatically non-zero as $B_i \neq 0$). If i is not a source, then there is a unique submodule M of P_i such that $B_i \subset M$ and M/B_i is simple, say isomorphic to L_s. By construction, s is not a sink and belongs to $\bar{i}$ with i being not a source. This implies that none of the B_t, where $t = 1, 2, \dots, n$, has P_s as a direct summand. Therefore, similarly to the above, $B' = B \oplus \Bbbk\langle \mathsf{a}_{si} \rangle$ is a strictly larger subbimodule and $B = J_sB'$ leading again to a contradiction.

Finally, consider the case when i is a source. If $\deg_Q(i) = 1$, then P_i is uniserial and there is a unique $s \in \bar{i}$ such that the radical of P_s is isomorphic to B_i. Then, similarly to the above, $B \oplus \Bbbk\{\mathsf{a}_{si}\}$ is easily checked to be a subbimodule of ${}_AA_A$ and $B = J_sB'$ by (11), a contradiction. If $\deg_Q(i) > 1$, then J_i decomposes according to (9). By intersecting with each direct summand of this decomposition we get a similar decomposition of B. For each of these direct summands the claim follows by induction on n provided that the corresponding generators for each $\Gamma^{(l)}$ can be obtained as products of elements in $\mathbf{Q}$. The latter follows easily by multiplying (several times) J_i on the left with all J_s with $s \notin \Gamma^{(l)}$. Claim (i) follows.

The fact that $\mathbf{B}$ generates $\mathcal{I}^{\mathrm{ind}}$ follows from claim (i) and Proposition 23 since $\mathbf{B}$ is exactly the set of indecomposable summands of elements in $\mathbf{Q}$. Uniqueness and irreducibility also follow from claim (i) and the easy observation that any of $J_i^{(q)}$ appearing in (9) cannot be obtained as a product of other elements in $\mathbf{B}$ looking at the supports of all involved bimodules. This completes the proof. □

7.2. **Relations.**

Proposition 27. *The ideals J_i, $i = 1, 2, \dots, n$, satisfy the following relations:*

(*a*) $J_i^2 = J_i$ *for all* i.

(*b*) $J_iJ_j = J_jJ_i$ *if there is no arrow between* i *and* j.

(*c*) $J_jJ_iJ_j = J_iJ_jJ_i = J_jJ_i$ *if there is an arrow* $i \to j$.

Proof. All this is a straightforward computation using (11). Alternatively, this also follows from Proposition 8 and properties of projection functors mentioned in Subsection 2.3. □

Proposition 28. *For $i, j \notin \mathbf{K}'(Q)$, $s, t \in \mathbf{K}'(Q)$, $q \in \{1, 2, \dots, m_s\}$ and $p \in \{1, 2, \dots, m_t\}$, the elements of $\mathbf{B}$ satisfy the following:*

(*a*) *Relations from Proposition 27*(a)-(c).

(*b*) $(J_s^{(q)})^2 = J_s^{(q)}$.

(*c*) $J_s^{(q)} J_s^{(q')} = 0$ *for any* $q' \in \{1, 2, \dots, m_s\}$, $q' \neq q$.

(*d*) $J_s^{(q)} J_t^{(p)} = J_t^{(p)} J_s^{(q)}$ *if there is no arrow between s and t.*

(*e*) $J_s^{(q)} J_i = J_i J_s^{(q)}$ *if there is no arrow between s and i.*

(*f*) $J_t^{(p)} J_s^{(q)} J_t^{(p)} = J_s^{(q)} J_t^{(p)} J_s^{(q)} = J_t^{(p)} J_s^{(q)}$ *if there is an arrow $s \to t$.*

(*g*) $J_t^{(p)} J_i J_t^{(p)} = J_i J_t^{(p)} J_i = J_t^{(p)} J_i$ *if there is an arrow $i \to t$.*

(*h*) $J_t^{(p)} J_i J_t^{(p)} = J_i J_t^{(p)} J_i = J_i J_t^{(p)}$ *if there is an arrow $t \to i$.*

(*i*) $J_t^{(p)} J_s^{(q)} J_t^{(p')} = 0$ *for any* $p' \in \{1, 2, \dots, m_t\}$, $p' \neq p$, *if there is an arrow between s and t.*

(*j*) $J_t^{(p)} J_i J_t^{(p')} = 0$ *for any* $p' \in \{1, 2, \dots, m_t\}$, $p' \neq p$, *if there is an arrow between t and i.*

(*k*) $J_s^{(q)} J_t^{(p)} = J_t^{(p)}$ *in case* $\mathrm{supp}(J_t^{(p)}) \subset \mathrm{supp}(J_s^{(q)})$.

(*l*) $J_s^{(q)} J_t^{(p)} = 0$ *in case* $\mathrm{supp}(J_t^{(p)}) \cap \mathrm{supp}(J_s^{(q)}) = \varnothing$.

(*m*) $J_s^{(q)} J_i = J_i J_s^{(q)} = J_s^{(q)}$ *if* $i \notin \mathrm{supp}(J_s^{(q)})$.

Proof. Relations (a) are clear. From Proposition 27(a) for $s \in \mathbf{K}'(Q)$ we have

$$\Big(\bigoplus_{q=1}^{m_s} J_s^{(q)}\Big)^2 \cong \bigoplus_{q=1}^{m_s} (J_s^{(q)})^2 \oplus \bigoplus_{q \neq q'} (J_s^{(q)} J_s^{(q')})^{\oplus 2} \cong \bigoplus_{q=1}^{m_s} J_s^{(q)}.$$

As $(J_s^{(q)})^2$ is the only direct summand in the middle with support in $\Gamma^{(q)}$, we get relations (b) and (c).

By Proposition 27(b), for $s, t \in \mathbf{K}'(Q)$ in case there is no arrow between s and t, we have

$$\Big(\bigoplus_{q=1}^{m_s} J_s^{(q)}\Big)\Big(\bigoplus_{p=1}^{m_t} J_t^{(p)}\Big) = \Big(\bigoplus_{p=1}^{m_t} J_t^{(p)}\Big)\Big(\bigoplus_{q=1}^{m_s} J_s^{(q)}\Big).$$

Opening brackets and matching summands with the same support on the left hand side and on the right hand side, we get relations (d). Relations (e) are obtained similarly from

$$\Big(\bigoplus_{q=1}^{m_s} J_s^{(q)}\Big) J_i = J_i \Big(\bigoplus_{q=1}^{m_s} J_s^{(q)}\Big)$$

which is again given by Proposition 27(b).

Relations (f)–(j) are obtained similarly from Proposition 27(c).

To prove relation (k) we compare $J_s^{(q)} J_t^{(p)} P_r$ with $J_t^{(p)} P_r$ for $r \in \{1, 2, \dots, n\}$ using (11). Note that the only P_i which appear as direct summands of $J_t^{(p)} P_r$ are those for which $i \in \mathrm{supp}(J_t^{(p)}) \setminus \{t\}$. If $\mathrm{supp}(J_t^{(p)}) \subset \mathrm{supp}(J_s^{(q)})$, then $J_s^{(q)} P_i = P_i$ for such i by (11). This implies relation (k). Similarly one checks relations (l) and (m). This completes the proof. $\square$

7.3. **The main results.**

Theorem 29. *Any relation between elements in the generating set* $\mathbf{Q}$ *of the monoid* $\mathcal{I}$ *is a consequence of the relations given in Proposition 27.*

Proof. Let S be the abstract monoid given by generators $\mathbf{Q}$ and relations from Proposition 27. Note that S is a Hecke-Kiselman monoid in the sense of [GM]. Then we have the canonical surjection $\psi : S \twoheadrightarrow \mathcal{I}$.

For any $I \in \mathcal{I}$ the additive functor Su_I acting on A-proj defines an endomorphism of the split Grothendieck group $[A\text{-proj}]_\oplus$. This gives a homomorphism φ from $\mathcal{I}$ to the monoid of all endomorphisms of $[A\text{-proj}]_\oplus$. Let T denote the image of φ. Combined together we have surjective composition $\varphi \circ \psi$ as follows: $S \twoheadrightarrow \mathcal{I} \twoheadrightarrow T$.

Consider the standard basis $\{[P_i] : i = 1, 2, \dots, n\}$ in $[A\text{-proj}]_\oplus$. From isomorphisms in (11), for $i, j = 1, 2, \dots, m$ we obtain

$$\varphi(J_i)[P_j] = \begin{cases} [P_j], & i \neq j; \\ \sum\limits_{i \to s} [P_s], & i = j. \end{cases}$$

This is exactly the linear representation of S considered in [Fo, Theorem 4.5] where it was proved that the corresponding representation map is injective, that is $S \cong T$. Consequently, because of the sandwich position of $\mathcal{I}$ between S and T, we obtain $S \cong \mathcal{I}$ and the proof is complete. $\square$

Theorem 30. *Any relation between elements in the generating set* $\mathbf{B}$ *of the monoid* $\mathcal{I}^{\mathrm{ind}}$ *is a consequence of the relations given in Proposition 28.*

Proof. Let S be the abstract monoid given by generators $\mathbf{B}$ and relations from Proposition 28. As usual we denote by $\mathbf{B}^+$ the set of all non-empty words in the alphabet $\mathbf{B}$. For simplicity, we call all elements in $\mathbf{B}$ of the form $J_s^{(q)}$ for $s \in \mathbf{K}'(Q)$ and $q \in \{1, 2, \dots, m_s\}$ the *split symbols*. For any $w \in \mathbf{B}^+$, let $J_{s_1}^{(q_1)}, J_{s_2}^{(q_2)}, \dots, J_{s_k}^{(q_k)}$ be the list of all split symbols which appear in w. If w has no split symbols, we set $\Omega = Q$. Otherwise, set

$$\Omega := \bigcap_{i=1}^{k} \mathrm{supp}(J_{s_i}^{(q_i)}).$$

If $\Omega = \varnothing$, then the fact that Q is a tree implies existence of $i, j \in \{1, 2, \dots, k\}$ such that $\mathrm{supp}(J_{s_i}^{(q_i)}) \cap \mathrm{supp}(J_{s_j}^{(q_j)}) = \varnothing$. We claim that in this case $w = 0$ in S. Without loss of generality we may assume that the indices i and j and the word w (in its equivalence class) are chosen such that $w = x J_{s_i}^{(q_i)} y J_{s_j}^{(q_j)} z$ with y shortest possible. Then y contains no J_r with $r \in \mathrm{supp}(J_{s_j}^{(q_j)})$ because otherwise we would take the leftmost occurrence of such element and use relations in Proposition 28(b) to move it past J_q with $q \notin \mathrm{supp}(J_{s_j}^{(q_j)})$. Note that, by the minimality of y and relations in Proposition 28(d), (e) and (k), there is no $J_{s_a}^{(q_a)}$ between $J_{s_i}^{(q_i)}$ and J_r, such that r and s_a are connected by an arrow. So J_r commutes with any split symbols between $J_{s_i}^{(q_i)}$ and J_r and so we can move it past $J_{s_i}^{(q_i)}$ making y shorter. Similarly, y does not contain any J_r with $r \in \mathrm{supp}(J_{s_i}^{(q_i)})$. Analogously (using also Proposition 28(c)) one shows that y does not contain any split symbol $J_r^{(f)}$

for $r \in \mathrm{supp}(J_{s_i}^{(q_i)}) \cup \mathrm{supp}(J_{s_j}^{(q_j)})$. Using similar arguments, it follows that y may contain only elements J_r where r belong to the unique (unoriented) path between s_i and s_j. Moreover, to avoid application of a similar argument, all vertices from this path must occur. But then one can use, if necessary, relations in Proposition 28(m) to make y shorter. Hence y is empty and we may use relations in Proposition 28(l) to conclude that $w = 0$.

The case when Ω has only one vertex is dealt with similarly using relations in Proposition 28(c), (i) and (j) and also results in $w = 0$.

If Ω has at least two vertices, then a similar commutation procedure as above combined with the relations in Proposition 28(k) shows that w can be changed to an equivalent word u with the property that the only split symbols in u are those $J_{s_i}^{(q_i)}$ for which $s_i \in \mathbf{K}'(Q)$ and $\deg_\Omega(s_i) = 1$, moreover, each of them occurs exactly one time. Furthermore, on can use relations in Proposition 28(m) to ensure that u contains only J_t for $t \in \Omega$.

Let Ω' be the full subgraph of Ω with vertex set $\Omega \setminus \mathbf{K}'(Q)$. If Ω' is empty, then the above implies that u is a product of split symbols $J_{s_i}^{(q_i)}$ for which $s_i \in \mathbf{K}'(Q)$ and $\deg_\Omega(s_i) = 1$. If Ω has two vertices, they are necessarily connected and we can use relation Proposition 28(f) to see that there are exactly two possibilities for u, namely $J_{s_1}^{(q_1)} J_{s_2}^{(q_2)}$ and $J_{s_2}^{(q_2)} J_{s_1}^{(q_1)}$. These two elements are different in $\mathcal{I}^{\mathrm{ind}}$ since from (11) it follows that their action on $P_{s_1} \oplus P_{s_2}$ are different. If Ω has more than two vertices, then all factors in u commute and thus define u uniquely.

It remains to consider the case when Ω' is non-empty. Since Q is admissible, Ω' is a disjoint union of graphs of the form (8). Let $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ be the connected components of Ω'. Using relations given by Proposition 28(d) and (e), we can write $u = u_1 u_2 \dots u_m$, where each u_r, $r = 1, 2, \dots, m$, is a product of J_i or $J_s^{(q)}$ with $i, s \in \Gamma_r$ and $\big(\mathrm{supp}(J_s^{(q)}) \setminus \{s\}\big) \cap \Gamma_r \neq \varnothing$.

Now, for each u_i, the remaining relations from Proposition 28 guarantee that the J_i's and $J_s^{(q)}$'s appearing in u_i satisfy all relations for the corresponding Hecke-Kiselman monoid of type A (see [GM, Fo]). Therefore we can apply the same arguments as in the proof of Theorem 29. The statement of the theorem follows. $\square$

References

[Ba] H. Bass. Algebraic K-theory. W. A. Benjamin, Inc., New York-Amsterdam 1968.

[CR] J. Chuang, R. Rouquier. Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification. Ann. of Math. (2) **167** (2008), no. 1, 245–298.

[Fo] L. Forsberg. Effective representations of Hecke-Kiselman monoids of type A. Preprint arXiv:1205.0676.

[Fr] P. Freyd. Representations in abelian categories. Proc. Conf. Categorical Algebra (1966), 95–120.

[GM] O. Ganyushkin, V. Mazorchuk. On Kiselman quotients of 0-Hecke monoids. Int. Electron. J. Algebra **10** (2011), 174–191.

[Gr] A.-L. Grensing. Monoid algebras of projection functors. J. Algebra **369** (2012), 16–41.

[GrMa] A.-L. Grensing, V. Mazorchuk. Categorification of the Catalan monoid. Semigroup Forum **89** (2014), no. 1, 155–168.

[KhMa] O. Khomenko, V. Mazorchuk. On Arkhipov's and Enright's functors. Math. Z. **249** (2005), no. 2, 357–386.

[KL] M. Khovanov, A. Lauda. A categorification of a quantum $\mathfrak{sl}_n$. Quantum Topol. **1** (2010), 1–92.

[KuMa] G. Kudryavtseva, V. Mazorchuk. On multisemigroups. Preprint arxiv:1203.6224, to appear in Portugaliae Mathematica.

[Le] T. Leinster. Basic bicategories. Preprint arXiv:math/9810017

[McL] S. Mac Lane. Categories for the working mathematician. Second edition. Graduate Texts in Mathematics, **5**. Springer-Verlag, New York, 1998.

[Ma] V. Mazorchuk. Lectures on algebraic categorification. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012.

[MM1] V. Mazorchuk, V. Miemietz. Cell 2-representations of finitary 2-categories. Compositio Math. **147** (2011), 1519–1545.

[MM2] V. Mazorchuk, V. Miemietz. Additive versus abelian 2-representations of fiat 2-categories. Moscow Math. J. **14** (2014), No. 3, 595–615.

[MM3] V. Mazorchuk, V. Miemietz. Endomorphisms of cell 2-representations. Preprint arxiv:1207.6236.

[MM4] V. Mazorchuk, V. Miemietz. Morita theory for finitary 2-categories. Preprint arXiv:1304.4698. To appear in Quantum Topol.

[MM5] V. Mazorchuk, V. Miemietz. Transitive 2-representations of finitary 2-categories. Preprint arXiv:1404.7589. To appear in Trans. Amer. Math. Soc.

[MM6] V. Mazorchuk, V. Miemietz. Isotypic faithful 2-representations of $\mathcal{J}$-simple fiat 2-categories. Preprint arXiv:1408.6102.

[Pa] A.-L. Paasch. Monoidalgebren von Projektionsfunktoren. PhD Thesis, Wuppertal University, 2011.

[Ro] R. Rouquier. 2-Kac-Moody algebras. Preprint arXiv:0812.5023.

[Sk] Ø. I. Skartsæterhagen. Quivers and admissible relations of tensor products and trivial extensions. Master Thesis, NTNU Trondheim. Available at: https://daim.idi.ntnu.no/masteroppgave?id=6346

[We] B. Webster. Knot invariants and higher representation theory. Preprint arXiv:1309.3796

A.-L. G.: Faculty of Mathematics, Bielefeld University PO Box 100 131, D-33501, Bielefeld, Germany, `apaasch@math.uni-bielefeld.de`

V. M.: Department of Mathematics, Uppsala University, Box 480, SE-75106, Uppsala, SWEDEN, `mazor@math.uu.se`; http://www.math.uu.se/∼mazor/